INCIDENCE RELATIONS IN MULTICOHERENT SPACES II

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Introduction. One standard method of studying the incidences of a system of sets $A_1, A_2, \ldots, A_n$ is to consider the nerve $\mathfrak{N}$ of the system. However, this gives no direct information as to the numbers of components of the various intersections of the sets—information which would be desirable in several geometrical problems. The object of the present paper is to modify the definition of the nerve so that these numbers of components can be taken into account, and to study this *modified nerve* $\mathfrak{M}$ for systems of sets in a connected, locally connected, normal T_1 space S of a given degree of multicoherence[1] $r(S)$. The principal result (Theorem 6, 6.4) is a refinement of a theorem of Eilenberg [4, p. 107], and asserts that, if $\bigcup A_i = S$, then under suitable hypotheses we have

$$r(\mathfrak{N}) \leqslant r(\mathfrak{M}) \leqslant r(S). \tag{1}$$

This theorem has several geometrical applications, but we shall have to leave these for subsequent treatment.

The proof proceeds as follows. After the necessary definitions (§**1**), we show (§**2**) that the modified nerve $\mathfrak{M}$ is conveniently related to the family of (continuous) mappings of S in the unit circle S^1. Next it is shown (§§**3-5**) that the *analytic degree of multicoherence*[2] $\rho(S)$ is equal to $r(S)$ even at the present generality; the proof, which makes frequent use of modified nerves, depends essentially on first obtaining (1) for the case in which $\mathfrak{M}$ and $\mathfrak{N}$ are 1-dimensional. The *analytic* technique of Borsuk and Eilenberg is then applied to deduce (1) in full generality, and to yield a few related results.

Though it will be clear that much of the work does not require the assumption of local connectedness, we shall use S throughout the paper to denote a non-empty, connected, locally connected, normal T_1 space. For notations in general we refer to [9] and [10].

1. The modified nerve

1.1. *Definitions, etc.* Let $A_1, A_2, \ldots, A_n$ be n given subsets of S. For each non-empty subset $J = \{i_1, i_2, \ldots, i_\nu\}$ of the set I of all integers from 1

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[1]Here $r(S) = \sup b_0(A \cap B)$, where A and B are closed connected sets such that $A \cup B = S$; the definition of b_0 is given below (footnote 4). For the fundamental properties of $r(S)$, see [3, 4, 12] in the bibliography at the end of the paper; for notations in general, see [9, 10]. In [10] the space S was assumed in addition to be completely normal; but as indicated in [10, 6.6(3)], this extra assumption is not needed for the results which will be quoted here.

[2]This notation follows [12, p. 229].

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to n, we shall write A_J as an abbreviation for $A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_\nu}$. By a *decomposition system* (abbreviated to d.s.) $\mathfrak{D} = \{A_J^\alpha\}$ of the system $A_1, A_2, \ldots, A_n$, we shall mean a decomposition of each A_J into a finite number[3] (possibly zero) of pairwise separated sets A_J^α with $\alpha = 1, 2, \ldots, \alpha(J)$ (so that, for each fixed J, we have $\bigcup A_J^\alpha = A_J$, $A_J^\alpha \cap A_J^\beta = 0$ if $\alpha \neq \beta$, and A_J^α is both open and closed relative to A_J), in such a way that the following "consistency" criterion is satisfied:

(1) Given α, J and J' such that $J' \subset J$, there exists α' such that $A_{J'}^{\alpha'} \supset A_J^\alpha$. (It follows that α' is unique, unless $A_J^\alpha = 0$.)

The sets $A_1, A_2, \ldots, A_n$ always have a *trivial* d.s. in which every $\alpha(J) = 1$ and $A_J^1 = A_J$. If further $A_1, A_2, \ldots, A_n$ satisfy[4] $b_0(A_J) < \infty$ for every J—or, as we shall say, if they are *of finite incidence*—they have a *natural* d.s., defined by taking the sets A_J^α to be the components of A_J. We shall be mainly interested in natural d.s.'s, though more general ones will sometimes have to be taken into account.

1.2. Corresponding to every d.s. $\mathfrak{D}$ of $A_1, A_2, \ldots, A_n$, we construct a complex $\mathfrak{M}(\mathfrak{D})$, the modified nerve of the decomposition, as follows. To each non-empty $A_{(j)}^\alpha$ we assign a vertex $a_{(j)}^\alpha$ of $\mathfrak{M}(\mathfrak{D})$ $(1 \leqslant j \leqslant n)$, and generally to each *non-empty* A_J^α we assign an open simplex a_J^α of $\mathfrak{M}(\mathfrak{D})$ having as vertices those points $a_{(j)}^{\alpha'}$ for which $j \in J$ and $A_J^\alpha \subset A_{(j)}^{\alpha'}$ (in accordance with (1) above). The *faces* of a_J^α are defined to be those simplexes $a_{J'}^{\alpha'}$ for which $J' \subset J$ and $A_{J'}^{\alpha'} \supset A_J^\alpha$; thus, for given α, J and J', there is exactly one face $a_{J'}^{\alpha'}$. With the obvious definition of incidence numbers, $\mathfrak{M}(\mathfrak{D})$ is a complex [6, p. 89] but not in general a simplicial complex [6, p. 92] (since several distinct simplexes may have identical vertices), though it becomes one on barycentric subdivision [8, p, 50]. We shall suppose $\mathfrak{M}(\mathfrak{D})$ to be realized geometrically ,and shall use $\mathfrak{M}(\mathfrak{D})$ to denote also the resulting (curved) polytope.

For the trivial d.s., $\mathfrak{M}(\mathfrak{D})$ reduces to the usual *nerve*, $\mathfrak{N}$ of $A_1, A_2, \ldots, A_n$. If the sets A_j have finite incidence and $\mathfrak{D}$ is the natural d.s., we shall write $\mathfrak{M}(\mathfrak{D})$ simply as $\mathfrak{M}$, and refer to $\mathfrak{M}$ as "the" modified nerve[5] of $A_1, A_2, \ldots, A_n$.

1.3. THEOREM 1. *Let $\mathfrak{N}$ be the nerve and $\mathfrak{M}$ the modified nerve of a system of connected sets $A_1, A_2, \ldots, A_n$ of finite incidence, and suppose that $A_j - A_k$ and $A_k - A_j$ are always separated*[6] $(1 \leqslant j, k \leqslant n)$. *Then $b_0(\mathfrak{M}) = b_0(\mathfrak{N}) = b_0(\bigcup A_j)$; and if $\bigcup A_j$, and therefore also $\mathfrak{M}$ and $\mathfrak{N}$, are connected, we have $r(\mathfrak{M}) \geqslant r(\mathfrak{N})$.*

Proof. We omit the easy argument showing that $b_0(\mathfrak{M}) = b_0(\bigcup A_j) = b_0(\mathfrak{N})$.

[3]It would be easy to extend these considerations to suitable infinite decompositions; cf. 5.3 below.

[4]Following [3], $b_0(X) + 1$ = number of components of X, if this number is finite, and $b_0(X) = \infty$ otherwise; in particular, $b_0(O) = -1$.

[5]Though $\mathfrak{M}$ consists, roughly, of $\mathfrak{N}$ with repeated cells, $\mathfrak{M}$ need not contain any subcomplex isomorphic with $\mathfrak{N}$.

[6]This condition (introduced in [11]) will always be satisfied if the sets A_j are all open, or all closed, relative to their union.

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To prove $r(\mathfrak{M}) \geqslant r(\mathfrak{N})$, let the vertices of $\mathfrak{M}$ (as in 1.2) be $a_1^1, a_2^1, \ldots, a_n^1$, and let those of $\mathfrak{N}$ be $a_1, a_2, \ldots, a_n$, a_j^1 and a_j both corresponding to the connected set A_j. There exists an obvious simplicial mapping f of $\mathfrak{M}$ onto $\mathfrak{N}$ such that $f(a_j^1) = a_j$, and it is easy to see that any closed edge-path in $\mathfrak{N}$ is the image under f of at least one closed edge-path in $\mathfrak{M}$. Thus f induces a homomorphism of $\pi_1(\mathfrak{M})$ onto $\pi_1(\mathfrak{N})$, π_1 denoting the fundamental group. By a theorem of Eilenberg [4, p. 110] there is a homomorphism of $\pi_1(\mathfrak{N})$, and thus also of $\pi_1(\mathfrak{M})$, onto the free (non-abelian) group with $r(\mathfrak{N})$ generators; and hence [4, p. 110] $r(\mathfrak{M}) \geqslant r(\mathfrak{N})$.

2. Mappings in S^1

2.1. In what follows, f, g, etc. will denote (continuous) mappings of some normal space X (usually a subset of S) in the space S^1 of complex numbers z with $|z| = 1$; and ϕ, ψ, etc. will similarly denote continuous real-valued functions on X. To save notation, we shall usually not distinguish between a mapping $f : X \to S^1$ and the "partial mapping" $f \mid X'$ (f restricted to X') where $X' \subset X$. For the convenience of the reader, we repeat the following definitions (cf. [2], [3], [12, ch. 11]).

The *product* fg is defined by $fg(x) = f(x)g(x)$, the multiplication on the right being that of ordinary complex numbers; and the powers f^q ($q = 0, \pm 1, \pm 2, \ldots$) are defined similarly. If there exists ϕ such that $f(x) = g(x) \exp (i\phi(x))$ for all $x \in X$, we write $f \sim g$ on X; in particular, if $f(x) = \exp (i\phi(x))$ we write $f \sim 1$ on X. Mappings $f_1, f_2, \ldots, f_n$ are said to be (linearly) *dependent* on X if integers $q_1, q_2, \ldots, q_n$ exist, positive or negative but not all zero, such that $f_1^{q_1} f_2^{q_2} \ldots f_n^{q_n} \sim 1$ on X; otherwise they are *independent* on X. If $X = S$, the qualifying phrases "on X" will generally be omitted.

