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# NETS OF CONICS IN THE EUCLIDEAN PLANE AND AN ASSOCIATED REPRESENTATIONAL GEOMETRY

BY
R. BLUM AND A. P. GUINAND

**Introduction.** The study of systems of conics and other algebraic curves was initiated in the middle of the nineteenth century by Cayley, Hesse, Cremona, and others. Most of the investigations from that time to the present have been concerned with extensions to algebraic varieties and systems of higher orders or dimensions, or with associated algebraic curves such as Jacobians and Hessians. By contrast, scant attention has been given to the details of internal structure of even the simplest systems of curves in the plane. References to much of the source material for any work on systems of curves can be found in references (1), (2), and (3).

In the present paper we discuss certain aspects of nets of conics in the euclidean plane. A net of conics here means a linear system of conics $\lambda S_1+\mu S_2+\nu S_3=0$, where $S_1=0$, $S_2=0$, $S_3=0$ are three independent co-planar conics.

The paper has three parts. In the first part we discuss the elementary properties of those nets of conics in the euclidean plane which we call general nets. We use a method of setting up the net in a standard form by taking as basis three degenerate parabolas of the net. This standard form leads to the consideration of analogies with the special cases where the net consists of all the conics circumscribing a fixed triangle. We call these latter nets "triangular nets". In particular, the nine-point circle of the fixed triangle is shown to be a degenerate case of a twelve-point circle associated with any more general net.

In the second part we discuss a representational "net-geometry" based on any general net of the type considered in Part I. The individual conics of the net become the point-elements of the net-geometry, and pencils of conics become line-elements. This representational geometry is, of course, a plane projective geometry which can be made euclidean by an arbitrary choice of the absolute elements (i.e. the line at infinity and the circular points). However, we show that in general a non-degenerate net of conics in the euclidean plane does possess distinctive elements whose choices as absolute elements seem specially appropriate. We are thereby led to a euclidean metric and a cartesian co-ordinate system which seem, by virtue of their mathematical elegance, to be the most natural for this net-geometry.

In the third part we examine the special or degenerate nets which were specifically excluded from consideration in the first two parts. For this purpose we use another

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standard form for the net, taking as basis two rectangular hyperbolas and a circle of the net. Again, the basis can be so set up that it is well suited to the net-geometry, and this latter form is more appropriate when considering nets which do not necessarily contain three distinct degenerate parabolas. We are thereby able to indicate a method of classifying the types of nets excluded from Parts I and II.

## PART I

**Circles associated with a net of conics.** A triangular net of conics contains one circle only, the circum-circle of the triangle which all the conics of the net circumscribe. Also, the locus of the centres of the rectangular hyperbolas of the net is the nine-point circle of this triangle (Salmon, reference (4), 215, Ex. 2).

Similar results hold for the general net. To show this, consider the net of conics

$$\lambda S_1+\mu S_2+\nu S_3 = 0 \tag{1}$$

where, in orthogonal cartesian coordinates,

$$S_r \equiv a_r x^2+2h_r xy+b_r y^2+2g_r x+2f_r y+c_r = 0 \qquad (r = 1, 2, 3)$$

are three linearly independent conics, none of which are circles or rectangular hyperbolas. If (1) is to be a circle, then

$$\lambda(a_1-b_1)+\mu(a_2-b_2)+\nu(a_3-b_3) = 0, \tag{2}$$

and

$$\lambda h_1+\mu h_2+\nu h_3 = 0. \tag{3}$$

In general these equations have only one solution for the ratio $\lambda:\mu:\nu$, so the net contains one circle only, which we will call the circumcircle of the net. (For the cases when (2) and (3) have more than one solution, see Part III).

Also, if (1) is to be a rectangular hyperbola then (reference (4), 169)

$$\lambda(a_1+b_1)+\mu(a_2+b_2)+\nu(a_3+b_3) = 0. \tag{4}$$

That is, there is a pencil of rectangular hyperbolas in the net. Further, the four points common to the conics of this pencil form an orthocentric set, since any line-pair through all four points must be a degenerate rectangular hyperbola.

