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VARIETIES OF GROUPS AND OF COMPLETELY SIMPLE SEMIGROUPS

MARIO PETRICH AND NORMAN R. REILLY

Completely simple semigroups form a variety if we consider them both with the multiplication and the operation of inversion. Denote the lattice of all varieties of completely simple semigroups by $L(CS)$ and that of varieties of groups by $L(G)$. We prove that the mappings $V \to V \cap G$ and $V \to V \vee G$ are homomorphisms of $L(CS)$ onto $L(G)$ and the interval $[G, CS]$, respectively. The homomorphism $V \to (V \cap G, V \vee G)$ is an isomorphism of $L(CS)$ onto a subdirect product. We explore different properties of the congruences on $L(CS)$ induced by these homomorphisms.

1. Introduction and summary

The class of completely simple semigroups is one of the most studied objects in semigroup theory. If considered as a class of universal algebras with the given binary operation and the unary operation of inversion it becomes a variety given by a simple set of identities:

$$x = xx^{-1}x \ , \quad x = \left(x^{-1}\right)^{-1} \ , \quad xx^{-1} = x^{-1}x \ , \quad xx^{-1} = (xyx)(xyx)^{-1} \ .$$

The recent construction of the free completely simple semigroup due to Clifford [*1*] and Rasin [*6*] raised the hope that the varieties of completely simple semigroups can be determined *via* a description of fully invariant congruences on a free completely simple semigroup on a countably infinite set. Indeed, Rasin [*6*] characterized fully invariant congruences in terms

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of certain endomorphisms of the structure group of the free completely simple semigroup.

The present work represents a study of the lattice of varieties of completely simple semigroups by means of two homomorphisms of this lattice:

$$V \to V \cap G , \quad V \to V \vee G ,$$

where G stands for the variety of all groups. We prove that the combination of the two homomorphisms is an isomorphism of the lattice of varieties of completely simple semigroups onto a precisely described subdirect product. Various properties of the above homomorphisms, and the congruences they induce, are discussed in some detail.

Section 2 contains most of the preliminary material needed in the later sections. A characterization of the variety $V \cap G$ is described in Section 3. The homomorphism $V \to V \cap G$ is discussed in Section 4, and the homomorphism $V \to V \vee G$ in Section 5. Finally, in Section 6, the homomorphism $V \to (V \cap G, V \vee G)$ is proved to be an isomorphism onto a subdirect product.

We note that Kleĭman [3] has performed an analogous analysis for the lattice of varieties of inverse semigroups. There is a remarkable difference between the case of varieties of inverse semigroups and the varieties of completely simple semigroups: the mapping $V \to (V \cap G, V \vee G)$ for inverse semigroup varieties is not one-to-one.

2. Preliminaries

In general, we use the notation and terminology of Howie [2] or Petrich [5]. In particular, we adopt the notation in [5] for Rees matrix semigroups, and use the description of congruences on a Rees matrix semigroup presented in [2]. In order to minimize the typographical complexity we modify the standard notation for a sandwich matrix and denote the (j, k)th entry by $[j, k]$.

We will consistently use the following notation:

G - the variety of all groups,

RB - the variety of all rectangular bands,

RG - the variety of all rectangular groups (orthodox completely simple),

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CS - the variety of all completely simple semigroups,

$F(G)$ - the lattice of fully invariant subgroups of the group G,

$[A, B]$ - the interval of a lattice with minimum A and maximum B,

T_X - the semigroup of all transformations on a set X,

$L(V)$ - the lattice of all subvarieties of a variety V of completely simple semigroups,

End S - the semigroup of all endomorphisms of a semigroup S.

The first result provides a form for endomorphisms of a Rees matrix semigroup expressed by means of three unique parameters.

LEMMA 2.1 ([6]). *Let* $S = M(I, G, \Lambda; P)$ *, where* P *is normalized. Let* $\varphi \in T_I$ *,* $\omega \in \text{End } G$ *,* $\psi \in T_\Lambda$ *be such that*

(1) $[\lambda, i]\omega = [1\psi, 1\varphi][\lambda\psi, 1\varphi]^{-1}[\lambda\psi, i\varphi][1\psi, i\varphi]^{-1} \quad (\lambda \in \Lambda, i \in I)$.

Then $\theta = \theta(\varphi, \omega, \psi)$ *defined by*

$$(i, g, \lambda)\theta = \left(i\varphi, [1\psi, i\varphi]^{-1}(g\omega)[1\psi, 1\varphi][\lambda\psi, 1\varphi]^{-1}, \lambda\psi\right)$$

is an endomorphism of S *. Conversely, every endomorphism of* S *can be so written uniquely.*

A construction of the Rees matrix representation of a free completely simple semigroup follows.

LEMMA 2.2 ([1], [6]). *Let* $X = \{x_i \mid i \in I\}$ *be a nonempty set, fix* $1 \in I$ *and let* $I' = I\backslash\{1\}$ *. Let*

$$Z = \{q_i \mid i \in I\} \cup \{[j, k] \mid j, k \in I'\} ,$$

F_Z *be the free group on* Z *, and let* $P = ([j, k])$ *with* $[1, k] = [j, 1] = 1$ *, the identity of* F_Z *. Then*

$$F = M(I, F_Z, I; P)$$

is a free completely simple semigroup over X *, with embedding* $x_i \to (i, q_i, i)$.