Given n sets $A_1, A_2, \ldots, A_n$, the greatest number of mappings f of $X = \bigcup A_j$ in S^1 which satisfy

$$(1) \qquad f \sim 1 \text{ on } A_j, \qquad 1 \leqslant j \leqslant n$$

and which are independent on X (or ∞ if there is no such greatest number) is written $p(A_1, A_2, \ldots, A_n)$.

Finally, the supremum of $p(F_1, F_2,)$ as F_1, F_2 range over all pairs of closed sets (not necessarily connected) such that $F_1 \cup F_2 = S$, is denoted by $\rho(S)$. It is known ([3, p. 172], [4, p. 113]) that $\rho(S) = r(S)$, provided that S is a Peano space or infinite polytope; we shall later be able to remove this proviso.

2.2. Many of the arguments and results in [2], [3] (in which the space X is assumed to be metric) apply here also with, at most, trivial changes. In particular:

(1) If f maps $A \cup B$ in S^1, where the sets $A - B$ and $B - A$ are separated and $A \cap B$ is connected, and if $f \sim 1$ on A and $f \sim 1$ on B, then $f \sim 1$ on $A \cup B$ [2, p. 64, (5)].

(2) If f maps X in S^1, where X is normal, and if A is a (relatively) closed

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subset of X on which $f \sim 1$, there exists a relatively open subset U of X such that $U \supset A$ and $f \sim 1$ on U([2, p. 65 (6)]; here the proof needs modification, and uses the fact that the real line is an AR [6, p. 28]).

(3) If f, g both map X in S^1 and $|f(x) - g(x)| < 1$ for each $x \in X$, then $f \sim g$ on X [3, p. 156, (2)].

(4) If f maps a closed simplex E in S^1, then $f \sim 1$ on E.

2.3. There is a close connection between modified nerves and mappings in S^1, as is shown by:

THEOREM 2. *Let $\mathfrak{M}$ be the modified nerve of a system of closed sets $A_1, A_2, \ldots, A_n$ of finite incidence. Then*[7] $b_1(\mathfrak{M}) = p(A_1, A_2, \ldots, A_n)$.

We prove (and shall need) a little more than this:

(1) If $A_1, A_2, \ldots, A_n$ are of finite incidence and such that $A_j - A_k$ and $A_k - A_j$ are always separated (but are not necessarily closed), then $b_1(\mathfrak{M}) \geqslant p(A_1, A_2, \ldots, A_n)$.

(2) If $A_1, A_2, \ldots, A_n$ are closed (but not necessarily of finite incidence), and if $\mathfrak{M} = \mathfrak{M}(\mathfrak{D})$ is the modified nerve corresponding to a d.s. $\mathfrak{D}$ of $A_1, A_2, \ldots, A_n$, then $b_1(\mathfrak{M}) \leqslant p(A_1, A_2, \ldots, A_n)$.

2.4. *Proof of* (1). First, to each mapping f of $\bigcup A_j$ in S^1 such that $f \sim 1$ on each A_j, we can assign a 1-cocycle class on $\mathfrak{M}$, as follows: We have $f(x) = \exp(i\phi_j(x))$ (say) for $x \in A_j$. For each 1-cell a_{jk}^α of $\mathfrak{M}$ (oriented from j to k), we pick $y \in A_{(j,k)^\alpha}$, and define $n_{jk}^\alpha = \{\phi_j(y) - \phi_k(y)\}/2\pi$; this number is an integer independent of the choice of y (because $A_{(j,k)^\alpha}$ is connected). It is easily verified that the 1-chain $c(f) = \sum n_{jk}^\alpha a_{jk}^\alpha$ is a cocycle, and that different choices of functions ϕ_j give rise to cocycles $c(f)$ differing only by coboundaries.

Now let μ such mappings f_λ $(1 \leqslant \lambda \leqslant \mu)$ be given, and suppose $\mu > b_1(\mathfrak{M})$. There exist integers $p_1, p_2, \ldots, p_\mu$, not all zero, such that $\sum p_\lambda c(f_\lambda) \sim 0$. Define $F = f_1^{p_1} f_2^{p_2} \ldots f_n^{p_n}$; thus we have $F \sim 1$ on each A_j, say $F = \exp(i\Phi_j)$ on A_j. Again, it readily follows that

$$c(F) = \sum N_{jk}^\alpha a_{jk}^\alpha,$$

say $\sim \sum p_\lambda c(f_\lambda) \sim 0$. Hence there exists a 0-cochain $\sum q_j^\beta a_j^\beta$ such that $N_{jk}^\alpha = q_j^\beta - q_k^\gamma$, where a_j^β, a_k^γ are the end-points of a_{jk}^α. Define a real-valued function Ψ on $\bigcup A_j$ by: $\Psi(x) = \Phi_j(x) - 2\pi q_j^\beta$ whenever $x \in A_j$. This definition is single-valued (and therefore continuous), since if $x \in A_j^\beta \cap A_k^\gamma$ we have $x \in A_{jk}^\alpha$ for some α, and then

$$(\Phi_j(x) - 2\pi q_j^\beta) - (\Phi_k(x) - 2\pi q_k^\gamma) = 2\pi(N_{jk}^\alpha - q_j^\beta + q_k^\gamma) = 0.$$

Since clearly $F = \exp(i\Psi)$ on $\bigcup A_j$, the mappings f_λ are not independent on $\bigcup A_j$ if $\mu > b_1(\mathfrak{M})$, and consequently $p(A_1, A_2, \ldots, A_n) \leqslant b_1(\mathfrak{M})$.

[7]Generalizing [2, p. 96]. Here b_1 denotes the 1-dimensional Betti number with (say) rational coefficients.

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2.5. *Proof of* (2). Now let c be a given 1-cocycle on $\mathfrak{M}$, its multiplicity on the oriented 1-cell a_{jk}^{α} being the integer m_{jk}^{α} say $(= -m_{kj}^{\alpha})$. We shall define, by recursion, real-valued continuous functions Ψ_k on $A_k \cap (A_1 \cup \ldots \cup A_{k-1})$ and ϕ_k on A_k, where $k = 1, 2, \ldots, n$, setting $\phi_1 \equiv 0$ on A_1, $\psi_2 = -2\pi m_{12}^{\alpha} + \phi_1$ on A_{12}^{α} $(\alpha = 1, 2, \ldots, \alpha(12))$, $\phi_2 =$ an extension of ψ_2 to A_2, and generally

$$\psi_k = -2\pi m_{jk}^{\alpha} + \phi_j \text{ on } A_{jk}^{\alpha} \qquad (1 \leqslant j < k, 1 \leqslant \alpha \leqslant \alpha(jk)),$$

and $\phi_k =$ an extension of ψ_k to A_k. To justify this definition, we must first show that the definition of ψ_k is consistent, i.e., that if $h < j < k$ and $x \in A_h \cap A_j \cap A_k$, say $x \in A_{hjk}^{\delta} \subset A_{jk}^{\alpha} \cap A_{hk}^{\beta} \cap A_{jk}^{\gamma}$, then

$$-2\pi m_{jk}^{\alpha} + \phi_j(x) = -2\pi m_{hk}^{\beta} + \phi_h(x).$$

This follows from the fact that $m_{jk}^{\alpha} + m_{kh}^{\beta} + m_{hj}^{\gamma} = 0$, c being a cocycle. Since ψ_k is thus a well-defined continuous function on the closed subset $A_k \cap (A_1 \cup \ldots \cup A_{k-1})$ of the normal space A_k, the extension ϕ_k exists [6, p. 28].

It follows that, whenever $x \in A_j \cap A_k$, we have $\exp(i\phi_j(x)) = \exp(i\phi_k(x))$; consequently the mapping f defined by

$$f = \exp(i\phi_j) \text{ on } A_j, \qquad 1 \leqslant j \leqslant n$$

is single-valued and continuous on $\bigcup A_j$. Further, even though the sets A_{jk}^{α} need not now be connected, we have $\phi_j(y) - \phi_k(y) = 2\pi m_{jk}^{\alpha}$ whenever $y \in A_{jk}^{\alpha}$, so that a cocycle $c(f)$ can still be associated with f as in 2.4 above, and is evidently simply c.

Now let $b_1(\mathfrak{M}) = \mu$, and choose μ 1-cocycles c_λ, $1 \leqslant \lambda \leqslant \mu$, linearly independent modulo cohomology in $\mathfrak{M}$. Corresponding to each c_λ, the above construction gives a mapping f_λ of $\bigcup A_j$ in S^1 such that

$$\text{(i)} \quad f_\lambda \sim 1 \text{ on each } A_j, \qquad \text{(ii)} \quad c(f_\lambda) = c_\lambda.$$

We have only to show that theses mappings f_λ are independent on $\bigcup A_j$. But if say $F \equiv f_1^{q_1} f_2^{q_2} \ldots f_\mu^{q_\mu} = \exp(i\Phi)$ on $\bigcup A_j$, where the q_j's are integers and Φ is a continuous real-valued function, we readily see that $c(F)$ exists and $\sum q_\lambda c_\lambda \sim c(F) \sim 0$; hence $q_1 = q_2 = \ldots = 0$.

2.6. Corollary. *If $A_1, A_2, \ldots, A_n$ are closed sets of finite incidence, no three of which have a common point, then*[8] $p(A_1, A_2, \ldots, A_n) = b_1(\mathfrak{M}) = h(A_1, A_2, \ldots, A_n)$.

For the definition of $h(A_1, A_2, \ldots, A_n)$ here reduces to

$$b_0(\bigcup A_j) + \sum_{j<k} (b_0(A_j \cap A_k) + 1) - n + 1 - \sum b_0(A_j).$$

Now $\mathfrak{M}$ is a linear graph having $b_0(\bigcup A_j) + 1$ components, $\sum(b_0(A_j \cap A_k) + 1)$ edges, and $\sum b_0(A_j) + n$ vertices; hence $h(A_1, A_2, \ldots A_n) = b_1(\mathfrak{M})$, by the Euler-Poincaré formula.

