Birationally Related Cubics.

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I.

Birational whole-plane correspondence.

1. The present paper is an extension to the general cubic of certain relations connecting two special cubics previously considered.* In what follows the term "birational" refers exclusively to the quadric transformation through three fixed points, the correspondence therefore being a whole-plane one for each transformation, and not confined to points on the transformed cubics. Also, the notation is that used in the paper referred to above.

Incidentally there is given a method—reducible to simple terms—of finding the ninth fixed point of a system of cubics through eight fixed points.

2. Let Π, Π' be two superposed planes, ABC, $A'B'C'$ two triangles of reference in Π and Π' with systems of co-ordinates $P(xyz)$, $p(x'y'z')$ respectively. Then the equations

$$\lambda xx' = \mu yy' = \nu zz' \quad \text{.......................} \quad \text{(A)}$$

establish a *birational correspondence* (b.c.) between P and p.† It is easily seen that a line in Π (or Π') corresponds to a conic in Π' (or Π) through $A'B'C'$ (or ABC). The conics become line-pairs if the lines pass through the vertices of the reference triangle. Thus AP in Π becomes the line-pair $A'p$, $B'C'$; we ignore, however,

* Vide *Proc. Edin. Math. Soc.*, Vol. XXXIII. (Pt. 2), 1914–15, "Two remarkable cubics associated with a triangle."

† *Cf.* Salmon, *Higher Plane Curves*, 3rd ed., p. 309.

$B'C'$ and regard $A'p$ as the corresponding locus in Π'. (See fig. 1.)

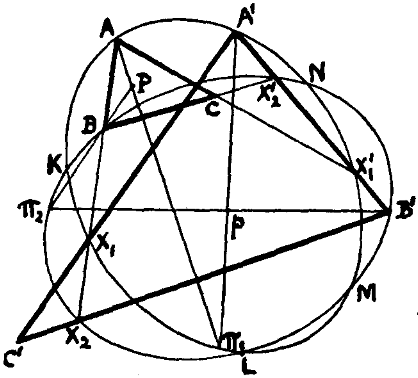

fig. 1.

3. *The four self-correspondents.*

Since the line $y=kz$ in Π is transformed by (A) into $z'=k'y'$ say...(1); the two are homographic, AB and AC corresponding to $A'C'$ and $A'B'$ respectively. Thus the "product" of AP and $A'p$ is a conic through AA'.

Similarly BP and $B'p$, or $z=lx$ and $x'=l'z'$...(2), generate a second conic, through BB'. These conics meet where $P=p$, *i.e. in four points* $KLMN$, say (K), which are the *self-correspondents* of the transformation. From (1) and (2) we have $y=klx$ transformed into $x'=k'l'y$, showing that CP, $C'p$ *generate a third conic*, through CC'. *All three conics pass through the four points* (K).

4. *Seven points in general sufficient to establish a (b.c.).*

Let $(K)\,ABC$ be seven given (independent) points. Take KLM for our triangle of reference for the two planes; and, *using absolute trilinears*, let $(x_1y_1z_1)$ be the point N. Choose (K) for the self-correspondents, and let ABC, $A'B'C'$ be the triangles of reference for the transformation (A).

Let $l_rx+m_ry+n_rz=0$; $l_r'x+m_r'y+n_r'z=0$... $[r=1, 2, 3]$ be the sides of ABC, $A'B'C'$ referred to KLM, all the coefficients being *absolute* in the sense that $(\Sigma l^2-2\Sigma mn\cos K)^{\frac{1}{2}}=+1$; and the signs of l, m, n being taken so that the origin is on the $+^{\text{ve}}$ side of all the lines.