NOTATION 2.3. We fix a countably infinite set X, and in addition to the above notation, introduce

$$F_q = \langle q_i \mid i \in I \rangle , \quad F_p = \langle [j, k] \mid j, k \in I' \rangle ,$$

the free subgroups of F_Z generated by the sets $\{q_i \mid i \in I\}$ and $\{[j, k] \mid j, k \in I'\}$, respectively. We will consistently use the notation $F = M(I, F_Z, I; P)$ introduced above.

Note that $F_Z = F_q * F_p$, the free product of F_q and F_p. As a consequence of Lemma 2.1, we have

COROLLARY 2.4. *If* $\theta(\varphi, \omega, \psi)$ *is an endomorphism of* F, *then* $F_p\omega \subseteq F_p$.

LEMMA 2.5 ([6]). *Any fully invariant congruence on* F *is either*

(i) idempotent separating or

(ii) a left group congruence or

(iii) a right group congruence or

(iv) a group congruence.

We will need only fully invariant idempotent separating congruences, for they are precisely the ones which correspond to the varieties in the interval $[RB, CS]$. In this context, the following special case of ([2], Lemma 4.19) is of particular interest.

LEMMA 2.6. *Let* $S = M(I, G, \Lambda; P)$. *If* N *is a normal subgroup of* G, *then* ρ_N *defined on* S *by*

$$(i, g, \lambda)\rho_N(j, h, \mu) \Leftrightarrow i = j , \quad gh^{-1} \in N , \quad \lambda = \mu ,$$

is an idempotent separating congruence on S, *and every such congruence is obtained in this way. Writing* P/N *for the* $\Lambda \times I$ *matrix with the* (j, k)th *entry equal to the* (j, k)th *entry of* P modulo N, S/ρ *is isomorphic to* $M(I, G/N, I; P/N)$.

NOTATION 2.7. We will consistently use the notation ρ_N introduced above. For a variety V of completely simple semigroups, we denote by $\rho(V)$ the fully invariant congruence on F corresponding to V. Also let

$$E(F_Z) = \{\omega \in \text{End } F_Z \mid \text{there exist } \varphi, \psi \in T_I \text{ such that (1) holds}\} ,$$

$$E(F_p) = \{\omega \in \text{End } F_p \mid \text{there exist } \varphi, \psi \in T_I \text{ such that (1) holds}\} .$$

Hence $E(F_Z)$ consists precisely of endomorphisms of F_Z that arise in association with endomorphisms of F. The latter are uniquely determined by the functions $\{q_i \mid i \in I\} \to F_Z$, $\varphi, \psi \in T_I$ independently. Furthermore, $E(F_p)$ consists precisely of endomorphisms of F_p that extend to elements of $E(F_Z)$.

LEMMA 2.8 ([6]). *Let* N *be a normal subgroup of* F_Z. *Then* ρ_N *is fully invariant if and only if* $N\omega \subseteq N$ *for all* $\omega \in E(F_Z)$.

DEFINITION 2.9. A normal subgroup of F_Z (respectively, F_p) is E-*invariant* if it is invariant under all $\omega \in E(F_Z)$ (respectively, $E(F_p)$). The set of all E-invariant subgroups of F_Z (respectively, F_p) will be denoted by N (respectively, N_p). For any $N \in \mathsf{N}$, let

$$N_q = N \cap F_q , \quad N_p = N \cap F_p .$$

It is clear that N (respectively, N_p) is a sublattice of the lattice of all normal subgroups of F_Z (respectively F_p), and that each element of N_p is the intersection with F_p of an element of N (for example, its normal closure in F_Z).

PROPOSITION 2.10 ([6]). *The interval* [RB, CS] *is a complete modular lattice anti-isomorphic to the lattice* N.

We take advantage of the basic results on varieties of groups as found in [4]. In particular, we recall that the lattice of group varieties is anti-isomorphic to the lattice of fully invariant subgroups of the free group F_X on a countable number of generators X.

NOTATION 2.11. If $\mathcal{U}$ is a group variety corresponding to the fully invariant subgroup N of F_X and G is any group, then the smallest normal subgroup H of G for which $G/H \in \mathcal{U}$ will be denoted by $N(G)$ or $\mathcal{U}(G)$.

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3. A characterization of $V \cap G$

We prove here some basic statements which will be used in later sections. In particular, we determine the fully invariant subgroup of F_q which corresponds to the variety $V \cap G$.

LEMMA 3.1. *Let* $N \in N$.

(i) N_q *is a fully invariant subgroup of* F_q .

Let U *be the corresponding group variety, so that* $U(F_q) = N_q$.

(ii) $U(F_Z) \subseteq N$.

(iii) $U(F_p) \subseteq N_p$.

Proof. *(i)* Any mapping of the free generators of F_Z into F_Z extends uniquely to an endomorphism of F_Z , and conversely, every endomorphism of F_Z is uniquely determined by its action on the free generators of F_Z . Condition (1) for membership in $E(F_Z)$ relates only to the free generators of F_p and is trivially satisfied if we choose φ and ψ to be the identity mappings.

It follows that any mapping of the free generators of F_q into F_q extends to an element of $E(F_Z)$. Consequently, any endomorphism of F_q extends to an element of $E(F_Z)$. Hence, N_q must be invariant under any endomorphism of F_q and is thus fully invariant in F_q .