If $(x, y)$ is the centre of (1), then (reference (5), 114)

$$(\textstyle\sum \lambda a)x+(\sum \lambda h)y+(\sum \lambda g) = 0, \tag{5}$$

$$(\textstyle\sum \lambda h)x+(\sum \lambda b)y+(\sum \lambda f) = 0, \tag{6}$$

where we write $\lambda a_1+\mu a_2+\nu a_3=(\sum \lambda a)$, and similarly for other coefficients. That is

$$\lambda(a_1x+h_1y+g_1)+\mu(a_2x+h_2y+g_2)+\nu(a_3x+h_3y+g_3) = 0,$$

$$\lambda(h_1x+b_1y+f_1)+\mu(h_2x+b_2y+f_2)+\nu(h_3x+b_3y+f_3) = 0.$$

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Hence the locus of centres of rectangular hyperbolas of the net is

(7) $$\begin{vmatrix} a_1x+h_1y+g_1 & a_2x+h_2y+g_2 & a_3x+h_3y+g_3 \\ h_1x+b_1y+f_1 & h_2x+b_2y+f_2 & h_3x+b_3y+f_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{vmatrix} = 0.$$

This reduces to

(8)

$$-(x^2+y^2)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + x\begin{vmatrix} a_1 & a_2 & a_3 \\ f_1 & f_2 & f_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + x\begin{vmatrix} g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{vmatrix} +$$

$$+y\begin{vmatrix} g_1 & g_2 & g_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} + y\begin{vmatrix} h_1 & h_2 & h_3 \\ f_1 & f_2 & f_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{vmatrix} + \begin{vmatrix} g_1 & g_2 & g_3 \\ f_1 & f_2 & f_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{vmatrix} = 0.$$

This is a circle provided that

(9) $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ h_1 & h_2 & h_3 \end{vmatrix} \neq 0.$$

Let us assume that this is so.* Then the circle (8) must pass through the three intersections of mutually perpendicular lines associated with the above-mentioned orthocentric set.

**The locus of centres of degenerate conics of the net.** For (1) to be degenerate it is necessary and sufficient that

(10) $$\begin{vmatrix} (\sum \lambda a) & (\sum \lambda h) & (\sum \lambda g) \\ (\sum \lambda h) & (\sum \lambda b) & (\sum \lambda f) \\ (\sum \lambda g) & (\sum \lambda f) & (\sum \lambda c) \end{vmatrix} = 0.$$

Eliminating $\lambda$, $\mu$, $\nu$ from (5), (6), and (10) we find that the locus of centres of degenerate conics of the net is the cubic

(11) $$\begin{vmatrix} a_1x+h_1y+g_1 & a_2x+h_2y+g_2 & a_3x+h_3y+g_3 \\ h_1x+b_1y+f_1 & h_2x+b_2y+f_2 & h_3x+b_3y+f_3 \\ g_1x+f_1y+c_1 & g_2x+f_2y+c_2 & g_3x+f_3y+c_3 \end{vmatrix} = 0.$$

This is the Jacobian of the net. (Sommerville, reference (5).)

**Parabolas of the net.** For (1) to be a parabola $\lambda$, $\mu$, $\nu$ must satisfy

(12) $$(\sum \lambda a)(\sum \lambda b)-(\sum \lambda h)^2 = 0.$$

* The cases when this determinant vanishes are considered in Part III.

That is, the parabolas form a quadratic family of conics of the net. The degenerate parabolas (or parallel line-pairs) correspond to those $\lambda$, $\mu$, $\nu$ which satisfy both (10) and (12). These equations are a cubic and a quadratic in $\lambda$, $\mu$, $\nu$, but as each solution is repeated, there cannot be more than three distinct solutions. Using (12) and putting

$$\frac{(\sum \lambda a)}{(\sum \lambda h)} = \frac{(\sum \lambda h)}{(\sum \lambda b)} = p,$$

then (10) becomes

$$\begin{vmatrix} p(\sum \lambda h) & p(\sum \lambda b) & (\sum \lambda g) \\ (\sum \lambda h) & (\sum \lambda b) & (\sum \lambda f) \\ (\sum \lambda g) & (\sum \lambda f) & (\sum \lambda c) \end{vmatrix} = 0,$$

whence

$$\begin{vmatrix} 0 & 0 & (\sum \lambda g)-p(\sum \lambda f) \\ (\sum \lambda h) & (\sum \lambda b) & (\sum \lambda f) \\ (\sum \lambda g) & (\sum \lambda f) & (\sum \lambda c) \end{vmatrix} = 0.$$