Then from (A), we have for the point K (δ, 0, 0) say,

and, for L and M,
$$\left.\begin{array}{l}\lambda\, l_1 l_1' = \mu\, l_2 l_2' = \nu\, l_3 l_3' \\ \lambda\, m_1 m_1' = \mu\, m_2 m_2' = \nu\, m_3 m_3' \\ \lambda\, n_1 n_1' = \mu\, n_2 n_2' = \nu\, n_3 n_3'\end{array}\right\} \quad\ldots\ldots\ldots \quad (\mathrm{B})$$

Also for N,

$$\lambda\,(l_1 x_1 + m_1 y_1 + n_1 z_1)\,(l_1' x_1 + m_1' y_1 + n_1' z_1) = \mu\,(l_2 x_1 \ldots)\,(l_2' x_1 \ldots)$$
$$= \nu\,(l_3 x_1 \ldots)\,(l_3' x_1 \ldots) \quad \ldots\ldots\ldots \quad (\mathrm{C})$$

Since also $(\Sigma\, l_r'^2 - 2\Sigma\, m_r' n_r' \cos K)^{\frac{1}{2}} = +1 \quad [r = 1, 2, 3]$ ...... (D) we have, in (B) (C) (D), 11 equations for finding l_r', m_r', n_r' and $\lambda : \mu : \nu$.

It may be shown without difficulty that the solutions are unique.

5. Using (B) in (C) we see that N lies on three conics through KLM.

Again, write (C) in the form

$$\lambda\, XX' = \mu\, YY' = \nu\, ZZ'$$

where (X) and (X') are the coordinates of the *same* variable point. Then $\mu\, YY' = \nu\, ZZ'$ is a conic—on which N lies—through the points YZ, $Y'Z'$, YZ', $Y'Z$; or A, A', (AC, $A'B'$), (AB, $A'C'$). The three conics are therefore (K) AA', (K) BB', (K) CC'.

6. *Geometrical constructions.*

(A.) *Given* (K) $AB\,A'B'$ *and* P; A' *and* B' *being any given points on the conics* (K) A, (K) B *respectively; to find* C, C' *and* p.

To find p (Fig. 1) draw $AP\pi_1$; $BP\pi_2$; then $(A'\pi_1, B'\pi_2) = p$, and conversely.

If $P = A$, π_1 is indefinite, $\pi_2 = X_2$, and p is any point on $B'X_2$; similarly if $P = B$, p is any point on $A'X_1$.

If P lies on AB, p is easily seen to be $(A'X_1, B'X_2)$, or C' (say), and conversely.

And if p lies on $A'B'$, $P = (AX_1', BX_2')$ or C (say), and conversely.

Hence the construction for C and C'. It remains to prove that they lie on a conic through $KLMN$.

Let the coordinates of P and p, referred to their own triangles, be (lmn) and ($l'm'n'$) respectively.

Then AP, $A'p$ are the lines $nY = mZ$, $n'Y' = m'Z'$.

But $(AP, A'p)$ lies on the given conic, say $\mu YY' = \nu ZZ'$ [Art. 5] (1)
whence $$\mu mm' = \nu nn'.$$

Similarly $(BP, B'p)$ lies on the given conic, say $\nu ZZ' = \lambda XX'$ (2)
whence $$\nu nn' = \lambda ll'.$$

$$\therefore \quad \lambda ll' = \mu mm' = \nu nn' \quad \ldots\ldots\ldots\ldots \quad (a)$$

showing that P and p are birationally related.

Also $\lambda XX' = \mu YY'$ (3) is a conic through the meets of (1) and (2), *i.e.* through $KLMN$, and $(CP, C'p)$ evidently lies on this conic, by (a).

Finally (3) passes through XY and $X'Y'$, or C and C'.

(B.) *Given* $(K)ABC$ *and* P, *to find* $A'B'C'$ *and* p.

Draw the conics $(K)A$, $(K)B$; ABX_1X_2; ACX_1'; BCX_2'; $X_1'X_2'A'B'$; then $(A'X_1, B'X_2) = C'$. Also p may be found as before.

The constructions are evidently unique, and may be made by ruler and pencil.*

II.

Two birationally related cubics through (K).

7. Let C_1 be any cubic, nodal or otherwise, in Π. Take 7 points on it, $(K)ABC$, none of which is at the node. These determine a birational system, and C_1 is transformed into a second cubic C_2, through $(K)A'B'C'$, as is easily seen by analysis on using equations (A).