(ii), *(iii)* In the same way, any mapping of the free generators of F_q into F_Z (or F_p) extends to an element of $E(F_Z)$. In particular, there exist κ, $\omega \in E(F_Z)$ which restrict to bijections of the free generators of F_q onto those of F_Z and F_p , respectively.

The hypothesis $N \in N$ implies

$$N_q\kappa \subseteq N \ , \quad N_q\omega \subseteq N \cap F_p = N_p \ .$$

The restrictions of κ and ω to F_q are isomorphisms of F_q onto F_Z

and F_p , respectively, and thus

$$U(F_Z) = U(F_q)\kappa = N_q\kappa \subseteq N ,$$

$$U(F_p) = U(F_q)\omega = N_q\omega \subseteq N_p ,$$

which completes the proof.

NOTATION 3.2. Let $V \in [RB, CS]$ and $\rho(V) = \rho_N$. Then $N_q = N \cap F_q$ is a fully invariant subgroup of F_q and so determines a variety of groups, which we denote by V_G .

We are now ready for the characterization theorem.

THEOREM 3.3. *If* $V \in [RB, CS]$ *and* $\rho(V) = \rho_N$ *, then* $V_G = V \cap G$.

Proof. The free group on a countable number of generators in V_G is simply F_q/N_q . Clearly, $F_q/N_q \in V \cap G$ and so $V_G \subseteq V \cap G$.

For the converse containment, let H be any group in $V \cap G$. Let $\{h_i \mid i \in I\}$ be any countable subset of H . Now $\{N_q q_i \mid i \in I\}$ is a set of relatively free generators of the relatively free group F_q/N_q . If we can show that there exists a homomorphism φ of F_q/N_q into H such that $(N_q q_i)\varphi = h_i$, for all $i \in I$, then we shall have, by the arbitrariness of H and the h_i , that every countably generated subgroup of any element of $V \cap G$ is a homomorphic image of F_q/N_q and therefore must satisfy all the laws of V_G . Hence, $V \cap G$ satisfies the laws of V_G and so $V \cap G \subseteq V_G$, as required.

We will show that such a homomorphism φ exists.

We start with the homomorphism θ of F into H defined on the generators by: $(i, q_i, i)\theta = h_i$. Since $(1, 1, 1)$ is an idempotent, we must have $(1, 1, 1)\theta = 1$. Hence

$$(1, q_i, 1)\theta = [(1, 1, 1)(i, q_i, i)(1, 1, 1)]\theta$$
$$= (i, q_i, i)\theta = h_i .$$

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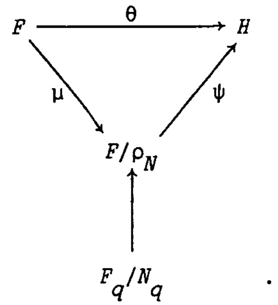
For ease of reference, let

$$F_{11} = \{(1, g, 1) \in F \mid g \in F_Z\} .$$

Since $H \in V$, we have $\theta \circ \theta^{-1} \supseteq \rho_N$, which implies that the homomorphism θ factors uniquely through F/ρ_N. Hence $\theta = \mu\psi$, where μ is the natural homomorphism of F onto F/ρ_N and ψ is a homomorphism of F/ρ_N into H. We illustrate this situation by the diagram



The homomorphism ψ is such that, for all $i \in I$,

$$(1, q_i, 1)\rho_N\psi = (1, q_i, 1)\mu\psi = (1, q_i, 1)\theta = h_i .$$

However, the mapping

$$\xi : N_q a \to (1, a, 1)\rho_N$$

of F_q/N_q into $F_{11}/\rho_N = F_{11}\mu$ is a monomorphism. Let $\varphi = \xi\psi$. Then φ is a homomorphism of F_q/N_q into H such that

$$(N_q q_i)\varphi = (1, q_i, 1)\rho_N\psi = h_i .$$

Thus φ is the required homomorphism.

4. The projection of $L(CS)$ onto $L(G)$

We explore here the relationship between varieties of completely simple semigroups and varieties of groups by considering the projection of $L(CS)$ onto $L(G)$ given by $V \to V \cap G$. First we introduce two mappings which will prove to be elements of $E(F_Z)$ and will play an important role in our discussion. Recall that $F_Z = F_q * F_p$.

NOTATION 4.1. Let π_q and π_p be the projections of F_Z onto F_q and F_p, respectively.

The next result summarizes the most salient features of these projections.

LEMMA 4.2. *(i)* $\pi_q, \pi_p \in E(F_Z)$.

(ii) If $N \in \mathsf{N}$, *then* $N\pi_q = N_q$, $N\pi_p = N_p$.

(iii) π_q *induces a lattice homomorphism of* N *onto* $\mathsf{F}(F_q)$

Proof. *(i)* It suffices to produce mappings $\varphi, \psi : I \to I$ in each case so that condition (1) is satisfied. For π_q take $i\varphi = i\psi = 1$ for all $i \in I$, and for π_p let $\varphi = \psi$ be the identity mapping on I. (In addition, one has $q_i\pi_q = q_i$ and $q_i\pi_p = 1$, for all $i \in I$.)