Hence

$$(\sum \lambda g) = p(\sum \lambda f),$$

or

$$(\sum \lambda h)(\sum \lambda f) = (\sum \lambda b)(\sum \lambda g)$$

In either case we have

$$p = \frac{(\sum \lambda a)}{(\sum \lambda h)} = \frac{(\sum \lambda h)}{(\sum \lambda b)} = \frac{(\sum \lambda g)}{(\sum \lambda f)},$$

or

(13) $$(\sum \lambda(a-hp)) = (\sum \lambda(h-bp)) = (\sum \lambda(g-fp)) = 0$$

Hence $p$ is a root of the cubic*

(14) $$\begin{vmatrix} a_1-h_1p & a_2-h_2p & a_3-h_3p \\ h_1-b_1p & h_2-b_2p & h_3-b_3p \\ g_1-f_1p & g_2-f_2p & g_3-f_3p \end{vmatrix} = 0.$$

Each root of this cubic in $p$ leads, in general, to one solution for the ratio $\lambda:\mu:\nu$, via the equations (13).

Thus, in general, a net contains three degenerate parabolas. Unlike the three degenerate rectangular hyperbolas, these three degenerate parabolas will not, in general, belong to a pencil, for there is no third parallel line-pair through the four points of intersection of two parallel line-pairs, unless the latter either have their axes parallel, or if at least one is a repeated line.

**A standard form for a general net of conics.** The preceding considerations suggest that for a general net the three degenerate parabolas can be selected as a "canonical

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* For the cases when this cubic is degenerate, see Part III.

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basis" for the net, which basis then depends only on the intrinsic properties of the net itself, and not on the co-ordinate system chosen.

We shall refer to the triangle formed by the axes of this triad of degenerate parabolas as the "axial triangle" of the net, and to its vertices as the "axial vertices" of the net. We adopt a "standard form" by choosing the orthocentre of the axial triangle as origin, writing $R$ for the circum-radius of the axial triangle, $A$, $B$, $C$ for its angles, and $\theta_1$, $\theta_2$, $\theta_3$ for the angles between the $x$-axis and the perpendiculars to $BC$, $CA$, $AB$ respectively. The equations of the sides $BC$, $CA$, $AB$ can then be written:

$$(15)\qquad \begin{aligned} m_1 &\equiv x\cos\theta_1+y\sin\theta_1-2R\cos(\theta_3-\theta_1)\cos(\theta_1-\theta_2) = 0,\\ m_2 &\equiv x\cos\theta_2+y\sin\theta_2-2R\cos(\theta_1-\theta_2)\cos(\theta_2-\theta_3) = 0,\\ m_3 &\equiv x\cos\theta_3+y\sin\theta_3-2R\cos(\theta_2-\theta_3)\cos(\theta_3-\theta_1) = 0.\end{aligned}$$

If the spacings of the three parallel line-pairs (or degenerate parabolas) are $2\alpha$, $2\beta$, $2\gamma$ respectively, then the canonical basis consists of the three conics

$$(16)\qquad \begin{aligned} S_1 &\equiv m_1^2-\alpha^2 = 0,\\ S_2 &\equiv m_2^2-\beta^2 = 0,\\ S_3 &\equiv m_3^2-\gamma^2 = 0.\end{aligned}$$

**Circumcircle of the net in standard form.** If, in the standard form (16), the conic

$$(17)\qquad S \equiv \lambda S_1+\mu S_2+\nu S_3 = 0$$

is to be a circle, then

$$(\textstyle\sum \lambda\cos^2\theta_1) = (\sum \lambda\sin^2\theta_1),$$

$$(\textstyle\sum \lambda\cos\theta_1\sin\theta_1) = 0.$$

That is

$$(\textstyle\sum \lambda\cos 2\theta_1) = (\sum \lambda\sin 2\theta_1) = 0.$$

Hence

$$(18)\qquad \lambda:\mu:\nu = \sin 2(\theta_2-\theta_3):\sin 2(\theta_3-\theta_1):\sin 2(\theta_1-\theta_2).$$

Setting these values in (17) we find, after some trigonometric manipulation, that the circum-circle is of the form

$$(19)\quad (x^2+y^2)\textstyle\sum\sin 2(\theta_2-\theta_3) = 2\sum\alpha^2\sin 2(\theta_2-\theta_3)-8R^2\sum\sin 2(\theta_2-\theta_3)\times\cos^2(\theta_1-\theta_2)\cos^2(\theta_3-\theta_1).$$