[8]By definition [9, p. 441], $h(A_1, \ldots, A_n) = \Sigma_1^n b_0(X_r) - \Sigma_1^n b_0(A_j)$, where X_r is the set of all points belonging to A_j for r or more values of j.

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Remark. For closed sets in general we have

$$p(A_1, A_2, \ldots, A_n) \leqslant h(A_1, A_2, \ldots, A_n).$$

This can be proved by induction over n, the case $n = 2$ being furnished by the above corollary.

3. Lemmas on linear graphs

3.1. In the next section we shall study "one-dimensional" coverings of S, whose modified nerves will be linear graphs; in preparation for this, we here collect the necessary graph-theoretic lemmas. In view of the applications, a (linear) *graph* G will here mean a finite 1-complex which may be "improper", i.e., in which two vertices may be joined by several edges (open 1-cells); but each edge is to have two distinct vertices. We denote the numbers of vertices and edges of G by $a_0(G)$, $a_1(G)$ respectively. The *order* $\nu(p, G)$ of a vertex p of G is the number of edges of G which are incident with p (have p as a vertex). A vertex p of order 1 is an *end-point* of G, and the single edge incident with p is then an *end-line.* An acyclic connected non-empty graph is a *tree.*

3.2. From the Euler-Poincaré formula, combined with the equality of b_1 and r for 1-dimensional Peano spaces [3, p. 162], we have:

(1) If G is a connected and non-empty graph, then

$$a_1(G) - a_0(G) + 1 = b_1(G) = r(G).$$

An elementary computation then gives:

(2) If G is a tree having exactly λ end-points and μ other vertices $q_1, q_2, \ldots, q_\mu$, then

$$\textstyle\sum_1^\mu \{\nu(q_j, G) - 2\} = \lambda - 2.$$

We note also the obvious property:

(3) If G is a connected graph having an end-point p with end-line C, then $G - C - (p)$ is connected.

3.3. Now let G be a graph having vertices $p_1, p_2, \ldots, p_m$ and edges $C_1, C_2, \ldots, C_n$, and suppose there exists a (continuous) monotone simplicial mapping ϖ of a graph H onto G. Thus $\varpi^{-1}(p_j)$ is a (closed) connected subgraph of H, $\varpi^{-1}(C_k)$ is a single (open) edge of H, and these inverse sets are pairwise disjoint, non-empty, and cover H. Suppose further that whenever C_j, C_k are distinct edges of G, the edges $\varpi^{-1}(C_j)$, $\varpi^{-1}(C_k)$ have disjoint closures (i.e., have no end-point in common). We shall then call ϖ^{-1} a *dispersion of* G, and shall also say that H is a dispersion of G. (Roughly speaking, the operation of "dispersing" G into H consists in replacing the vertices p_j of G by disjoint connected graphs $\varpi^{-1}(p_j)$, and reattaching the 1-cells of G in such a way that no two of them have a common vertex.)

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3.4. In what follows, we suppose that H is a dispersion of a *connected* graph G. Since ϖ is monotone,

(1) $$H \text{ is connected;}$$

and from 3.2(1) we readily obtain

(2) $$b_1(H) \geqslant b_1(G).$$

A dispersion of G will be called *minimal* if it satisfies: (a) the sub-graphs $\varpi^{-1}(p_j)$ are all trees, (b) each end-point of each $\varpi^{-1}(p_j)$ is incident with at least one (and therefore exactly one) edge $\varpi^{-1}(C_k)$. From 3.2(1) we see that:
(3) If H is a minimal dispersion of G, then

$$b_1(H) = b_1(G).$$

Further,

(4) Given a dispersion H of G, and a subgraph G^* of G, there exists a subgraph H^* of H which is a minimal dispersion of G^*.

In fact, $H_1 = \varpi^{-1}(G^*)$ is a subgraph of H which is a dispersion of G^*. Of those subgraphs of H_1 which are dispersions of G^*, let H^* be one having as few edges as possible. It is easy to see that H^* will be a minimal dispersion of G^*.

Now assume that H is a *minimal* dispersion of a connected graph G, and let the vertices of the subgraph $\varpi^{-1}(p)$ of H (p being a given vertex of G) be $q_1, q_2, \ldots, q_h$. An easy calculation, based on 3.2(2), gives $\sum_1^h \{\nu(q_j, H) - 2\} = \nu(p, G) - 2$, whence, since (with trivial exceptions) each summand is non-negative:

(5) $$\nu(q_j, H) \leqslant \nu(p, G) \qquad (1 \leqslant j \leqslant h);$$

and if for some j we have $\nu(q_j, H) = \nu(p, G)$, then

$$\nu(q_k, H) = 2 \text{ for all } k \neq j \qquad (1 \leqslant k \leqslant h).$$

We shall say that a minimal dispersion H of a connected graph G is *non-trivial* if there exists a vertex p of G for which the vertices q_j of $\varpi^{-1}(p)$ all satisfy $\nu(q_j, H) < \nu(p, G)$, and that it is *trivial* otherwise. From (5) we have:

(6) *If $G_1, G_2, \ldots$ is an infinite sequence of connected graphs such that G_{n+1} is a minimal dispersion of G_n ($n = 1, 2, \ldots$), then, for all large enough n, G_{n+1} is a trivial dispersion of G_n.*

Further, (5) shows that a trivial minimal dispersion of G is essentially a "subdivision" of G. In fact, we have:

(7) *Let $G_1, G_2, \ldots, G_n$ ($n \geqslant 2$) be connected graphs such that G_{j+1} is a trivial minimal dispersion of G_j ($1 \leqslant j \leqslant n-1$). Then each non-zero 1-cycle of G_n contains (i.e., has non-zero multiplicity on) a sequence of edges $E_1, E_2, \ldots, E_m$, where $m = 2^{n-2} + 1$, such that*

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(i) *E_j and E_{j+1} have exactly one common end-point, which is moreover of order* 2 *in* G_n $(1 \leqslant j \leqslant m-1)$, *and*[9]

(ii) $$\mathrm{Cl}(E_j) \cap \mathrm{Cl}(E_k) = 0 \text{ if } |j-k| \geqslant 2 \qquad (1 \leqslant j, k \leqslant m).$$

The proof of (7) is straightforward by induction over n, using (5).

4. One-dimensional coverings

4.1. THEOREM 3. *Let $r(S)$ be finite, and let $A_1, A_2, \ldots, A_n$ be n non-empty closed connected sets covering S, no three of which have a common point. Then the sets A_j are of finite incidence; and if $\mathfrak{M}$ is their modified nerve, we have $r(\mathfrak{M}) \leqslant r(S)$.*

We have, if $j \neq k$,

$$\begin{aligned}\mathrm{Fr}(A_j) \cap \mathrm{Fr}(A_k) \cap \mathrm{Fr}(A_j \cup A_k) &\subset A_j \cap A_k \cap \mathrm{Cl}(\mathrm{Co}(A_j \cup A_k)) \\ &\subset A_j \cap A_k \cap \bigcup \{A_m | m \neq j, k\} = 0;\end{aligned}$$

hence [10, 7.3] $b_0(A_j \cap A_k) \leqslant r(S) < \infty$. Thus the sets A_j are of finite incidence, and $\mathfrak{M}$ is defined (and is evidently a graph). In accordance with the notation of 1.1, we write A_{jk}^{α} $(1 \leqslant \alpha \leqslant \alpha(j,k))$ for the components of $A_{jk} = A_j \cap A_k$. Since

$$\mathrm{Fr}(A_j) \subset A_j \cap \bigcup \{A_k | k \neq j\} = \bigcup_{k,\alpha} A_{jk}^{\alpha},$$

a union of pairwise disjoint closed connected (non-empty) sets, there exist [10, 3.4], for each fixed j, closed connected sets $H_{jk}^{\alpha} \supset A_{jk}^{\alpha}$ such that $\bigcup_{k,\alpha} H_{jk}^{\alpha} = A_j$, no three of the sets H_{jk}^{α} have a common point, and the intersection of every two of them is contained in $A_j - \bigcup \{A_k | k \neq j\}$. (Note that $H_{jk}^{\alpha} \neq H_{kj}^{\alpha}$, though of course $A_{jk}^{\alpha} = A_{kj}^{\alpha}$.)

It readily follows that no three of *all* the sets H_{jk}^{α} can have a common point, even if j varies. Thus if we renumber the sets H_{jk}^{α}, say as $A_1(1), A_2(1), \ldots, A_{n_1}(1)$, the sets $A_j(1)$ have all the properties which were postulated for the sets A_j; hence they are also of finite incidence. Let the nerve and modified nerve of $\{A_j(1)\}$ be G_1 and H_1 respectively; both are graphs. We assert:

(1) $$G_1 \text{ is a dispersion of } \mathfrak{M}.$$

In fact, we can map G_1 on $\mathfrak{M}$ as follows. Each vertex q of G_1 corresponds to some set H_{jk}^{α}; we define $\varpi(q) = a_j$, the vertex of $\mathfrak{M}$ corresponding to A_j. Each edge of G_1 corresponds to a non-empty intersection $H_{jk}^{\alpha} \cap H_{lm}^{\beta}$. If $j = l$, we map the whole edge on a_j; if $j \neq l$, we must have $m = j$, $k = l$ and $\alpha = \beta$, and map the edge "linearly" onto the edge a_{jk}^{α} of $\mathfrak{M}$. The resulting mapping ϖ is easily seen to be continuous. Further, it is monotone, since ϖ^{-1} is clearly $1-1$ on a_{jk}^{α}, while $\varpi^{-1}(a_j)$ is precisely the nerve of the sets H_{jk}^{α} with fixed j, and is connected since A_j is connected. And it is not hard to see that if a_{jk}^{α} and a_{lm}^{β} are distinct edges of $\mathfrak{M}$, their inverse images under ϖ cannot have a common end-point. Thus (1) is established.