(ii) Let $N \in \mathsf{N}$. Then $N\pi_q \subseteq N$ since $\pi_q \in E(F_Z)$, and $N\pi_q \subseteq F_q$ by the definition of π_q, so that $N\pi_q \subseteq N \cap F_q$. On the other hand, since $N_q \subseteq F_q$, we have $N_q = N_q\pi_q \subseteq N\pi_q$, and thus $N\pi_q = N_q$. The same type of argument can be used to prove that $N\pi_p = N_p$.

(iii) Let $M, N \in \mathsf{N}$. By part *(ii)* and Lemma 3.1 *(i)*, we have $N\pi_q = N_q$ which is a fully invariant subgroup of F_q; thus $\pi_q : \mathsf{N} \to \mathsf{F}(F_q)$. In addition,

$$(M \cap N)\pi_q = (M \cap N)_q = M \cap N \cap F_q = (M \cap F_q) \cap (N \cap F_q)$$
$$= M\pi_q \cap N\pi_q ,$$

$$(M \vee N)\pi_q = (MN)\pi_q = (M\pi_q)(N\pi_q) = M\pi_q \vee N\pi_q ,$$

and thus π_q determines a lattice homomorphism of N into $\mathsf{F}(F_q)$. Since F_q is a free group on a countable number of generators, any $N \in \mathsf{F}(F_q)$ determines a variety of groups U, say. Then $M = \mathsf{U}(F_Z) \in \mathsf{N}$ and $M\pi_q = N$. Consequently, the homomorphism induced by π_q maps N onto $\mathsf{F}(F_q)$.

NOTATION 4.3. For any $V \in L(CS)$, let

$$\bar{V} = \{S \in CS \mid \text{all subgroups of } S \text{ are in } V\} .$$

It is readily verified that $\bar{V}$ is a variety of completely simple semigroups. We are now ready for the principal result of this section.

THEOREM 4.4. *The mapping*

$$\chi : V \to V \cap G \quad (V \in L(CS))$$

is a homomorphism of $L(CS)$ *onto* $L(G)$. *Denote by* α *the congruence induced by* χ . *For any* $V \in L(CS)$, *we have*

$$V\alpha = [V \cap G, \bar{V}] .$$

Proof. Let $V \in [RB, CS]$ be determined by the fully invariant congruence ρ_N on F . By Theorem 3.3, the variety of groups determined by the fully invariant subgroup N_q of F_q is just $V \cap G$. Combining the mappings

$$V \to \rho_N \to N \to N_q \to V \cap G$$

we obtain, by Lemma 4.2 *(iii)*, a homomorphism of $[RB, CS]$ onto $L(G)$. It is then straightforward to verify that this homomorphism extends to a homomorphism of $L(CS)$ onto $L(G)$.

The statement concerning $V\alpha$ needs no formal argument.

The rest of the section is devoted to characterizations of the maxima of α-classes in terms of identities and subgroups of F_Z . In the context of completely simple semigroups, group identities are written in the form $u = v$ with $u \neq 1 \neq v$. In the interest of simplicity, we will frequently abbreviate expressions of the form $u(x_1, \ldots, x_n)$ for words in the variables x_i to $u(x_i)$. However, for an identity $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$, it need not be the case that all variables appear on both sides of the identity.

LEMMA 4.5. *Let* $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ *be a group identity and* S *be a completely simple semigroup. Then all subgroups of* S *satisfy* $u(x_i) = v(x_i)$ *if and only if* S *satisfies* $u(\bar{x}_i) = v(\bar{x}_i)$, *where*

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$\bar{x}_i = \left(x_1x_1^{-1}\right)x_i\left(x_1x_1^{-1}\right)$, $1 \leq i \leq n$.

Proof. Assume that all subgroups of S satisfy the identity $u(x_i) = v(x_i)$. If we assign any value to the variables $x_1, x_2, \ldots, x_n$, all the variables $\bar{x}_i$ will be contained in the maximal subgroup of S containing x_1 . Since $u(x_i) = v(x_i)$ is valid in that subgroup, $u(\bar{x}_i) = v(\bar{x}_i)$ is valid in S .

Conversely, assume that S satisfies the identity $u(\bar{x}_i) = v(\bar{x}_i)$. If all variables $x_1, x_2, \ldots, x_n$ assume values in the same subgroup G of S , then $\bar{x}_i = x_i$ for $1 \leq i \leq n$, and G satisfies the identity $u(x_i) = v(x_i)$.

COROLLARY 4.6. *If* U *is a variety of groups given by the set of identities* $\{u_\alpha(x_i) = v_\alpha(x_i)\}_{\alpha\in A}$, *then* $\bar{U}$ *is determined by the system of identities* $\{u_\alpha(\bar{x}_i) = v_\alpha(\bar{x}_i)\}_{\alpha\in A}$, *where* $\bar{x}_i = (xx^{-1})x_i(xx^{-1})$ *and* x *is a fixed variable.*

NOTATION 4.7. For any $S \in CS$, let $\langle S\rangle$ denote the subvariety of CS generated by S .

PROPOSITION 4.8. *For any* $U \in L(G)$, *we have* $\bar{U} = \langle F/\rho_M\rangle$ *where* $M = U(F_Z)$.