This can be expressed more simply in terms of the angles $A$, $B$, $C$, of the axial triangle, since, modulo $2\pi$,

$$(20)\qquad A\equiv\pi+\theta_2-\theta_3,\ B\equiv\pi+\theta_3-\theta_1,\ C\equiv\pi+\theta_1-\theta_2.$$

The equation (19) becomes

$$(x^2+y^2)(\textstyle\sum\sin 2A) = 2\sum\alpha^2\cos 2A-8R^2\sum\sin 2A\cos^2 B\cos^2 C.$$

Since

$$\textstyle\sum\sin 2A = 4\sin A\sin B\sin C$$

and

$$\sum \cot B \cot C = 1,$$

this can be written

$$x^2+y^2 = r^2,$$

where

$$(21)\qquad r^2 = \frac{\alpha^2 \cos A}{\sin B \sin C} + \frac{\beta^2 \cos B}{\sin C \sin A} + \frac{\gamma^2 \cos C}{\sin A \sin B} - 4R^2 \cos A \cos B \cos C.$$

Thus the centre, of the circum-circle of the net is the orthocentre of the axial triangle, and the radius $r$ is given by (21).

**The twelve-point circle of a net of conics.** In the standard form the condition (4) for rectangular hyperbolas becomes $\lambda+\mu+\nu=0$, and the equation (7) of the locus of their centres becomes

$$\begin{vmatrix} m_1 \cos \theta_1 & m_2 \cos \theta_2 & m_3 \cos \theta_3 \\ m_1 \sin \theta_1 & m_2 \sin \theta_2 & m_3 \sin \theta_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

That is

$$(22)\qquad C_N \equiv m_2 m_3 \sin A + m_3 m_1 \sin B + m_1 m_2 \sin C = 0.$$

This passes through the points where two of $m_1$, $m_2$, $m_3$ vanish; that is, through the axial vertices of the net. In the case of a triangular net the axial vertices are the mid-points of the sides of the triangle which all the conics of the net circumscribe; but this is not true of a general net. Thus the locus of the centres of the rectangular hyperbolas of a net is a circle through twelve points, namely the three centres of the orthogonal line-pairs of the net, the six mid-points of the associated orthocentric set, and the three axial vertices.

**The Euler axis of a general net.** For a triangle the Euler axis passes through the circum-centre, the centroid, the orthocentre, and the nine-point centre. For a general net there does not seem to be an unambiguous analogue of the orthocentre, but we can draw the following analogies.

| Point related to a triangular net | Analogous point for a general net | Point as related to the axial triangle | Designation |
|---|---|---|---|
| Nine-point centre of the circumscribed triangle $\triangle$ | Twelve-point centre | Circum-centre | $N$ |
| Circum-centre of $\triangle$ | Circum-centre | Orthocentre | $O$ |
| Centroid of $\triangle$ = Centroid of its mid-points triangle | Centroid of the axial triangle | Centroid | $G$ |

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With these analogies the Euler axis of the net coincides with the Euler axis of the axial triangle, and $OG=2GN$, just as for the Euler axis of a triangle. For a triangle the nine-point radius is equal to half the circum-radius. For a general net this is replaced by the formula (21).

**Some special cases.**

(i) Triangular net.

If $\alpha=2R\sin B\sin C$, $\beta=2R\sin C\sin A$, $\gamma=2R\sin A\sin B$, then the parallel line-pairs of the canonical basis all pass through three points $D$, $E$, $F$, where $D$ is the point $(2R\cos(\theta_2+\theta_3-\theta_1), 2R\sin(\theta_2+\theta_3-\theta_1))$, and similarly for $E$ and $F$. The net consists of all the conics through $D$, $E$, and $F$, and (21) reduces to $r=2R$.

(ii) If the half spacings $\alpha$, $\beta$, $\gamma$ are so chosen that the conics $S_1$, $S_2$, $S_3$ have a point of concurrency, then all conics of the net must pass through this point, including the circum-circle of the net. Then $r$ is equal to the distance between the orthocentre of the axial triangle and the point of concurrency. In particular, if $\alpha=2R\cos B\cos C$, $\beta=2R\cos C\cos A$, $\gamma=2R\cos A\cos B$, then there is a point of concurrency at the orthocentre of the axial triangle, and $r=0$, as may also be verified by (21).