[9]We use the customary abbreviations Cl for *closure*, Co for *complement*, Fr for *frontier*.

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Clearly also, to within isomorphism,

(2) $$G_1 \text{ is a subgraph of } H_1.$$

The whole process is now repeated, starting with the sets $A_j(1)$; and so on. We thus obtain, for each λ (= 1, 2, . . .), a covering of S by closed connected sets $A_j(\lambda)$ $(1 \leqslant j \leqslant n_\lambda)$, no three of which have a common point, having nerve G_λ and modified nerve H_λ, such that $G_{\lambda+1}$ is both a dispersion of H_λ and a subgraph of $H_{\lambda+1}$.

From 3.4(4), we obtain recursively a sequence of graphs K_λ such that $K_1 = G_1$ and K_λ is a subgraph of G_λ which is a minimal dispersion of $K_{\lambda-1}$ $(\lambda \geqslant 2)$. By 3.4(6), there exists an integer $N > 3$ such that $K_{\lambda+1}$ is a trivial minimal dispersion of K_λ whenever $\lambda \geqslant N - 2$. On applying 3.4(7) to K_{N-2}, K_{N-1}, K_N, we see that every non-zero 1-cycle of K_N contains a sequence C_1, C_2, C_3 of three edges, such that:

(i) $\bar{C}_1 \cap \bar{C}_2 =$ a single vertex p_1 of K_N,

(ii) $\bar{C}_2 \cap \bar{C}_3 =$ a single vertex p_2 of K_N,

(iii) $\nu(p_1, K_N) = 2 = \nu(p_2, K_N)$,

(iv) $\bar{C}_1 \cap \bar{C}_3 = 0$.

For short we shall call such a sequence of three edges a "triad".

The graph K_N is connected (3.4(1) and Theorem 1, 1.3); hence if it is not already a tree it contains a cycle containing a triad (C_1^1, C_2^1, C_3^1). The subgraph $K_N - C_2^1$ is clearly connected, and has C_1^1 and C_3^1 among its end-lines. Hence if $K_N - C_2^1$ is not a tree it contains a triad (C_1^2, C_2^2, C_3^2) disjoint from the first. After a finite number of steps, say r, we obtain r mutually exclusive triads (C_1^s, C_2^s, C_3^s), $1 \leqslant s \leqslant r$, in K_N, such that $K_N - \bigcup C_2^s = T$, say, is a tree having all the edges C_1^s, C_3^s among its end-lines.

From 3.2(1) we obtain

(3) $$r = r(K_N).$$

Let U_i denote the subgraph of T formed by omitting from T all the edges C_i^s and the corresponding end-points $\mathrm{Cl}(C_i^s) \cap \mathrm{Cl}(C_2^s)$ $(i = 1, 3)$. From 3.2(3), U_1 and U_3 are connected subgraphs of K_N, and thus *a fortiori* of G_N; further, $U_1 \cap U_3 \neq 0$, and we note that G_N also contains the r distinct edges C_2^s, no two of which have a common end-point, and each of which joins a vertex in $U_1 - U_3$ to a vertex in $U_3 - U_1$.

For each vertex p of $G_N - (U_1 \cup U_3)$, join p to a vertex of $U_1 \cup U_3$ by a simple edge-path $W(p)$ in G_N (this is possible since G_N is connected, by 1.3), and further choose $W(p)$ to have as few edges as possible. Define $V_i =$ union of U_i with all those paths $W(p)$ whose ends (other than p) are in U_i $(i = 1, 3)$. Clearly $V_1 - V_3 \supset U_1 - U_3$ and $V_3 - V_1 \supset U_3 - U_1$; and moreover $V_1 \cup V_3$ contains all the vertices of G_N. Now G_N is the nerve of the closed connected sets $A_j(N)$ covering S. Let $X_i =$ union of those sets $A_j(N)$ which correspond to vertices in V_i $(i = 1, 3)$. It readily follows that X_1, X_3 are closed connected sets which cover S, and hence

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(4) $$b_0(X_1 \cap X_3) \leqslant r(S).$$

We may suppose the notation so chosen that A_j corresponds to a vertex in $V_1 \cap V_3$ if $1 \leqslant j \leqslant \mu$, in $V_1 - V_3$ if $\mu < j \leqslant \nu$, and in $V_3 - V_1$ if $\nu < j \leqslant n_N$. Write $D = A_1(N) \cup A_2(N) \cup \ldots \cup A_\mu(N)$. Then clearly

(5) $$X_1 \cap X_3 = D \cup \bigcup\{A_j(N) \cap A_k(N) | \mu < j \leqslant \nu < k \leqslant n_N\}.$$

The sets D, $A_j(N) \cap A_k(N)$ appearing here are closed and pairwise disjoint (for no three of the sets $A_j(N)$ have a common point). Further, $D \neq 0$ (for $V_1 \cap V_3 \neq 0$), and at least r of the sets $A_j(N) \cap A_k(N)$ are non-empty —namely those corresponding to the edges $C_2{}^s$ of G_N. Hence (5) shows that $b_0(X_1 \cap X_3) \geqslant r$, and so, from (3) and (4), we have $r(K_N) \leqslant r(S)$. But 3.4(2) and 3.4(3) show that

$$r(\mathfrak{M}) \leqslant r(G_1) = r(K_1) = r(K_2) = \ldots = r(K_N),$$

and consequently $r(\mathfrak{M}) \leqslant r(S)$.

4.2. Corollary. *If $r(S) < \infty$, there exists a covering of S by a finite number of closed connected sets A_j, no three of which have a common point, such that* (i) *their nerve $\mathfrak{N}$ satisfies $r(\mathfrak{N}) = r(S)$, and* (ii) *every intersection $A_j \cap A_k$ is connected.*

4.3. We next derive, for later use, a related property of *open* sets (which need not necessarily cover S).

Lemma. *Let $A_1, A_2, \ldots, A_n$ be n non-empty closed connected sets such that* $\mathrm{Fr}(A_j) \cap \mathrm{Fr}(A_k) \cap \mathrm{Fr}(A_j \cup A_k) = 0$ *whenever $j \neq k$, and no three of which have a common point. Then*

$$b_0(\bigcup A_j) + b_0(\bigcup\{A_j \cap A_k | j \neq k\}) \leqslant r(S) + n - 2.$$

We may evidently assume $n > 1$ and $r(S) < \infty$. Write $U = \mathrm{Co}(\bigcup A_j)$ and $F_j = \mathrm{Fr}(U) \cap \mathrm{Fr}(A_j)$; thus $\bigcup F_j = \mathrm{Fr}(U)$ and the sets F_j are pairwise disjoint. By [10, 3.4], there exist closed sets H_j $(1 \leqslant j \leqslant n)$ such that $H_j \supset F_j$, H_j is connected relative[10] to F_j, $\bigcup H_j = \bar{U}$, $H_j \cap F_k = 0$ if $j \neq k$, $H_j \cap H_k \subset U$ if $j \neq k$, and no three of the sets H_j have a common point.

Write $A_j \cup H_j = B_j$; thus the n sets B_j are closed, connected, and cover S, and no three of them have a common point. Hence, from Theorem 3 (4.1) and 3.2(1), the modified nerve $\mathfrak{M}$ of $B_1, B_2, \ldots, B_n$ exists and satisfies

(1) $$r(\mathfrak{M}) = a_1(\mathfrak{M}) - a_0(\mathfrak{M}) + 1 \leqslant r(S), \quad a_0(\mathfrak{M}) = n.$$

Now if $j \neq k$ we have $A_j \cap H_k = A_j \cap \bar{U} \cap H_k \subset F_j \cap H_k = 0$, and similarly $A_k \cap H_j = 0$. Thus

(2) $$B_j \cap B_k = (A_j \cap A_k) \cup (H_j \cap H_k);$$

and since $H_j \cap H_k \subset U$, the closed sets $A_j \cap A_k$ and $H_j \cap H_k$ are disjoint.

[10]For the definition and elementary properties of relative connectedness, see [9, p. 428] and [10, 3.3].

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Thus the modified nerve $\mathfrak{M}_0$ of $A_1, A_2, \ldots, A_n$ exists and can be obtained from $\mathfrak{M}$ merely by deleting certain edges of $\mathfrak{M}$ (corresponding to the components of the sets $H_j \cap H_k$). Since $\mathfrak{M}$ is connected, while $b_0(\mathfrak{M}_0) = b_0(\bigcup A_j)$ (Theorem 1, 1.3), the number of edges so deleted must be at least $b_0(\bigcup A_j)$. Thus we have $b_0(\bigcup A_j) + a_1(\mathfrak{M}_0) \leqslant a_1(\mathfrak{M})$; and since the sets $A_j \cap A_k$ $(j < k)$ are pairwise disjoint, $a_1(\mathfrak{M}_0)$ = number of components of $\bigcup(A_j \cap A_k) = b_0(\bigcup(A_j \cap A_k)) - 1$. The lemma now follows from (1).

4.4. THEOREM 4. *Let U, V be open subsets of S which satisfy* $\text{Fr}(U) \cap \text{Fr}(V) \cap \text{Fr}(U \cap V) = 0$. *Then* $h(U, V) \leqslant r(S)$ (i.e., $b_0(U \cup V) + b_0(U \cap V) \leqslant b_0(U) + b_0(V) + r(S)$).