Proof. First note that $M \in N$ and that $M_q = U(F_q)$ is the fully invariant subgroup of F_q corresponding to U . Theorem 3.3 then gives that $\langle F/\rho_M\rangle \cap G = U$. Let $V \in L(CS)$ be such that $V \cap G = U$. If V is a variety of groups, left or right groups, then clearly $V \subseteq \langle F/\rho_M\rangle$. Otherwise, let V be determined by the fully invariant congruence ρ_N on F . Since $V\chi = U$, we must have $M = U(F_Z) \subseteq N$. But then $\rho_M \subseteq \rho_N$ and thus $V \subseteq \langle F/\rho_M\rangle$. By the maximality of $\bar{U}$, the result follows.

Indeed, we see from Proposition 4.8 that

$$F/\rho_M \cong M(I, F_Z/M, I; P/M)$$

is a relatively free object in $\overline{U}$.

COROLLARY 4.9. *Let* $V \in [RB, CS]$ *and* $\rho(V) = \rho_N$. *Then* V *is the maximum element of its* α*-class if and only if* N *is a fully invariant subgroup of* F_Z .

Proof. That N is fully invariant if V is the maximum element of its α-class follows immediately from Proposition 4.8. For the converse, suppose that N is fully invariant and let $\rho(\overline{V}) = \rho_M$. By Proposition 4.8, M is also fully invariant so that, by Theorem 3.3, we have

$$\langle F_q/N_q \rangle = V \cap G = \overline{V} \cap G = \langle F_q/M_q \rangle .$$

Thus, in the notation of 2.11,

$$N(F_q) = N_q = M_q = M(F_q)$$

from which it follows that $N = M$ and $V = \overline{V}$, as required.

COROLLARY 4.10. *The varieties that are maximum (respectively, minimum) in their* α*-classes form a sublattice of* $L(CS)$.

Proof. The varieties that are minimum in their α-classes are simply the group varieties and so constitute a sublattice. By Corollary 4.9, the maximum elements correspond to the fully invariant subgroups of F_Z . Since these form a sublattice of N , it follows that the maximum elements form a sublattice of $L(CS)$.

5. The projection of $L(CS)$ onto $[G, CS]$

We now turn to the study of the relationship of the lattice $L(CS)$ and its interval $[G, CS]$ *via* the homomorphism $V \to V \vee G$. We then characterize the maximal elements of the congruence on $L(CS)$ induced by this homomorphism in two different ways.

NOTATION 5.1. Let $\hat{F}_p$ denote the normal closure in F_Z of F_p . For any $N \in N$, let $N_p^* = N \cap \hat{F}_p$ and let

$$N_p^* = \{N_p^* \mid N \in N\} .$$

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The following lemma supplies the necessary information for the main result of this section.

LEMMA 5.2. *(i)* $N_p^* \subseteq N$.

(ii) $\rho(RG) = \rho_{\hat{F}_p}$.

(iii) $F_Z = F_q \hat{F}_p$.

(iv) $F_q \cap \hat{F}_p = \{1\}$.

(v) For any $N \in N$ *, we have* $N = N_q N_p^*$.

Proof. *(i)* Since any endomorphism in $E(F_Z)$ maps F_p into F_p , it must also map $\hat{F}_p$ into itself. But then it must map $N \cap \hat{F}_p$ into itself, for any $N \in N$.

(ii) For $N = \hat{F}_p$, it is clear that F/ρ_N is a rectangular group. On the other hand, if $N \in N$ is such that F/ρ_N is a rectangular group, then $F_p \subseteq N$, and since N is normal, we have $\hat{F}_p \subseteq N$.

(iii) This is a consequence of the fact that $F_Z = F_q * F_p$.

(iv) Consider the projection $\pi_q : F_Z \to F_q$. Its kernel is $\hat{F}_p$ and it maps F_q identically, whence the assertion.

(v) Let $N \in N$ and $n \in N$. By part *(iii)*, $n = ab$ for some $a \in F_q$, $b \in \hat{F}_p$. Hence

$$a = (ab)\pi_q \in N \cap F_q = N_q$$

so that

$$b = a^{-1}(ab) \in N \cap \hat{F}_p = N_p^* .$$

Thus $N \subseteq N_q N_p^*$, and the opposite inclusion is trivial.

We first deduce two interesting consequences.

COROLLARY 5.3. *For* $V \in [RB, CS]$ *with* $\rho(V) = \rho_N$ *, we have*

$\rho(V \vee G) = \rho_{N_p^*}$. *Consequently,* $RG \subseteq V$ *if and only if* $N \subseteq \hat{F}_p$.

Proof. The hypothesis $V \supseteq RB$ yields

$$V \vee G = V \vee RB \vee G = V \vee RG ,$$

whence, by Lemma 5.2 *(ii)*, we get

$$\rho(V \vee G) = \rho(V \vee RG) = \rho(V) \cap \rho(RG)$$
$$= \rho_N \cap \rho_{\hat{F}_p} = \rho_{N \cap \hat{F}_p} = \rho_{N_p^*} .$$

COROLLARY 5.4. *For* $U, V \in [RB, CS]$ *with* $\rho(U) = \rho_M$ *and* $\rho(V) = \rho_N$ *, we have*

$$U \vee G = V \vee G \Leftrightarrow M_p^* = N_p^* .$$

We are now ready for the principal result of this section.

THEOREM 5.5. *The mapping*

$$\theta : V \to V \vee G \quad (V \in L(CS))$$

is a homomorphism of $L(CS)$ *onto* $[G, CS]$.