(iii) If $\alpha=\beta=\gamma$, then any two of the parallel line-pairs of the canonical basis form a rhombus. The diagonals of each rhombus form an orthogonal line-pair which is a conic of the net, their centre being at the associated axial vertex, and the line-pair being the internal and external bisectors of the angle of the axial triangle there. Hence in this case also the twelve-point circle of the net degenerates into a circle through only nine points.

(iv) If $\alpha=\beta=\gamma=0$, then the net consists of the conics for which the axial triangle is self-conjugate.

**The locus of centres of conics of the net which have eccentricity $e$.** For a general conic

(23) $$ax^2+2hxy+by^2+2gx+2fy+c = 0$$

the eccentricity $e$ satisfies the equation

$$(2-e^2)^2(ab-h^2) = (1-e^2)(a+b)^2.$$

For the conic (17) of a general net in standard form this becomes

$$(2-e^2)^2((\sum \lambda\cos^2\theta_1)(\sum \lambda\sin^2\theta_1)-(\sum \lambda\cos\theta_1\sin\theta_1)^2) = (1-e^2)(\lambda+\mu+\nu)^2$$

This can also be written

(24) $$(2-e^2)^2\sum \mu\nu\sin^2 A = (1-e^2)(\lambda+\mu+\nu)^2.$$

Now the centre $(x, y)$ of (17) satisfies the equations

$$\frac{\partial S}{\partial x} = \frac{\partial S}{\partial y} = 0.$$

That is, by (15) and (16),

$$(\sum \lambda m_1 \cos \theta_1) = (\sum \lambda m_1 \sin \theta_1) = 0.$$

Hence

$$\frac{\lambda m_1}{\sin A} = \frac{\mu m_2}{\sin B} = \frac{\nu m_3}{\sin C}. \tag{25}$$

Substituting this in (24) and re-arranging, we obtain

$$(2-e^2)^2 \sin A \sin B \sin C\, m_1 m_2 m_3 (\sum m_1 \sin A) = (1-e^2)(\sum m_2 m_3 \sin A)^2 \tag{26}$$

for the locus of centres of conics of the net of eccentricity $e$. Now

$$\begin{aligned} \sum m_1 \sin A &= x \sum \cos \theta_1 \sin(\theta_2-\theta_3) + y \sum \sin \theta_1 \sin(\theta_2-\theta_3) \\ &\quad -2R \sum \sin(\theta_2-\theta_3)\cos(\theta_3-\theta_1)\cos(\theta_1-\theta_2) \\ &= -2R \sum \sin A \cos B \cos C \\ &= -2R \sin A \sin B \sin C. \end{aligned} \tag{27}$$

By (22) and (27) we can write (26) in the form

$$(1-e^2)C_N^2 + 2R(2-e^2)^2 \sin^2 A \sin^2 B \sin^2 C\; m_1 m_2 m_3 = 0. \tag{28}$$

That is, the locus of centres is a bicircular quartic passing through the vertices of the axial triangle, and having nodes there. If $e^2=1$ (for parabolas) this degenerates into the line at infinity and the sides of the axial triangle. If $e^2=2$ (for rectangular hyperbolas) it degenerates into the twelve-point circle, repeated.

Also, the locus is independent of the half spacings $\alpha$, $\beta$, $\gamma$. If we consider a particular conic of a particular net, keeping $\theta_1$, $\theta_2$, $\theta_3$, $\lambda$, $\mu$, $\nu$ constant, and vary the $\alpha$, $\beta$, $\gamma$ of the canonical basis, then the centre, orientation, and eccentricity of the conic remain unchanged; only its size alters. Accordingly, given any general non-degenerate net we can find an associated triangular net with corresponding conics having the same centres, orientations, and eccentricities (but not necessarily the same sizes) as the corresponding conics of the original net.

Further, the axial vertices are singular points of the net in the sense that for each vertex there is a pencil of conics of the net whose centres are at the vertex. For other points there is only one conic of the net whose centre is at the point.