Proof. We may assume that $r(S)$, $b_0(U)$ and $b_0(V)$ are all finite. Let U, V have components $U_1, \ldots, U_m$, $V_1, \ldots, V_n$ respectively. From [10, 7.4], each of the sets $U_j \cap V_k$ has only a finite number of components, say W_{jk}^{α} $(1 \leqslant \alpha \leqslant \alpha(jk))$. Pick points $x_j \in U_j$, $y_k \in V_k$, $z_{jk}^{\alpha} \in W_{jk}^{\alpha}$. Since U_j is open and connected, there exists a closed connected set joining x_j and z_{jk}^{α} in U_j; let the union of these closed connected sets, as k and α vary, be denoted by A_j. Similarly we construct a closed connected set $B_k \subset V_k$ containing all the points z_{jk}^{α}(for each fixed k). Write $\bigcup A_j = A$, $\bigcup B_k = B$. Then $\text{Co}(A)$ and $\text{Co}(B)$ are open sets containing $\text{Co}(U)$ and $\text{Co}(V)$ respectively; and [10, 6.3] gives the existence of open sets C, D such that $\text{Co}(A) \supset C \supset \text{Co}(U)$, $\text{Co}(B) \supset D \supset \text{Co}(V)$, and $\text{Fr}(C) \cap \text{Fr}(D) = 0$. Thus $A \subset \text{Co}(C) \subset U$, which shows that each component A_j of A is contained in a component C_j (say) of $\text{Co}(C)$, and that $C_j \subset U_j$. Similarly we obtain n distinct components D_k of $\text{Co}(D)$ such that $B_k \subset D_k \subset V_k$. We have $\text{Fr}(C_j) \cap \text{Fr}(D_k) \subset \text{Fr}(C) \cap \text{Fr}(D) = 0$, so that the sets $C_1, \ldots, C_m, D_1, \ldots, D_n$ satisfy the hypotheses of the lemma (4.3), and therefore

$$(1) \qquad b_0(\bigcup C_j \cup \bigcup D_k) + b_0(\bigcup(C_j \cap D_k)) \leqslant r(S) + m + n - 2.$$

Now the different sets $C_j \cap D_k$ are pairwise disjoint, and, since $z_{jk}^{\alpha} \in C_j \cap D_k \subset U_j \cap V_k$, each set $C_j \cap D_k$ has at least as many components as $U_j \cap V_k$. Thus

$$b_0(\bigcup(C_j \cap D_k)) \geqslant b_0(U \cap V).$$

Similarly $b_0(\bigcup C_j \cup \bigcup D_k) \geqslant b_0(U \cup V)$; and the theorem now follows from (1).

4.5. *Remark.* A similar argument will apply to any finite number of open sets, no three of which have a common point, and every two of which satisfy the frontier relation of Theorem 4. Further, if S is completely normal, the "approximation" method [10, 6.5] can be carried a step farther [10, 7.5] to yield the following theorem:

THEOREM 4a. *If S is completely normal, and $E_1, E_2, \ldots, E_n$ are n sets, no three of which have a common point, and every two of which satisfy* (i) *$E_j - E_k$ and $E_k - E_j$ are separated,* (ii) *$E_j \cap E_k$ and $\text{Co}(E_j \cup E_k)$ are separated* $(j \neq k)$, *then*

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$$\sum b_0(E_j) + n - 2 \leqslant b_0(\bigcup E_j) + b_0(\bigcup \{E_j \cap E_k | j \neq k\}) \leqslant \sum b_0(E_j) + r(S) + n - 2.$$

5. The analytic definition of $r(S)$

5.1. The number $\rho(S)$, defined (2.1) in terms of mappings of S in S^1, is known to equal $r(S)$ for e.g. Peano spaces. We shall now show that this equality holds for all connected, locally connected, normal T_1 spaces, without any requirements of compactness or completeness.

THEOREM 5. $$\rho(S) = r(S).$$

5.2. *Proof.* It is easy to see that

(1) $$r(S) \leqslant \rho(S).$$

In fact, let A_1, A_2 be closed connected sets which cover S, and suppose $b_0(A_1 \cap A_2) \geqslant n$. We can write $A_1 \cap A_2$ as a union of $n + 1$ disjoint closed non-empty sets $A_{12}{}^{\alpha}$; and this defines a d.s. of A_1, A_2 for which the corresponding modified nerve $\mathfrak{M}$ has 2 vertices and $n + 1$ edges, so that $n = b_1(\mathfrak{M})$. But (2.3 (2)) $b_1(\mathfrak{M}) \leqslant p(A_1, A_2) \leqslant \rho(S)$; thus $n \leqslant \rho(S)$, and (1) follows.

5.3. Now suppose

(2) $$r(S) < \rho(S);$$

we shall derive a contradiction. From (2), $r(S) = n$ say $< \infty$, and there exist closed (but not necessarily connected) sets F_1, F_2 and $n + 1$ independent (continuous) mappings f_j of S in S^1 $(1 \leqslant j \leqslant n + 1)$ such that $f_j \sim 1$ on each of F_1, F_2. There exist (2.2 (2)) open sets $A \supset F_1$, $B \supset F_2$, and continuous real-valued functions ϕ_j, ψ_j, such that

(3) $$f_j = \exp(i\phi_j) \text{ on } A, \text{ and } f_j = \exp(i\psi_j) \text{ on } B \qquad (1 \leqslant j \leqslant n + 1).$$

Let A, B have components $\{A_\lambda\}$, $\{B_\mu\}$, repectively. Each of these components is open; further, we have

$$\text{Fr}(A_\lambda) \cap \text{Fr}(B_\mu) \subset \text{Fr}(A) \cap \text{Fr}(B) \subset \text{Co}(A) \cap \text{Co}(B) = 0.$$

Hence for any finite unions $\mathfrak{A} = A_{\lambda_1} \cup A_{\lambda_2} \cup \ldots \cup A_{\lambda_h}$ and $\mathfrak{B} = B_{\mu_1} \cup B_{\mu_2} \cup \ldots \cup B_{\mu_k}$ we have $\text{Fr}(\mathfrak{A}) \cap \text{Fr}(\mathfrak{B}) = 0$ and therefore (Theorem 4, 4.4)

(4) $$h(\mathfrak{A}, \mathfrak{B}) \leqslant n.$$

In particular,

(5) $$b_0(A_\lambda \cap B_\mu) \leqslant n.$$

Now form a "graph" $\mathfrak{M}$ (which however will be infinite, in general) by taking vertices a_λ, b_μ corresponding to the sets A_λ, B_μ, and joining a_λ to b_μ by as many edges as $A_\lambda \cap B_\mu$ has components. (Thus $\mathfrak{M}$ is the "modified nerve" of A and B except that it is formed with respect to an infinite decompo-

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sition, in general.) From (4) and 3.2 (1) we have $b_1(G) \leqslant n$ whenever G is a subgraph of $\mathfrak{M}$ generated by a finite number of vertices of $\mathfrak{M}$, and hence also whenever G is any finite subgraph of $\mathfrak{M}$. Thus there is a finite subgraph G_1 of $\mathfrak{M}$ for which $b_1(G_1)$ is as large as possible; say $b_1(G_1) = N$, where $N \leqslant n$. Next, since $\mathfrak{M}$ is connected (for S is), there exists a connected finite subgraph G_2 of $\mathfrak{M}$ containing G_1 (obtained by adding to G_1 a finite number of edge-paths connecting the vertices of G_1 in $\mathfrak{M}$). Let $a_{\lambda_1}, \ldots, a_{\lambda_h}, b_{\mu_1}, \ldots, b_{\mu_k}$ be the vertices of G_2, and let G_3 be the subgraph of $\mathfrak{M}$ which they generate. Thus G_3 is a connected finite graph, and since $G_3 \supset G_2 \supset G_1$ we have $b_1(G_3) \geqslant b_1(G_1) \geqslant N$, and therefore $b_1(G_3) = N$. Write $\mathfrak{A} = A_{\lambda_1} \cup \ldots \cup A_{\lambda_h}$, $\mathfrak{B} = B_{\mu_1} \cup \ldots \cup B_{\mu_k}$; then clearly $\mathfrak{A}$ and $\mathfrak{B}$ are of finite incidence, and their modified nerve is G_3.

We shall next assign a "rank" to each vertex of $\mathfrak{M}$, as follows. Let $p (= a_\lambda$ or $b_\mu)$ be a given vertex of $\mathfrak{M}$. If $p \in G_3$, its rank is zero. If p non $\in G_3$, join p to G_3 by a finite edge-path $W(p)$ in $\mathfrak{M}$ such that $W(p)$ contains no edge in G_3 (e.g., take $W(p)$ to be as short as possible). We assert that $W(p)$ is now unique. In fact, if $W'(p)$ were a different edge-path satisfying these requirements, the subgraph $G_3 \cup W(p) \cup W'(p)$ would (as is easy to see) contain a closed path not lying entirely in G_3, so that $b_1(G_3 \cup W(p) \cup W'(p)) > b_1(G_3) = N$, contradicting the definition of N. The rank of p is now defined to be the number of edges in $W(p)$.

The "rank" of a component A_λ or B_μ of A or B is defined to be the rank of the corresponding vertex of $\mathfrak{M}$, and we write C_ν = union of all sets (A_λ or B_μ) of rank $\leqslant \nu$ ($\nu = 0, 1, 2, \ldots$). Thus C_ν is open, $\mathfrak{A} \cup \mathfrak{B} = C_0 \subset C_1 \subset C_2 \subset \ldots$, and $\bigcup C_\nu = S$. Further, the construction shows that the sets of fixed rank $\nu > 0$ are pairwise disjoint, while each set of rank $\nu > 0$ intersects one and only one set of rank $\nu - 1$, and this intersection is always connected.