The following properties, some of which were proved for the special cubics,† are summarised here:—

(1) If P is a fixed point on C_1, PQR a variable line cutting C_1 in Q and R; then, for C_2, pqr lie on a variable conic through $A'B'C'p$; and qr passes through a fixed point p_1' on C_2, called the *cross-correspondent* (c.c.) of P. The relation is reciprocal.

(2) If $P = A$, then $p_1' = A'$ [Art. 2]. *Hence* $A'B'C'$ *are the c.c. of* ABC, and may be written a_1', b_1', c_1' respectively.

(3) If $Q = R$, then $q = r$ and P, p_1' are the tangentials of two corresponding tetrads.‡

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* *Vide* Cremona, *Proj. Geom.*, trans. by C. Leudesdorf, 2nd ed., p. 176.

† *Op. Cit.* ‡ For non-singular cubics.

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(4) *If the join of PQ, two points on C_1, meets the curve in a fixed point R, the join of their c.c. $p_1'q_1'$ will meet C_2 in a fixed point s_2, and conversely.*

Let t be the tangential of r. Then since qrp_1', rpq_1', pqr_1', rrt, $q_1'p_1's_2$ are collinear triads, we have the scheme

$$\begin{matrix} r & r & t \\ p & q & r_1' \\ q_1' & p_1' & s_2 \end{matrix}$$

whence, by Maclaurin's property, s_2 is collinear with t and r_1', which are fixed. The converse is similarly proved. Hence *if PQRS form a tetrad on C_1, with a common tangential, their c.c. $p_1'q_1'r_1's_1'$ will form a tetrad on C_2.* This admits of a simple direct proof.

(5) The pencils PQR, $p_1'qr$ are homographic and generate a conic through $(K)\,Pp_1'$.

If the cubics do *not* pass through (K) the properties (1) to (4) still hold; but in (5) the conic does not pass through (K).

8. *The joins of the correspondents of two birationally related cubics through (K) pass through a fixed point $H' = t'$ on both cubics.*

Let Qq be any pair of correspondents.

Draw a line cutting C_1C_2 in $PQR\,p_1\,qr_1$.

(1) If p_1' is the c.c. of P, then $p_1'qr$, PQR correspond; hence q is on the conic $(K)\,Pp_1'$ [Art. 7 (5)]. Since this cuts C_2 in the six

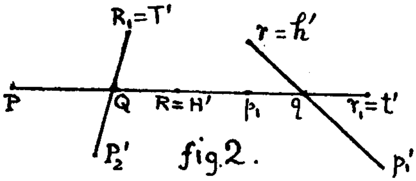

fig.2.

points $(K)\,p_1'q$ of which (K) are fixed, r is a fixed point ($= h'$ say) [Fig. 2]. Hence also R is fixed ($= H'$ say).

(2) If P_2' is the c.c. of p_1, then $P_2'QR_1$, p_1qr_1 correspond, and it easily follows that R_1 is a fixed point ($= T'$ say); hence also r_1 is fixed ($= t'$ say).

But Q and q are variable: hence $r_1 = R = H' = t'$. Q.E.D. Evidently also $p_1 = p$ the correspondent of P.

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9. The point $H'(=t')$ will be called the *centre of the system.* It is evidently a self-c.c. It will be seen later that there is in general only one such point. It is evident that *H' is the tangential of four common tangents to C_1 and C_2, the points of contact for each tangent being correspondents.*

10. *Def.* (*a*) If PQR are three collinear points on a cubic, in any order, R is the *P-conjugate* of Q.

(*b*) If P and p are correspondents on C_1 and C_2, then the t'-conjugate of p (viz. p_1) is called the *conjugate-correspondent* (conj.-corr.) of P; so also P_1 and p are a pair of conj.-corr.

Let Pp_1' be a pair of c.c., and let $H'P$, $H'p_1'$ cut $C_1 C_2$ as in Fig. 3.

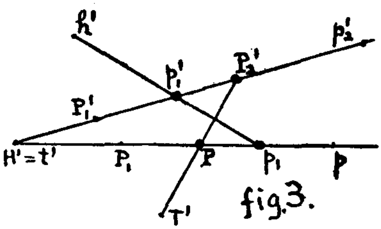

fig.3.