**The Jacobian of the standard form of a net.** In standard form the equation (11) of the Jacobian becomes

$$\begin{vmatrix} m_1 \cos \theta_1 & m_2 \cos \theta_2 & m_3 \cos \theta_3 \\ m_1 \sin \theta_1 & m_2 \sin \theta_2 & m_3 \sin \theta_3 \\ \begin{matrix} 2Rm_1 \cos B \cos C \\ +\alpha^2 \end{matrix} & \begin{matrix} 2Rm_2 \cos C \cos A \\ +\beta^2 \end{matrix} & \begin{matrix} 2Rm_3 \cos A \cos B \\ +\gamma^2 \end{matrix} \end{vmatrix} = 0.$$

This reduces to

(29) $$2Rm_1m_2m_3 \sin A \sin B \sin C = \sum \alpha^2 m_2 m_3 \sin A.$$

That is, a cubic through the axial vertices which touches the conic $\sum \alpha^2 m_2 m_3 \sin A = 0$ at those points. If $\alpha=\beta=\gamma$, this conic is the twelve-point circle of the net.

## PART II

**A geometry on a net.** A plane geometry can be set up whose point-elements are the conics of the net. We shall use the following nomenclature and notations to distinguish the elements of this "net-geometry".

| Plane geometry | Net-geometry | Co-ordinates of equation in $\lambda$, $\mu$, $\nu$ |
|---|---|---|
| Conic of the net | Net-point | $\lambda$, $\mu$, $\nu$ (homogeneous) |
| Pencil of conics of the net | Net-line | $p\lambda+q\mu+r\nu = 0$ |
| Quadratic family of conics of the net | Net-conic | $A\lambda^2+B\mu^2+C\nu^2 +2F\mu\nu+2G\nu\lambda+2H\lambda\mu = 0$ |

Thus two net-lines meet in a net-point, two distinct net-points determine a net-line, and a net-line meets a net-conic in two net-points. In order to develop this projective geometry into a euclidean plane geometry we must choose a "net-line at infinity" and two "circular net-points" on it (reference (5), Chapter XIV). These absolute elements may, of course, be chosen arbitrarily, but there is one particular choice of them which appears peculiarly natural to this net-geometry. There are even natural choices for setting up a system of cartesian co-ordinates in the net-geometry.

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We make the following choices:

| Plane geometry | Net-geometry | Coordinates or equation in $\lambda$, $\mu$, $\nu$ |
|---|---|---|
| Pencil of rectangular hyperbolas of the net. | Net-line at infinity. | $\lambda+\mu+\nu = 0.$ |
| Quadratic family of parabolas of the net. | Net-unit-circle about the origin. | $\sum \mu\nu \sin^2 A = 0.$ (30) |
| Intersections of the above pencil and quadratic family. | Net-circular points, $I$ and $J$. | $(e^{-iB} \sin A, e^{iA} \sin B, -\sin C)$<br>$(e^{iB} \sin A, e^{-iA} \sin B, -\sin C)$ |
| Unique circle or circum-circle of the net. | Net-origin for orthogonal cartesian coordinates. | $(\sin 2A, \sin 2B, \sin 2C)$ |
| Pencils of conics including the circum-circle of the net. | Net-axes of orthogonal cartesian coordinates. | $\sum \lambda \cos 2\theta_1 = 0,$<br>$\sum \lambda \sin 2\theta_1 = 0.$ |

The reasons for describing these choices as natural are, firstly, that the pencil of rectangular hyperbolas is the only unique net-line immediately determined by the nature of the conics constituting it, and, secondly, that the quadratic family of parabolas is the only net-conic similarly determined (the equation of the latter follows on putting $e=1$ in (24)). Next, the net-circular points must be the intersections of the net-line at infinity and the net-unit-circle. Furthermore, these intersections certainly correspond to imaginary conics of the net in that they are simultaneously parabolas and rectangular hyperbolas, so that in the form (23) we have $a=-b=\pm ih$. Their essentially imaginary nature thus makes them appropriate for choice as $I$ and $J$.