Proof. We first consider θ on the interval $[RB, CS]$. Recall that this interval is anti-isomorphic to $\mathcal{N}$. In the light of Corollary 5.3, for any $V \in [RB, CS]$ with $\rho(V) = \rho_N$, we have $\rho(V \vee G) = \rho_{N_p^*}$. Hence it suffices here to show that the mapping

$$\mu : N \to N_p^* = N \cap \hat{F}_p$$

is a homomorphism of $\mathcal{N}$ onto N_p^* .

Obviously μ maps $\mathcal{N}$ into N_p^* and preserves meets. The onto part is a consequence of Lemma 5.2 *(i)*. It remains to show that for any $M, N \in \mathcal{N}$, we have

$$(M \vee N)\mu = M\mu \vee N\mu ,$$

that is, $(MN)_p^* = M_p^* N_p^*$.

Let $a \in (MN)_p^*$, say $a = mn$, where $m \in M$, $n \in N$. By Lemma 5.2 *(v)*, we have $m = m_1 m_2$ and $n = n_1 n_2$ with $m_1, n_1 \in F_q$,

$m_2, n_2 \in \hat{F}_p$. Then

(2) $$mn = m_1m_2n_1n_2 = \left(m_1m_2m_1^{-1}\right)\left[(m_1n_1)n_2(m_1n_1)^{-1}\right]m_1n_1 .$$

Note that in (2), $m_2, n_2 \in \hat{F}_p$, which is normal, and also $mn = a \in (MN)_p^*$, which implies that $m_1n_1 \in \hat{F}_p$. But then, by Lemma 5.2 *(iv)*, we get $m_1n_1 \in F_q \cap \hat{F}_p = \{1\}$. Consequently,

$$a = mn = \left(m_1m_2m_1^{-1}\right)n_2 \in M_p^*N_p^* .$$

This proves that $(MN)_p^* \subseteq M_p^*N_p^*$; the opposite inclusion is obvious. Therefore θ is a homomorphism on the interval $[RB, CS]$.

To see that θ is a homomorphism on the entire lattice $L(CS)$, we consider $U \in L(RG)$ and $V \in L(CS)$. It is straightforward to verify that the following is true:

$$\begin{aligned}(U \cap V) \vee G &= \{S \in CS \mid S \cong G \times R,\ G \in G,\ R \in (U \cap V) \cap RB\} \\ &= (U \vee G) \cap (V \vee G) .\end{aligned}$$

Therefore θ is indeed a homomorphism of $L(CS)$ onto $[G, CS]$.

We have seen in Theorem 4.4 that the congruence α induced by the homomorphism $V \to V \cap G$ has the property that its classes are intervals of $L(CS)$. We conjecture that the congruence β induced on $L(CS)$ by the homomorphism $V \to V \vee G$ also has this property. We are unable to prove the existence of the least element of the β-class containing an arbitrary variety V , but observe that the greatest element is obviously $V \vee G$. In Corollary 5.3, we have already characterized the corresponding element of N . We now turn to the description of $V \vee G$ in terms of the system of identities it satisfies.

NOTATION 5.6. In any group G , denote by x^a the conjugate $a^{-1}xa$ of x . For $i_t \neq 1$, $j_t \neq 1$, $q_k \in F_q$,

(3) $$v = v\left([i_1, j_1]^{f_1(q_1,\ldots,q_n)}, \ldots, [i_m, j_m]^{f_m(q_1,\ldots,q_n)}\right) \in \hat{F}_p ,$$

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(4) $$\hat{v} = v\left(z_{i_1,j_1}^{f_1(\bar{x}_1,\ldots,\bar{x}_n)}, \ldots, z_{i_m,j_m}^{f_m(\bar{x}_1,\ldots,\bar{x}_n)}\right)$$

where

(5) $$\bar{x}_k = xx^{-1}x_kxx^{-1}, \quad z_{i_t,j_t} = (xy_{i_t})(xy_{i_t})^{-1}(y_{j_t}x)(y_{j_t}x)^{-1}$$

for variables x, x_k, y_{i_t}, y_{j_t} $(k = 1, 2, \ldots, n, \quad t = 1, 2, \ldots, m)$.

LEMMA 5.7. *Let* $V \in [RB, CS]$, $\rho(V) = \rho_N$; *then*

$$N_p^* = \left\{v \in \hat{F}_p \mid \hat{v}^2 = \hat{v} \text{ is a law in } V\right\}.$$

Proof. Let $v \in N_p^*$ be given by (3). Consider any substitution of the variables in $\hat{v}$, see (4), into F:

$$x \to a, \quad x_k \to a_k, \quad y_{i_t} \to b_{\alpha_t}, \quad y_{j_t} \to b_{\beta_t}.$$

Let $\hat{v}^\sigma$, z_{i_t,j_t}^σ and so on, denote the elements obtained from $\hat{v}$, z_{i_t,j_t} and so on, see (5), by making these substitutions. Without loss of generality, we may assume that $b_{\alpha_t} \in H_{-\alpha_t}$, $b_{\beta_t} \in H_{\beta_t-}$, where $H_{-\alpha_t}$, H_{β_t-} need not all be distinct even for distinct α_t or β_t.