We have $n + 1 > N = b_1(G_3) \geqslant p(\mathfrak{A}, \mathfrak{B})$, from 2.3 (1); hence, in view of (3) above, there must exist integers $q_1, q_2, \ldots, q_{n+1}$, not all zero, and a continuous real-valued function θ on $\mathfrak{A} \cup \mathfrak{B}$, such that

(6) $$F \equiv f_1^{q_1} f_2^{q_2} \ldots f_{n+1}^{q_{n+1}} = \exp(i\theta) \text{ on } \mathfrak{A} \cup \mathfrak{B}.$$

Using (3), we define

$$\Phi = \sum q_j \phi_j \text{ on } A, \Psi = \sum q_j \psi_j \text{ on } B;$$

thus $F = \exp(i\Phi)$ on A, and $F = \exp(i\Psi)$ on B.

We now extend θ to a continuous function Θ, defined for all $x \in S$, and such that $F = \exp(i\Theta)$, as follows. On C_0, define $\Theta = \theta$. Now suppose Θ has been defined with the desired properties on C_ν. If A_λ is a set of rank $\nu + 1$, it intersects a unique set of rank ν, necessarily of the form B_μ, and $A_\lambda \cap C_\nu = A_\lambda \cap B_\mu$ which is connected. Hence on $A_\lambda \cap C_\nu$ we have $\Theta = \Phi - 2\pi m_\lambda$ where m_λ is a (constant) integer; and we define $\Theta = \Phi - 2\pi m_\lambda$ on A_λ. Similarly Θ is defined on each B_μ of rank $\nu + 1$ (using the function Ψ). Since the sets of rank $\nu + 1$ are pairwise disjoint open sets, Θ is single valued and con-

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tinuous on $C_{\nu+1}$, and clearly $\exp(i\Theta) = F$ on $C_{\nu+1}$. This process defines Θ with the above properties on all of S; but this contradicts the independence of the mappings f_j, and the proof is complete.

6. Finite coverings in general

6.1. LEMMA 1. *Given a d.s.* $\{A_J^\alpha\}$ *of* n *closed sets* $A_1, A_2, \ldots, A_n$, *and given open sets* $U(J, \alpha) \supset A_J^\alpha$, *there exist open* F_σ *sets* $B_1, B_2, \ldots, B_n$ *and a d.s.* $\{B_J^\alpha\}$ *of* $B_1, B_2, \ldots, B_n$, *such that* (i) $A_J^\alpha \subset B_J^\alpha \subset \mathrm{Cl}(B_J^\alpha) \subset U(J, \alpha)$, (ii) B_J^α *is connected*[10] *relative to* A_J^α, (iii) $\mathrm{Cl}(B_J^\alpha) \cap \mathrm{Cl}(B_{J'}^{\alpha'}) = 0$ *whenever* $A_J^\alpha \cap A_{J'}^{\alpha'} = 0$, (iv) $B_J^\alpha \subset B_{J'}^{\alpha'}$ *whenever* $A_J^\alpha \subset A_{J'}^{\alpha'}$, *and* (v) $\mathrm{Cl}(B_J) = \bigcap \{\mathrm{Cl}(B_j) | j \in J\}$.

Remark. It follows that $\{\mathrm{Cl}(B_J^\alpha)\}$ will be a d.s. of the sets $\overline{B_j}$, and that if the sets A_j have finite incidences then so do the sets B_j and the sets $\overline{B_j}$, and all three systems of sets have then the same modified nerve.

Proof. Let k be the greatest number of different suffixes j, $1 \leqslant j \leqslant n$, for which the intersection of the corresponding sets A_j is not empty. The proof will go by induction over k (n remaining fixed). If $k = 1$, the result follows easily from the following two well-known properties:

(1) Given $F \subset U$, where F is closed and U open, there exists an open F_σ set V such that $F \subset V \subset \overline{V} \subset U$.

(2) If E is an open F_σ set, so is every union of components of E.

Now assume the lemma holds whenever every intersection of k of the sets A_j is empty ($k > 1$). In what follows, K and K' will always denote sets of k suffixes j ($1 \leqslant j \leqslant n$) for which the corresponding intersections A_K, A_K', are not empty; J, J', etc. denote (as hitherto) arbitrary non-null sets of suffixes; and, except where the contrary is stated, all suffixes and superscripts run over all their admissible values.

From the definition of k and the properties of a d.s., we have

(3) $$A_K^\beta \cap A_J^\alpha = 0 \quad \text{unless } J \subset K \text{ and } A_J^\alpha \supset A_K^\beta.$$

In particular, the sets A_K^β are all pairwise disjoint. Hence there exist open sets $V(K, \beta)$ such that

(4) $$\begin{aligned} &A_K^\beta \subset V(K, \beta), \\ &V(K, \beta) = 0 \text{ whenever } A_K^\beta = 0, \\ &\mathrm{Cl}(V(K, \beta)) \subset U(J, \alpha) \text{ whenever } A_K^\beta \subset A_J^\alpha, \\ &\mathrm{Cl}(V(K, \beta)) \cap A_J^\alpha = 0 \text{ whenever } A_K^\beta \cap A_J^\alpha = 0, \text{ and} \\ &\mathrm{Cl}(V(K, \beta)) \cap \mathrm{Cl}(V(K', \beta')) = 0 \text{ unless } K = K' \text{ and } \beta = \beta'. \end{aligned}$$

From (1) and (2), we may further suppose that each $V(K, \beta)$ is an open F_σ and is connected relative to A_K^β.

Now write

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(5) $$W = \bigcup V(K, \beta),\ A'_j = A_j - W,\ A'_{J^\alpha} = A_{J^\alpha} - W.$$

Clearly the sets A'_j are closed, $\{A'_{J^\alpha}\}$ is a d.s. of $\{A'_j\}$, and no k of the sets A'_j have a common point. Again, in view of (3), there exist open sets $U'(J,\alpha)$ such that

(6) $$A'_{J^\alpha} \subset U'(J, \alpha) \subset U(J, \alpha),$$
$$\mathrm{Cl}(U'(J, \alpha)) \cap \mathrm{Cl}(V(K, \beta)) = 0 \text{ whenever } A_{K^\beta} \cap A_{J^\alpha} = 0, \text{ and}$$
$$\mathrm{Cl}(U'(J, \alpha)) \cap A_{J'^{\alpha'}} = 0 \text{ whenever } A_{J^\alpha} \cap A_{J'^{\alpha'}} = 0.$$

Applying the hypothesis of induction to the system $\{A'_J\}$ and open sets $U'(J, \alpha)$, we obtain open F_σ sets $B'_1, \ldots, B'_n$, with a d.s. $\{B'_{J^\alpha}\}$, having the properties corresponding to (i)—(v) of the lemma. Define

(7) $$B_j = B'_j \cup \bigcup\{V(K, \beta) | j \in K\}, \text{ and}$$
$$B_{J^\alpha} = B'_{J^\alpha} \cup \bigcup\{V(K, \beta) | A_{K^\beta} \cap A_{J^\alpha} \neq 0\}$$
$$= B'_{J^\alpha} \cup \bigcup\{V(K, \beta) | K \supset J \text{ and } A_{K^\beta} \subset A_{J^\alpha}\}$$

(as follows from (3) and (4)). Clearly B_j is an open F_σ, and B_{J^α} is connected relative to A_J (for B'_{J^α} is connected relative to $A'_{J^\alpha} \subset A_{J^\alpha}$, and each $V(K, \beta)$ occurring is connected relative to $A_{K^\beta} \subset A_{J^\alpha}$). It follows easily from (4), (6), and the hypothesis of induction that $\mathrm{Cl}(B_{J^\alpha}) \subset U(J, \alpha)$. To prove $A_{J^\alpha} \subset B_{J^\alpha}$, suppose $x \in A_{J^\alpha} - B_{J^\alpha}$; then x non $\in A'_{J^\alpha}$ (else $x \in B'_{J^\alpha} \subset B_{J^\alpha}$), and so, from (5), $x \in W$, say $x \in V(K, \beta)$. From (4), $A_{K^\beta} \cap A_{J^\alpha} \neq 0$; hence, from (7), $V(K, \beta) \subset B_{J^\alpha}$, contradicting x non $\in B_J$. Thus properties (i) and (ii) are established.

Property (iii) is proved as follows. Suppose $A_{J^\alpha} \cap A_{J'^{\alpha'}} = 0$ and

$$x \in \mathrm{Cl}(B'_{J^\alpha} \cup V(K, \beta)) \cap \mathrm{Cl}(B'_{J'^{\alpha'}} \cup V(K', \beta')),$$

where (from (7)) $K \supset J$, $K' \supset J'$, $A_{K^\beta} \subset A_{J^\alpha}$ and $A_{K'^{\beta'}} \subset A_{J'^{\alpha'}}$; we must derive a contradiction. The hypothesis of induction gives $\mathrm{Cl}(B'_{J^\alpha}) \cap \mathrm{Cl}(B'_{J'^{\alpha'}}) = 0$, while from (4) we obtain $\mathrm{Cl}(V(K, \beta)) \cap \mathrm{Cl}(V(K', \beta')) = 0$. Hence we may assume

$$x \in \mathrm{Cl}(V(K, \beta)) \cap \mathrm{Cl}(B'_{J'^{\alpha'}}) \subset \mathrm{Cl}(V(K, \beta)) \cap \mathrm{Cl}(U'(J', \alpha')).$$

From (6) we must have $A_{K^\beta} \cap A_{J'^{\alpha'}} \neq 0$, and therefore (from (3)) $A_{K^\beta} \subset A_{J'^{\alpha'}}$. But this contradicts the assumption $A_{J^\alpha} \cap A_{J'^{\alpha'}} = 0$.