The centre of the net-unit-circle must then be the pole, in the $(\lambda, \mu, \nu)$ plane, of the net-line $\lambda+\mu+\nu=0$ with respect to the net-conic (30). This pole is the net-point $(\sin 2A, \sin 2B, \sin 2C)$, which corresponds to the circum-circle of the net, by (18) and (20). This unique circle of the net makes a natural choice as net-origin $O$. The equations of the net-lines $OI$ and $OJ$ are then

$$\sum \lambda e^{2i\theta_1} = 0, \qquad \sum \lambda e^{-2i\theta_1} = 0. \tag{31}$$

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If these are to be regarded as essentially conjugate imaginary net-lines, then a natural choice of real net-lines through the net-origin as net-axes of co-ordinates is to take the sum and the difference of the equations (31). That is, we choose the net-lines

$$\sum \lambda \cos 2\theta_1 = 0, \tag{32}$$

$$\sum \lambda \sin 2\theta_1 = 0 \tag{33}$$

as net-axes of co-ordinates. This choice, unlike the previous choices, does not depend only on the intrinsic structure of the net; it also depends on the orientation of the axes of reference taken for the canonical basis of the net. The net-line (32) is the pencil of conics of the net whose axes are at 45° to the original axes, and (33) is the similar pencil with axes parallel to the originals. These net-axes (32), (33) are orthogonal in the net-geometry since they form a harmonic pencil of net-lines with *OI*, *OJ*.

Unit net-distances along the net-axes will be cut off where the net-unit-circle meets them, and the tangents to the net-unit-circle at these net-points should be the net-lines whose equations, in cartesian net-coordinates $(\xi, \eta)$, are $\xi=\pm 1$ and $\eta=\pm 1$. Now

$$(\sum \lambda \cos 2\theta_1)^2+(\sum \lambda \sin 2\theta_1)^2 = (\lambda+\mu+\nu)^2-4\sum \mu\nu \sin^2 A,$$

so the two tangents where (33) cuts (30) are

$$\sum \cos 2\theta_1 = \pm(\lambda+\mu+\nu).$$

We choose (32) as the net $\eta$-axis- $\xi=0$, and (33) as the net $\xi$-axis, $\eta=0$. We choose the sense of these axes so that the net-line

$$\sum \lambda \cos 2\theta_1 = \lambda+\mu+\nu$$

is, in cartesian net-coordinates, $\xi=1$, and similarly

$$\sum \lambda \sin 2\theta_1 = \lambda+\mu+\nu$$

as $\eta=1$. Then the net-lines $\xi=a$ and $\eta=b$ correspond to

$$\sum \lambda \cos 2\theta_1 = a(\lambda+\mu+\nu), \tag{34}$$

and

$$\sum \lambda \sin 2\theta_1 = b(\lambda+\mu+\nu), \tag{35}$$

respectively. Solving these equations for $\lambda$, $\mu$, $\nu$ we find that the conic of the standard net corresponding to the net-point with cartesian net-coordinates $(a, b)$ is

$$\alpha\{\sum S_1(\sin 2\theta_2-\sin 2\theta_3)\}+b\{\sum S_1(\cos 2\theta_2-\cos 2\theta_3)\}+\{\sum S_1 \sin 2(\theta_2-\theta_3)\} = 0.$$

2

**Distance in the net-geometry.** If two net-points $P$, $Q$ have cartesian net-coordinates $(a_1, b_1)$ and $(a_2, b_2)$ respectively, and if corresponding $(\lambda, \mu, \nu)$ coordinates are $(\lambda_1, \mu_1, \nu_1)$ and $(\lambda_2, \mu_2, \nu_2)$ then we can find the latter in terms of the former by solving (34) and (35). Since

$$PQ^2 = (a_1-a_2)^2+(b_1-b_2)^2$$

in cartesians, we find that in the $(\lambda, \mu, \nu)$ coordinates

$$PQ^2 = \frac{4\{\sum (\lambda_1\mu_2-\lambda_2\mu_1)^2 \sin^2 C-2\sum (\lambda_1\mu_2-\lambda_2\mu_1)(\mu_1\nu_2-\mu_2\nu_1)\sin A \sin B \cos C\}}{(\lambda_1+\mu_1+\nu_1)^2(\lambda_2+\mu_2+\nu_2)^2}$$

In particular, the canonical basis can be regarded as three net-points $S_1$, $S_2$, $S_3$ forming a net-triangle. By (34) and (35) their cartesian net-coordinates are $(\cos 2\theta_1, \sin 2\theta_1)$, $(\cos 2\theta_2, \sin 2\theta_2)$, and $(\cos 2\theta_3, \sin 2\theta_3)$, so in the net-geometry $S_1S_2S_3$ is a triangle with angles $A$, $B$, $C$ equal to those of the axial triangle, and with unit circum-radius.