Then by the def. of c.c., since $H'P_1 P$ collinear, $\therefore h'p_1 p_1'$ collinear; and Pp_1, $p_1'p_1$ are corresponding rays. Hence p_1 lies on the conic (K) Pp_1'. Similarly, $t'p_1'p_2'$, and $\therefore T'PP_2'$, are collinear; and P_2' lies on the conic.

Also $p_1 p_1'h'$ and $P_2'P_1'H'$ are collinear, whence $P_2'p_1$ are c.c.

Hence (1) *every conic* (K) *cuts* C_1C_2 (i) *in two pairs of c.c.* (Pp_1'; $P_2'p_1$) (ii) *in two pairs of conj.-corr.* (Pp_1; $P_2'p_1'$).

(2) *Every line through H' is cut by two conics (K) in pairs of conj.-corr.* (Pp_1; P_1p).

(3) The cross joins PP_2', p_1p_1' pass through the (fixed) points (T', h').

11. *Every line in the plane* $(\Pi = \Pi')$ *contains in general two pairs (only) of correspondents, and if the line passes through a fixed point the locus of the points is a pair of cubics.*

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(1) Let l [Fig. 4] be any line in the plane; denote it by l_1 or l_2 according as it is regarded as a line in Π or Π', so that $l_1 = l_2 = l$.

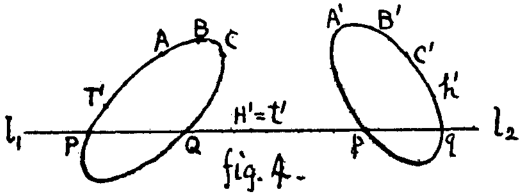

fig. 4.

Draw the conics Σ_1, Σ_2 corresponding to l_2, l_1 respectively, and meeting l in $PQpq$. These are the pairs of correspondents, for P and Q are (l_1, Σ_1) and this corresponds to (Σ_2, l_2) or p and q.

(2) Let the variable line l pass through the fixed point $H' = t'$. Then $\Sigma_1 \Sigma_2$ pass through the fixed points T', h' respectively, and we have two four-point systems. One member only of the system Σ_1 will pass through H', which is therefore a simple point on the locus. Hence every line through H' meets the locus in three points including H'; the locus is therefore a cubic which is easily seen to pass through the nine points $(K) ABCH'T'$. Similarly for p and q; and the two cubics meet at $H' = t'$.

12. *Geometrical construction for finding the ninth fixed point of a system of cubics through eight fixed points.*

Draw any cubic C_1 through seven points $(K) ABC$, and transform it into C_2. Fix an eighth point P on C_1; then p is an eighth fixed point on C_2. Now (Art. 11) the line Pp only contains two pairs of correspondents. If Qq are the second pair, these must lie on the cubics, since Pp passes through H', And Q, q being fixed, they must be the ninth points. The point H' is of course variable with $C_1 C_2$. Hence:—Choose any four of the points, say (K), for self-correspondents and any three of the remainder for the triangle ABC, P being the eighth point. Obtain the triangle $A'B'C'$ and p by Art. 6 (B). Join Pp and, using Fig. 1, construct the conic Σ_1 in Π, corresponding to Pp regarded as a line in Π' cutting the line in PQ; whence Q is found.

13. *Double infinity of transformations from C_1 to C_2.*

Referring to Fig. 1, and using the same letters for convenience —except that we shall write a_1' for A', etc.—suppose that AB, instead of being vertices of a triangle of reference, are *any* two points on C_1. Draw the conics $(K)A$, $(K)B$, respectively meeting C_1C_2 in two pairs of c.c., Aa_1', Bb_1' [see Art. 7 (2)]. We can then find a third pair of points Cc_1' exactly as in Art. 6, and we have to show that these lie on the cubics, and that they are c.c.