Property (iv) is immediate from (7), (5) and the hypothesis of induction. Thus all that remains to be proved, apart from (v), is that $\{B_{J^\alpha}\}$ is in fact a d.s. of $\{B_j\}$; and in virtue of (iii) and (iv) it will suffice to verify that

(8) $$\bigcup_\alpha B_{J^\alpha} = B_J, \text{ where } B_J = \bigcap\{B_j | j \in J\}.$$

First suppose $x \in B_{J^\alpha}$. If $x \in B'_{J^\alpha}$, then $x \in \bigcap\{B'_j | j \in J\} \subset B_j$; hence we may suppose $x \in V(K, \beta)$ where (from (7)) $A_{K^\beta} \subset A_{J^\alpha}$ and $K \supset J$. Thus (7) gives $V(K, \beta) \subset B_j$ whenever $j \in J$, so again $x \in B_J$. This proves $\bigcup_\alpha B_{J^\alpha} \subset B_J$.

Conversely, suppose $x \in B_J$. If for every $j \in J$ we have $x \in B'_j$, then

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$x \in B'_J = \bigcup_\alpha B'_J{}^\alpha \subset \bigcup_\alpha B_J{}^\alpha$, as desired. Thus we may assume (from (7)) that $x \in V(K, \beta)$, where $j \in K$, for at least one $j \in J$. We assert $J \subset K$. For if say $j' \in J - K$, then $x \in B'_{j'}$, since otherwise $x \in V(K', \beta')$ with $j' \in K'$, and then $V(K', \beta') \cap V(K, \beta) \neq 0$ though $K \neq K'$, contradicting (4). Thus $x \in \bigcup_\gamma B'_{j'}{}^\gamma \subset \bigcup U'(j', \gamma)$, and so for some γ we have $x \in U'(j', \gamma) \cap V(K, \beta)$, which from (6) implies $A_K{}^\beta \cap A_{J'}{}^\gamma \neq 0$, whence (by (3)) $j' \in K$, a contradiction. Thus $J \subset K$; and the definition of a d.s. now gives the existence of an α' such that $A_K{}^\beta \subset A_J{}^{\alpha'}$. From (7), we have $V(K, \beta) \subset B_J{}^{\alpha'}$, and so $x \in \bigcup_\alpha B_J{}^\alpha$, completing the proof of (8).

Finally, the verification of (v) is along similar lines, and is left to the reader.

6.2. *Strictly canonical mappings.* Let $U_1, U_2, \ldots, U_n$ be a given covering of S, with a given d.s. $\mathfrak{D} = \{U_J{}^\alpha\}$. For each $x \in S$, let $J(x)$ be the set of all suffixes j for which $x \in U_j$; thus $x \in U_{J(x)}$, and so $x \in U_{J(x)}{}^\alpha$ for one and only one value of α, say for $\alpha = \alpha(x)$. The corresponding (open) simplex $u_{J(x)}{}^{\alpha(x)}$ of $\mathfrak{M}(\mathfrak{D})$ will be denoted by $\sigma(x)$.

A continuous mapping h of S in $\mathfrak{M}(\mathfrak{D})$ will be called *strictly canonical*[11] if it satisfies

$$(1) \qquad h(x) \in \sigma(x), \ \text{all } x \in S.$$

It is easy to see that (1) is equivalent[12] to

$$(2) \qquad h^{-1}(\text{St } u_J{}^\alpha) = U_J{}^\alpha,$$

St $u_J{}^\alpha$ denoting the (open) star of the simplex $u_J{}^\alpha$ in $\mathfrak{M}(\mathfrak{D})$.

The proof of the standard existence theorem for mappings in ordinary nerves can readily be extended to give:

LEMMA 2. *Let $U_1, U_2, \ldots, U_n$ be open F_σ sets which cover S and let $\mathfrak{D} = \{U_J{}^\alpha\}$ be a d.s. of $\{U_j\}$. Then there exists a strictly canonical mapping h of S in $\mathfrak{M}(\mathfrak{D})$.*

6.3. The fundamental lemma is the following analogue of a lemma of Eilenberg [4, p. 105], and the idea of the proof is essentially the same, though with some complications.

LEMMA 3. *Let $B_1, B_2, \ldots, B_n$ be a covering of S by open F_σ sets of finite incidence, with $\{B_J{}^\alpha\}$ as natural d.s., and suppose that $\{\text{Cl}(B_J{}^\alpha\}$ is a d.s. of $\{\overline{B_j}\}$. Let h be a strictly canonical mapping of S in the modified nerve $\mathfrak{M}$ of $\{B_j\}$, and let f be a mapping of $\mathfrak{M}$ in S^1 such that $fh \sim 1$ on S. Then $f \sim 1$ on $\mathfrak{M}$.*

Suppose not. Then, as in [4, p. 105], there exists a simple closed edge-path in $\mathfrak{M}$ on which f non ~ 1; let $\mathfrak{C}$ be such a closed edge-path having as few edges as possible. There is no loss of generality in assuming the sets B_j to be connected (otherwise we replace them by their components); hence the nota-

[11] Compare [1, p. 210].

[12] For ordinary nerves it is enough to require only that (2) hold for vertex-stars; but this reduction is no longer valid for modified nerves, in general.

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tion may be chosen so that $\mathfrak{C}$ consists of the edges $b_{12}^1, b_{23}^1, \ldots, b_{(s-1)s}^1, b_{s1}^1$ joining successive vertices $b_1, b_2, \ldots, b_s$. (Note that here s may well equal 2.) As in [4], it follows from 2.2(1) and the choice of $\mathfrak{C}$ that $B_j \cap B_k = 0$ $(1 \leqslant j < k \leqslant s)$ unless j, k are consecutive in the cyclic order 12 . . . sl; and thence it follows, if $s > 3$, that no three of the sets B_j $(1 \leqslant j \leqslant s)$ can have a common point. Further, this holds even if $s = 3$. For otherwise b_1, b_2, b_3 are the vertices of a 2-cell b_{123}^α in $\mathfrak{M}$, which will have edges say $b_{23}^\beta, b_{31}^\gamma, b_{12}^\delta$; but $f \sim 1$ on $\mathrm{Cl}(b_{123}^\alpha)$ (from 2.2(4)), and also $f \sim 1$ on $\mathrm{Cl}(b_{23}^1) \cup b_{23}^\beta$ (which is either an arc or a closed edge-path shorter than $\mathfrak{C}$), and similarly $f \sim 1$ on $\mathrm{Cl}(b_{31}^1) \cup b_{31}^\gamma$ and on $\mathrm{Cl}(b_{12}^1) \cup b_{12}^\delta$, so that (from 2.2(1)) $f \sim 1$ on $\mathfrak{C}$, which is absurd. Hence, in view of the postulates on the sets B_j, we have:

(1) No three of the sets $\overline{B_j}$ have a common point $(1 \leqslant j \leqslant s)$.

Write $S' = \bigcup B_j$ $(1 \leqslant j \leqslant s)$; evidently S' is connected, and further, as an open F_σ subset of S, S' is also locally connected and normal. In the next paragraph, all considerations will be relative to S', and we use dashes to indicate relative closures and frontiers. The suffixes j, k, will run between 1 and s, and will be taken modulo s.

For each fixed j we have

$$\mathrm{Fr}'(B_j) \subset \mathrm{Cl}'(B_j) \cap \mathrm{Cl}'(B_{j-1} \cup B_{j+1}) = \bigcup_\alpha \mathrm{Cl}'(B_{(j-1)j}^\alpha) \cup \bigcup_\beta \mathrm{Cl}'(B_{j(j+1)}^\beta),$$

the union of a finite number of pairwise disjoint and (relatively) closed connected non-empty sets. On applying [10, 3.4] in S', we obtain connected sets $H_j^\alpha \supset \mathrm{Cl}'(B_{(j-1)j}^\alpha)$, $K_j^\beta \supset \mathrm{Cl}'(B_{j(j+1)}^\beta)$, no three of which have a common point (j being fixed), such that the intersection of every two of these sets is contained in $B_j - \bigcup\{\mathrm{Cl}'(B_k) | k \neq j\}$. Moreover, the sets H_j^α, K_j^β, so obtained will in the first instance satisfy $\bigcup_\alpha H_j^\alpha \cup \bigcup_\beta K_j^\beta = \mathrm{Cl}'(B_j)$, and will be closed (relative to S'); but we replace them (using 6.1(1) and 6.1(2)) by slightly larger sets to make them open F_σ's (relative to S' and thus also relative to S) without introducing any further intersections. For convenience, we introduce the symbol L_j^α to stand for either H_j^α or K_j^α. If now j is allowed to vary, we see that, while $H_j^\alpha \cap K_{j-1}^\alpha \supset B_{(j-1)j}^\alpha \neq 0$, all other intersections of the form $L_j^\alpha \cap L_k^\beta$ $(j \neq k)$ are empty, and consequently no three of the sets L_j^α, $1 \leqslant j \leqslant s$, can have a common point.

Let $\mathfrak{N}$ denote the (unmodified) nerve of the sets L_j^α; clearly $\mathfrak{N}$ is a linear graph. We use $h_j^\alpha, k_j^\alpha, l_j^\alpha$ for the vertices of $\mathfrak{N}$ corresponding to the sets H_j^α, K_j^α, L_j^α respectively. Since B_j is connected, there exists a simple edge-path C_j in $\mathfrak{N}$, joining h_j^1 to k_j^1 via vertices of the form l_j^α (j fixed) only; and since $B_{j(j-1)}^1 \neq 0$ there exists a 1-cell $(k_{j-1}^1 h_j^1)$ in $\mathfrak{N}$. The sequence

$$\mathfrak{K} = (k_s^1 h_1^1),\ C_1,\ (k_1^1 h_2^1),\ C_2, \ldots, (k_{s-1}^1 h_s^1),\ C_s,$$

constitutes a simple closed curve in $\mathfrak{N}$.