Let $a \in H_{rs}$ and note that

$$\xi : (r, h, s) \to [s, r]h \quad (h \in F_Z)$$

is an isomorphism of H_{rs} onto F_Z. Then

(6) $$\bar{x}_k^\sigma \in H_{rs}, \quad g_k = \bar{x}_k^\sigma\xi \in F_Z,$$

where the latter part of (6) defines g_k, and

$$\begin{aligned} z_{i_t,j_t}^\sigma &= (ab_{\alpha_t})(ab_{\alpha_t})^{-1}(b_{\beta_t}a)(b_{\beta_t}a)^{-1} \\ &= \left(r, [\alpha_t, r]^{-1}, \alpha_t\right)\left(\beta_t, [s, \beta_t]^{-1}, s\right) \\ &= \left(r, [\alpha_t, r]^{-1}[\alpha_t, \beta_t][s, \beta_t]^{-1}, s\right) \end{aligned}$$

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so that

(7) $$z^{\sigma}_{i_t, j_t}\xi = [s, r][\alpha_t, r]^{-1}[\alpha_t, \beta_t][s, \beta_t]^{-1} .$$

Taking into account (6) and (7), we obtain

(8) $$\hat{v}^{\sigma}\xi = \left(\left\{[s, r][\alpha_1, r]^{-1}[\alpha_1, \beta_1][s, \beta_1]\right\}^{f_1(g_1,\ldots,g_n)} \ldots \left\{[s, r][\alpha_m, r]^{-1}[\alpha_m, \beta_m][s, \beta_m]\right\}^{f_m(g_1,\ldots,g_n)}\right) .$$

Let $\omega \in E(F_Z)$ be defined by (1) and

$$q_k\omega = g_k , \quad i_t\psi = \alpha_t , \quad j_t\varphi = \beta_t , \quad 1\psi = r , \quad 1\varphi = s ,$$

and let ψ and φ be defined arbitrarily elsewhere. Using (1), we obtain

$$[i_t, j_t]^{f_t(q_1,\ldots,q_n)}\omega = \left([i_t, j_t]\omega\right)^{f_t(q_1\omega,\ldots,q_n\omega)}$$
$$= \left([s, r][\alpha_t, r]^{-1}[\alpha_t, \beta_t][s, \beta_t]^{-1}\right)^{f_t(g_1,\ldots,g_n)} .$$

Comparing this with (8), we conclude that $\hat{v}^{\sigma}\xi = v\omega$, where $v\omega \in N_p^*$ since $v \in N_p^*$ and $N_p^* \in N$, by Lemma 5.2 *(i)*. Since

$$[s, r]^{-1}(v\omega)^2\left([s, r]^{-1}(v\omega)\right)^{-1} = [s, r]^{-1}(v\omega)[s, r] \in N_p^* ,$$

Lemma 2.6 yields

$$\left(r, [s, r]^{-1}(v\omega)^2, s\right)\rho_{N_p^*}\left(r, [s, r]^{-1}(v\omega), s\right) .$$

Using $N_p^* \subseteq N$ and the definition of ξ , we deduce $\left(\hat{v}^{\sigma}\right)^2\rho_N\hat{v}^{\sigma}$ and thus $\hat{v}^2 = \hat{v}$ is a law in V .

Conversely, suppose that $\hat{v}^2 = \hat{v}$ is a law in V . Consider the substitution

$$\sigma : x \to (1, 1, 1) , \quad x_i \to (1, q_i, 1) ,$$
$$y_{i_t} \to (i_t, 1, i_t) , \quad y_{j_t} \to (j_t, 1, j_t) .$$

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Then

$$\bar{x}_i^{\sigma} = (1, q_i, 1) \ , \quad z_{i_t, j_t} = (1, |i_t, j_t|, 1)$$

and thus $\hat{v}^{\sigma} = (1, v, 1)$. Since $\hat{v}^2 = \hat{v}$ is a law in V , we get $(1, v, 1)^2 \rho_N (1, v, 1)$ and so $Nv^2 = Nv$. But then $v \in N \cap \hat{F}_p = N_p^*$, as required.

PROPOSITION 5.8. *Let* $V \in [RB, CS]$, $\rho(V) = \rho_N$. *Then* $\{\hat{v}^2 = \hat{v} \mid v \in N_p^*\}$ *is a basis of laws for* $V \vee G$.

Proof. This is immediate from Corollary 5.4 and Lemma 5.7.

It is a simple consequence of Proposition 5.8 that

$$F/\rho_{N_p^*} \cong M(I, F_Z/N_p^*, I; P/N_p^*)$$

is a relatively free object in $V \vee G$.

6. Embedding of $L(CS)$ into a subdirect product

We combine here the homomorphism χ of Theorem 4.4 with the homomorphism θ of Theorem 5.5 and prove that the resulting mapping $V \to (V \cap G, V \vee G)$ is actually an isomorphism of $L(CS)$ onto a subdirect product of $L(G)$ and $[G, CS]$.