Let AB meet the cubic C_1 in C_1'; then, if c_1' is the C_2-correspondent, since Aa_1' are c.c., AC_1', $a_1'c_1'$ meet on the A-conic; hence c_1' lies on $a_1'X_1$ (as also does b). Similarly, c_1' and a lie on $b_1'X_2$.

Thus $c_1' = (A'X_1, B'X_2)$ as before; and similarly for C. We may show that $A_1'B_1'$, the correspondents of $a_1'b_1'$, lie on CB, CA respectively. Hence $CA_1'B$, $c_1'a_1'b$ correspond, *i.e.* C and c_1' are c.c.

It is easily seen that the *same* p on C_2 is found from a given point P on C_1 as in Art. 6, however we vary A and B.

Hence there is a double infinity of transformations. But, *points on C_1 (or C_2) alone have the same correspondents for all transformations, and these are the only pairs collinear with H'.*

14. *Second system of transformations.*

Let q_1r_1 be the conj.-corr. of QR. Then if QR be a variable line through a fixed point P, so that qr passes through the fixed point p_1', it is evident from Maclaurin's property that q_1r_1 passes through a fixed point p_1'', which we term the *cross-conjugate-correspondent* (c.c.c.) of P. The relation is reciprocal; and the pencils PQR, $p_1''q_1r_1$ are homographic and generate a conic through Pp_1'' and the self-conjugate-correspondents $K'L'M'N'$. The points (K') are the remaining four of the nine points common to C_1C_2.

The other relations are analogous to those previously considered. Thus:—

(1) *H' is a self-c.c.c.*, and for the general cubic there is only one such point.

(2) Conics through (K') cut C_1C_2 in two pairs of c.c.c., say Pp_1'', $P_1''p$, such that the cross-joins Pp, $P_1''p_1''$ pass through H'; *or, any pair of (direct) correspondents lie on a conic (K').*

(3) *A doubly-infinite system of transformations may be effected* by choosing two points AB on C_1, and drawing two conics (K') meeting C_1C_2 in Aa_1'', Bb_1''; the points Cc_1'' being found as before. P of course transforms into the conj.-corr. p_1.

15. It may be noted that any line through H' contains four points PP_1pp_1, which may be paired off in three ways, giving rise to pairs (PP_1, pp_1) on the cubics; conjugate-correspondents (Pp_1, P_1p) on two conics (K); and (direct) correspondents (Pp, P_1p_1) on two conics (K').

16. *Case of nodal cubics.*

It is clear that a node P corresponds to a node p. Also Pp are oth direct- and conjugate-correspondents. Hence

(1) *The line of nodes passes through the centre of the system, and is a common chord of two conics* (K), (K').

(2) The nodes are c.c. and also c.c.c. for each branch [Arts. 10 (1) and 14 (2)]

17. *A special transformation.*

Suppose the two conics $(K)Aa_1'$, $(K)Bb_1'$ to coincide. Then, referring to Fig. 1, X_1 and X_2 coincide with A or B, X_1' and X_2' coincide with a_1' or b_1'.

Now if one conic (K) be drawn cutting C_1C_2 in $ABa_1'b_1'$, then $H'Ab_1'$, $H'Ba_1'$ are collinear. Hence (Fig. 5) taking $X_1 = B$, $X_2 = A$,

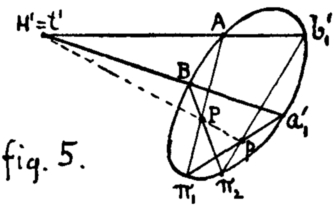

fig. 5.

we have $c_1' = H'$. Similarly if $X_1' = b_1'$, $X_2' = a_1'$, we have $C = H'$. Thus ABH', $a_1'b_1'H'$ are the triangles of reference.

If P be any point in Π, the construction for finding p reduces to the following: $AP\pi_1$; $BP\pi_2$; $(a_1'\pi_1, b_1'\pi_2) = p$.

Taking $Ab_1'\pi_2Ba_1'\pi_1$ for the Pascal hexagon it is seen that $H'Pp$ are collinear. This is true for all pairs of correspondents whether on C_1 and C_2 or not.