Now consider the (continuous) simplicial mapping ϖ of $\mathfrak{N}$ in $\mathfrak{M}$ defined as follows: ϖ maps each vertex l_j^α and edge $l_j^\alpha l_j^\beta$ on the vertex b_j of $\mathfrak{M}$, and maps

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each edge of the form $k_{j-1}^a h_j^a$ "linearly" on the edge $b_{(j-1)j}^a$ of $\mathfrak{M}$. Clearly $\varpi(C_j) = b_j$ and so ϖ maps $\mathfrak{K}$ on $\mathfrak{C}$ with degree 1. From this and the uniform continuity of f, we obtain a sequence of mappings $\varpi = \varpi_0, \varpi_1, \ldots, \varpi_\mu$ of $\mathfrak{K}$ on $\mathfrak{C}$ such that

(i) ϖ_μ is a homeomorphism of $\mathfrak{K}$ on $\mathfrak{C}$,

(ii) $|f(\varpi_{\lambda-1}(x)) - f(\varpi_\lambda(x))| < 1$ for all $x \in \mathfrak{K}$ $(1 \leqslant \lambda \leqslant \mu)$.

Thus, from 2.2(3), $f\varpi \sim f\varpi_1 \sim \ldots f\varpi_\mu$ on $\mathfrak{K}$; and from the fact that f non ~ 1 on $\mathfrak{C}$, we readily deduce $f\varpi_\mu$ non ~ 1 on $\mathfrak{K}$, and consequently $f\varpi$ non ~ 1 on $\mathfrak{K}$.

Thus there exist simple closed edge-paths in $\mathfrak{N}$ on which $f\varpi$ non ~ 1; let $\mathfrak{K}_0$ be one having as few edges as possible, and let the corresponding sets L_j^a be renamed $L(1), L(2), \ldots, L(p), L(1)$, following the cyclic order of $\mathfrak{K}_0$. (Note that now $p \geqslant 3$.) As before, two sets $L(j)$, $L(k)$ meet if and only they are consecutive in this cyclic order; hence the nerve of $L(1), L(2), \ldots, L(p)$ is precisely $\mathfrak{K}_0$. Write $Q = \bigcup L(j)$ $(1 \leqslant j \leqslant p)$, and let h' be a strictly canonical mapping of Q in $\mathfrak{K}_0$. It is easy to see that, for each $x \in Q$, the point $\varpi h'(x)$ of $\mathfrak{M}$ belongs to the closure of the simplex $\sigma(x)$ of $\mathfrak{M}$ which contains $h(x)$. Let $h(x) = h_0(x)$, $h_1(x), \ldots, h_N(x) = \varpi h'(x)$ be points dividing the "straight" segment joining $h(x)$ to $\varpi h'(x)$, in $\mathrm{Cl}(\sigma(x))$, into N equal parts. One readily verifies that each h_k is a continuous mapping of Q in $\mathfrak{M}$, and that, from 2.2(3), $fh_0 \sim fh_1 \sim \ldots \sim fh_N$ if N is large enough. Thus $f\varpi h' \sim fh \sim 1$ on Q.

The argument can be concluded as in [4, p. 106]; alternatively, by the theorem there proved, we must have $f\varpi \sim 1$ on $\mathfrak{K}_0$. contradicting the definition of $\mathfrak{K}_0$.

6.4. THEOREM 6. *Let $A_1, A_2, \ldots, A_n$ be non-empty closed connected sets of finite incidence which cover S; let $\mathfrak{N}$ be their nerve and $\mathfrak{M}$ their modified nerve. Then*

$$r(\mathfrak{N}) \leqslant r(\mathfrak{M}) \leqslant r(S).$$

That $r(\mathfrak{N}) \leqslant r(\mathfrak{M})$ has been proved in 1.3 Suppose $r(\mathfrak{M}) \geqslant m$; from Theorem 5 (5.1) it will suffice to prove $\rho(S) \geqslant m$. There exist closed subsets M, N of $\mathfrak{M}$, and m independent mappings f_j $(1 \leqslant j \leqslant m)$ of $\mathfrak{M}$ in S^1, such that $M \cup N = \mathfrak{M}$, $f_j \sim 1$ on M, and $f_j \sim 1$ on N. By Lemma 1 (6.1), we can enlarge the sets A_j to open F_σ sets B_j having the same modified nerve $\mathfrak{M}$ and satisfying the hypotheses of Lemma 3 (6.3). By Lemma 2(6.2), there exists a strictly canonical mapping h of S in $\mathfrak{M}$. let $X = h^{-1}(M)$ and $Y = h^{-1}(N)$; X and Y are closed sets covering S, and each of the m mappings f_jh of S in S^1 evidently satisfies $f_jh \sim 1$ on X and $f_jh \sim 1$ on Y. But Lemma 3 (6.3) shows that these mappings are independent on S; hence $\rho(S) \geqslant m$, and the theorem is proved.

6.5. THEOREM 7. *Let $A_1, A_2, \ldots, A_n$ be non-empty, connected, locally connected, normal sets of finite incidence, which cover S and are such that $A_j - A_k$ and $A_k - A_j$ are always separated*[6]. *Let $\mathfrak{M}$ be their modified nerve. Then $r(S) \leqslant b_1(\mathfrak{M}) + \sum r(A_j)$.*

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We may assume $r(A_j) = r_j < \infty$. Suppose there exist N independent mappings $f_1, f_2, \ldots, f_N$ of S in S^1, and closed sets X, Y such that $X \cup Y = S$, $f_j \sim 1$ on X, and $f_j \sim 1$ on $Y (1 \leqslant j \leqslant n)$; we must prove (in view of Theorem 5, 5.1) that $N \leqslant b_1(\mathfrak{M}) + \sum r_j$.

Since $f_j \sim 1$ on $X \cap A_1$ and on $Y \cap A_1$, Theorem 5 shows that at most r_1 of the mappings f_j can be independent on A_1. Let the greatest number of independent mappings f_j on A_1 be $s_1 \leqslant r_1$; we may suppose the notation so chosen that $f_1, \ldots, f_{s_1}$ are independent on A_1, and obtain for each $j > s_1$ a relation, say

$$g_j \equiv f_j^{p_j} f_1^{q_{j1}} \ldots f_{s_1}^{q_{js_1}} \sim 1 \text{ on } A_1,$$

where the exponents are integers and clearly $p_j \neq 0$. It readily follows that the $N - s_1$ mappings g_j are independent on S, and satisfy $g_j \sim 1$ on X and on Y.

By repeating this argument, applying it to $A_2, \ldots, A_n$ in turn, we obtain $N - \sum s_k$ independent mappings (say) h_j of S in S^1 (expressible as power-products of the N given mappings f_j), where $s_k \leqslant r_k$, such that $h_j \sim 1$ on each A_k $(1 \leqslant k \leqslant n)$. Hence, from 2.3(1),

$$N - \sum s_k \leqslant p(A_1, A_2, \ldots, A_n) \leqslant b_1(\mathfrak{M}),$$

and the theorem follows.

COROLLARY. *If further the sets A_j are closed and unicoherent, and no three of them have a common point, then $r(S) = r(\mathfrak{M})$.*

For Theorem 7 gives $r(S) \leqslant b_1(\mathfrak{M}) = r(\mathfrak{M})$, since $\mathfrak{M}$ is now a graph; and on the other hand Theorem 6 (6.4) gives $r(S) \geqslant r(\mathfrak{M})$.

6.6. It is natural to ask whether, in Theorem 7 above, the term $b_1(\mathfrak{M})$ can be replaced by $r(\mathfrak{M})$. The answer is negative, as is shown by the following example: Let T be a 2-manifold of genus k, simplicially subdivided, and let $B_1, B_2, \ldots, B_n$ denote the closed stars of the vertices of T in the barycentric subdivision. Let C be a small circular region interior to B_1, and define $S = T - C$, $A_1 = B_1 - C$, and $A_j = B_j$ $(j \geqslant 2)$. It follows immediately from known theorems that $r(S) = 2k$, $r(A_1) = 1$, and $r(A_j) = 0$ $(j \geqslant 2)$. But the modified nerve $\mathfrak{M}$ of $A_1, A_2, \ldots, A_n$ is simply the nerve of $B_1, B_2, \ldots, B_n$—i.e., is T. Hence $r(\mathfrak{M}) = r(T) = k$.

However, the replacement of $b_1(\mathfrak{M})$ by $r(\mathfrak{M})$ in Theorem 7 is justified (under reasonable conditions) provided all the sets A_j are unicoherent. For simplicity we consider only the polyhedral case (though the generalization to ANR's would be easy), and in stating the result do not distinguish between "complex" and "polytope".

THEOREM 8. *Let $A_1, A_2, \ldots, A_n$ be closed, connected, non-empty unicoherent subcomplexes of a complex S, which cover S, and let $\mathfrak{M}$ be their modified nerve. Then $r(S) = r(\mathfrak{M})$.*

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Sketch of proof. Choose points $p_J{}^{\alpha} \in A_J{}^{\alpha}$, $\{A_J{}^{\alpha}\}$ being the natural d.s. of $\{A_j\}$, and for each pair $A_J{}^{\alpha}$, $A_K{}^{\beta}$ with $K \supset J$ and $A_J{}^{\alpha} \supset A_K{}^{\beta}$, join $p_K{}^{\alpha}$ to $p_J{}^{\beta}$ by an arc in $A_J{}^{\alpha}$. These arcs form a graph G. There is an obvious mapping ϕ of the edge-paths in $\mathfrak{M}$ onto paths in $G \subset S$. In general, ϕ need not induce a homomorphism of $\pi_1(\mathfrak{M})$. However, if $r(S) = r$, there exists [4, p. 110] a homomorphism ψ of $\pi_1(S)$ onto F_r, the free (non-abelian) group on r generators. Using the fact that the sets A_j are unicoherent, one can show that $\psi\phi$ induces a homomorphism of $\pi_1(\mathfrak{M})$ onto F_r. Hence [4, p. 110] $r(\mathfrak{M}) \geqslant r$. But $r(\mathfrak{M}) \leqslant r$, by Theorem 6 (6.4); and Theorem 8 is established.

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