THEOREM 6.1. *The mapping*

$$\xi : V \to (V \cap G, V \vee G) \quad (V \in L(CS))$$

is an isomorphism of $L(CS)$ *onto the subdirect product of* $L(G)$ *and* $[G, CS]$ *consisting of the pairs* (U, V) *such that* $V \subseteq \overline{U} \vee G$. *Moreover, for* $W \in L(CS)$,

$$W \cap G = U \ , \quad W \vee G = V \Leftrightarrow W = \overline{U} \cap V \ .$$

Proof. Since χ and θ are homomorphisms (Theorems 4.4 and 5.5) so also is ξ . Let $V, W \in [RB, CS]$ and $\rho(V) = \rho_M$, $\rho(W) = \rho_N$. Then, by Corollary 5.3,

$$\rho(V \vee G) = \rho_{M_p^*} \ , \quad \rho(W \vee G) = \rho_{N_p^*} \ . \tag{9}$$

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On the other hand, by Theorem 3.3,

(10) $$V \cap G = \langle F_q/M_q \rangle \ , \quad W \cap G = \langle F_q/N_q \rangle \ .$$

If $V\xi = W\xi$, then from (9) and (10), we have

$$M_q = N_q \ , \quad M_p^* = N_p^* \ ,$$

and from Lemma 5.2 *(v)* it follows that

$$M = M_q M_p^* = N_q N_p^* = N \ .$$

Therefore $V = W$ and ξ is one-to-one on $[RB, CS]$.

If either of V or W does not contain RB , then it must be a variety of left groups or a variety of right groups (or a variety of groups) and a simple case-by-case argument will again show that $V\xi = W\xi$ implies that $V = W$. Therefore ξ is an isomorphism.

For $W \vee L(CS)$, let $U = W \cap G$ and $V = W \vee G$. Then $W \subseteq \overline{U}$ and so $V = W \vee G \subseteq \overline{U} \vee G$.

Conversely, let $(U, V) \in L(G) \times [G, CS]$ with $V \subseteq \overline{U} \vee G$, and let $W = \overline{U} \cap V$. We have

$$W \cap G = \overline{U} \cap V \cap G = \overline{U} \cap G = U \ .$$

Now consider $W \vee G$. Then

$$W \vee G = (\overline{U} \cap V) \vee G \subseteq V$$

since $G \subseteq V$. For the opposite inclusion, first assume that $RG \subseteq V$. Then clearly $RB \subseteq W$. Let $\rho(\overline{U}) = \rho_M$ and $\rho(V) = \rho_N$; also let $m \in M$, $n \in N$ be such that $mn \in \hat{F}_p$ so that $mn \in (MN)_p^*$. The hypothesis $RG \subseteq V$ implies that $n \in N \subseteq \hat{F}_p$, and $V \subseteq \overline{U} \vee G$ implies that $\rho_{M_p^*} = \rho_{M \cap \hat{F}_p} = \rho(\overline{U} \vee G) \subseteq \rho_N$ so that $M_p^* \subseteq N$. Consequently

$$m = (mn)n^{-1} \in M \cap \hat{F}_p = M_p^* \subseteq N$$

and thus $mn \in N$, which proves that $(MN)_p^* \subseteq N$. But then

$$\rho(W \vee G) = \rho(W \vee RG) = \rho_{MN \cap \hat{F}_p} = \rho_{(MN)_p^*} \subseteq \rho_N = \rho(V)$$

and hence $V \subseteq W \vee G$.

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If V does not contain RG, then we must have V equal to G, or to the variety of left groups or the variety of right groups. Particular, but simpler, arguments will show that $W \vee G = V$ in each of these cases.

We have thus established the converse part of the implication in the statement of the theorem. This establishes that ξ maps $L(CS)$ onto the sublattice $\{(U, V) \mid U \in L(G),\ G \subseteq V \subseteq \overline{U} \vee G\}$.

The direct part of the implication follows from the fact that ξ is one-to-one.

COROLLARY 6.2. *For any* $V \in L(CS)$, *we have*

$$V = \overline{V \cap G} \cap (V \vee G) .$$

REMARK 6.3. By Theorem 3.3 and Corollary 5.3, we have the following associations for any $V \in L(CS)$:

$$V \to \rho(V) = \rho_N \nearrow N_q = N \cap F_q \to V \cap G$$

$$V \to \rho(V) = \rho_N \searrow N_p^* = N \cap \hat{F}_p \to V \vee G .$$

References

[1] A.H. Clifford, "The free completely regular semigroup on a set", *J. Algebra* 59 (1979), 434-451.

[2] J.M. Howie, *An introduction to semigroup theory* (London Mathematical Society Monographs, 7. Academic Press, London, New York, San Francisco, 1976).

[3] Е.И. Клейман [E.I. Kleĭman], "Некоторые сзойства структуры многообразий инверсных полугрупп" [Some properties of the lattice of varieties of inverse semigroups], *Ural. Gos Univ. Mat. Zap.* 10 (1977), no. 3 *Issled. po Sovremen. Algebre*, 56-72, 216-217. "Замечание к статье Е.И. Клеймана 'Некоторые свойства структуры многообразий инверсных полугрупп'" [Correction to Some properties of the lattice of varieties of inverse semigroups], *Ural. Gos. Univ. Mat. Zap.* 11 (1979), 207.

[4] Hanna Neumann, *Varieties of groups* (Ergebnisse der Mathematik und ihrer Grenzgebiete, 37. Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[5] Mario Petrich, *Introduction to semigroups* (Charles E. Merrill, Columbus, Ohio, 1973).

[6] В.В. Расин [V.V. Rasin], "Свободные вполне простые полугруппы" [Free completely simple semigroups], *Ural. Gos. Univ. Mat. Zap.* 11 (1979), no. 3, 140-151.

Department of Mathematics,
Simon Fraser University,
Burnaby,
British Columbia V5A 1S6,
Canada.