If $P = p$, each lies on the conic. Hence *all points on the conic* (K) *are self-correspondents for the whole-plane transformation.* We may note that every line in the plane, not passing through H', still contains two pairs of correspondents, *but they are self*

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corresponding. Also, the locus of pairs of correspondents whose join passes through a fixed point O is the line OH' and the conic, twice over; when $O = H'$ we have the whole plane for the locus.

For simplicity we may take for the conic the line-pair KM, LN. The birational relation may be shown to be, in trilinears,

$$XX' = YY' = \nu ZZ',$$

(XYZ) $(X'Y'Z')$ being the coords. of P and p referred to ABH', $a_1'b_1'H'$ respectively.

18. *Case of more than a single self-c.c.; double cubics.*

A self-c.c. other than H' must be one of the points (K'), say K'. Then $K'PQ$, $K'pq$ correspond, hence if $P = p = K$, we have $Q = q = L$ say; *i.e.* K' lies on KL; similarly on MN.

Hence (1) *there are four self-c.c.*, viz., H' and the diagonal points of $KLMN$, say $K'L'M'$; (2) the cubics are not general. Also we may prove that (3) *if* $K'L'M'$ be chosen for the (double) triangle of reference (Art. 13) the cubics coincide; (4) if H' is (a_0, b_0, c_0), and (K) is $(f, \pm g, \pm h)$, then N' is (f^2/a_0); the transformation being $xx'/f^2 =$ etc. (5) H' is the tangential of (K), and N' that of the four self-c.c. (6) A double cubic can also occur when there are four self-c.c c., H' being the tangential of (K'); etc.

III.

Involution pencil of birationally related cubics.

19. *System of cubics* $C_1 + \Lambda C_2$.

Let $H'X$ be any *fixed* line through H' and not passing through any of the points (K) (K'), meeting $C_1 C_2$ in PP_1, pp_1. Then we have seen that conics $(K) Pp_1$, $(K) P_1 p$, $(K') Pp$, $(K') P_1 p_1$ can be drawn. Let the p's always refer to $C_1 C_2$.

Now let two cubics of the system $C_1 + \Lambda C_2$, say $C_1 + \lambda C_2$, $C_1 + \mu C_2$, or (λ) (μ), be drawn cutting $H'X$ in points $QQ_1 qq_1$. Then one condition for a (b.c.) is that four conics $(K) Qq_1$, etc., can be drawn.

Now by varying Λ, since all the cubics of the system pass through H', we have an involution of points on $H'X$, defined by

PP_1pp_1). Projecting on to a conic Σ (Fig. 6)—and using the same letters—these determine a vertex O. Let $OO'O''$ be the diagonal triangle and therefore self-conjugate to the conic.

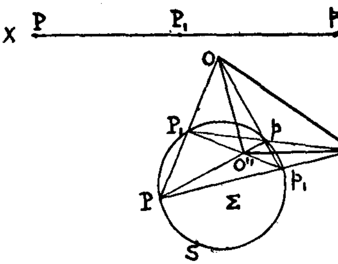

fig. 6.

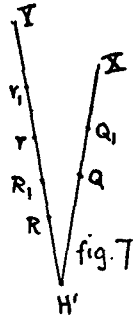

fig. 7

Keeping $OO'O''$ fixed we may vary P (say $=Q$), the other three points being determined by the fact that $OO'O''$ is always the diagonal triangle of QQ_1qq_1. Thus $O'O''$ are vertices giving two other involutions (Pp_1, P_1p) and (Pp, P_1p_1), which are those given by the conics (K) and (K'). Hence if the cubic (λ) be drawn giving QQ_1, then qq_1 are uniquely found, and the group (Q) possesses the *four-conics property* above.

Also if λ is given, μ is uniquely found, and conversely. The cubics (λ) and (μ) are therefore mates of an involution pencil through (K) (K') H'. But (0) and (∞) are mates, hence $\lambda\mu=$ constant $=a^2$ say, and a group (Q) is given by the cubics (λ) (a^2/λ).

20. Let a second line $H'Y$ (Fig. 7) be drawn, cut by (λ) in RR_1. These determine rr_1, points on a second cubic (μ') say, where $\lambda\mu'=b^2$. We shall show that $b=a$, $\mu'=\mu$.

Choose λ so that $Q=q$ or q_1 on $H'X$; then $Q_1=q_1$ or q. The cubics (λ) (a^2/λ) have therefore eleven common points, hence they coincide and $\lambda=\pm a$. Let the cubic ($+a$) give the points QQ_1, RR_1 as in the figure, so that RR_1 lie on a conic (K), since H' is the *point opposé** for (K). But Rr_1 lie on a conic (K), as also R_1r. Hence r and r_1 coincide with R and R_1; therefore $b=a$, and $\mu'=\mu$.

* See a paper by Lieut. Edward Press in Part 1 of the present volume.

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We have therefore shown that all lines through H'—and we may now include those through (K) (K')—meet the cubics (λ) (a^2/λ) in groups (Q) possessing the four-conics property. Changing λ (or μ) into $a\lambda$ (or $a\mu$), and aC_2 into C_2 we may write $\lambda\mu=1$; suppose this done. Then for all λ's we may say that the cubics $C_1+\lambda C_2$, λC_1+C_2 possess the four-conics property.

21. The involution pencil of cubics may be correlated to the involution pencil O, of which OO'', OO' are the double lines. Assume the order of the points (P) to be as in the figure. Then only one of the lines OO', OO'' cuts the conic Σ in real points; hence only one of the double cubics cuts $H'X$ in two real points other than H', viz., that in which $P=p_1$, $P_1=p$.

22. *Proof of birationality.*

We shall now prove that (λ) $(1/\lambda)$ are birationally related. Draw conics $(K)\,Aa_1'$, $(K)\,Bb_1'$ cutting (λ) $(1/\lambda)$ in AB, $a_1'b_1'$. Let these points determine a (b.c.) as in Art. 6. Then the locus of pairs of correspondents whose join passes through the fixed point H' is a pair of cubics $C_3\,C_4$ through seven points on (λ) and seven on $(1/\lambda)$, viz., $(K)\,ABH'$, $(K)\,a_1'b_1't'$.

Now since $C_3\,C_4$ are birationally related the conics (K) determine pairs of conj.-corr. exactly as for $C_1\,C_2$.

Join $H'A$; then $(K)A$ is a definite conic, and cuts $H'A$ in a point a_1 on C_4. But, since A is on (λ), a_1 is on $(1/\lambda)$, by the four-conics property. Hence a_1 is common to C_4 and $(1/\lambda)$. Similarly, if $b_1\,A_2'B_2'$ are the conj.-corr. of $Ba_1'b_1'$, then b_1 lies on C_4 and $(1/\lambda)$; $A_2'B_2'$ each lie on C_3 and (λ).

Hence C_3 passes through nine points on (λ), viz., $(K)\,ABA_2'B_2'H'$; and C_4 through nine on $(1/\lambda)$, viz., $(K)\,a_1'b_1'a_1\,b_1t'$. But AA_2', BB_2' are chords of two conics (K), hence they each pass through the same point T'' (say) on (λ) and on C_3. Hence C_3 and (λ) have more than nine points in common, *i.e.* $C_3\equiv(\lambda)$; similarly $C_4\equiv(1/\lambda)$. We might also have proceeded from (K').

23. It has been proved, then, that an involution pencil of cubics $C_1+\Lambda C_2$ through $(K)\,(K')\,H'$ exists, the mates of which are birationally related. The double cubics are $C_1\pm C_2$. Writing

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$C_1 + C_2 = D_1$, $C_1 - C_2 = D_2$, and changing $(1-\lambda)/(1+\lambda)$ into λ, then $D_1 \pm \lambda D_2$ represents a pair of mates, and D_1 D_2 are the double cubics.

The tangents at each of the points (K) (K') H' of course form an involution; the tangents at (K) and (K') to the double cubics respectively pass through H'. [Art. 18 (5) and (6)].

Finally, the twelve nodal cubics of the system consist of six pairs of mates, the nodes lying in pairs on six lines through H'.

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