**Polar coordinates in the net-geometry.** The patterns formed in the net-geometry by the conics of the original net can be very simply related to a system of polar coordinates in the net-geometry. Consider a system of polar coordinates $(\rho, \psi)$, where

$$\xi = \rho \cos \psi,\ \eta = \rho \sin \psi.$$

Then by (34) and (35)

$$\rho = \xi^2+\eta^2 = \frac{\{(\sum \lambda \cos 2\theta_1)^2+(\sum \lambda \sin 2\theta_1)^2\}}{(\lambda+\mu+\nu)^2}.$$

Hence by (20)

$$\lambda^2+\mu^2+\nu^2+2\sum \mu\nu \cos 2A = \rho^2(\lambda+\mu+\nu)^2.$$

That is

(36) $$4\sum \mu\nu \sin^2 A = (1-\rho^2)(\lambda+\mu+\nu)^2.$$

If we now put

$$\rho^2 = e^4(2-e^2)^{-2}$$

then

$$\tfrac{1}{4}(1-\rho^2) = (1-e^2)(2-e^2)^{-2},$$

and (36) is the necessary condition (24) that the conic $\lambda S_1+\mu S_2+\nu S_3=0$ should have the eccentricity $e$. That is, conics of the original net with eccentricity $e$ correspond to net-points at a radial net-distance

$$\rho = |e^2(2-e^2)^{-1}|$$

from the net-origin. Note that conics of conjugate eccentricity $e_1$ also have the same $\rho$, since

$$e^{-2}+e_1^{-2} = 1,$$

and consequently

$$e_1^2/(2-e_1^2) = -e^2/(2-e^2).$$

For the angular net-coordinate $\psi$,

$$(37)\qquad \begin{aligned}\tan\psi = \eta/\xi &= \{\textstyle\sum\lambda\sin 2\theta_1\}/\{\textstyle\sum\lambda\cos 2\theta_1\}\\ &= 2\{\textstyle\sum\lambda\cos\theta_1\sin\theta_1\}/\{\textstyle\sum\lambda(\cos^2\theta_1-\sin^2\theta_1)\}.\end{aligned}$$

Now the angles $\phi$ between the axes of a conic in the form (23) and the axes of coordinates satisfy

$$\tan 2\phi = \frac{2h}{a-b}.$$

For the conic (17) of the standard net this is equal to (37), so $\tan\psi=\tan 2\phi$. Hence $\psi\equiv 2\phi$ modulo $\pi$. That is, those conics of the original net which constitute a net-line through the net-origin all have axes making the same angles $\frac{1}{2}\psi\pm\frac{1}{2}\pi n$ with the original axes, where $\psi$ is the angular net-coordinate of this net-line.

Note that since the net-line passes through the net-origin, it corresponds to a pencil of conics including the unique circle, and any pencil of conics through four concyclic points consists necessarily of conics with parallel axes.

**The Jacobian of the net in the $\lambda$, $\mu$, $\nu$ plane.** The equation (10) is a cubic in $\lambda$, $\mu$, $\nu$ corresponding to the Jacobian of the net. For the standard form of the net this becomes, after some trigonometric manipulation,

$$(38)\qquad \{\textstyle\sum\mu\nu\sin^2 A\}\{\lambda\alpha^2+\mu\beta^2+\nu\gamma^2\}-4R^2\lambda\mu\nu\sin^2 A\sin{}^2B\sin^2 C = 0.$$

Now

$$(39)\qquad \lambda\alpha^2+\mu\beta^2+\nu\gamma^2 = 0$$

corresponds to a pencil through $(0, \gamma^2, -\beta^2)$, and two other similar net-points. That is, the pencil includes the conic

$$\gamma^2 S_2 = \beta^2 S_3,$$

which reduces to the line-pair

$$(m_2/\beta) = \pm(m_3/\gamma).$$

These two lines are the diagonals of the parallelogram formed by the two parallel-line-pairs $S_2=0$ and $S_3=0$. Thus the pencil (39) consists of the conics through the four points determined by

$$\pm(m_1/\alpha) = \pm(m_2/\beta) = \pm(m_3/\gamma)$$

with all possible sign combinations. These are the four centres of perspective of the four pairs of parallel-sided triangles formed by the canonical basis.

Hence the Jacobian (38) is a cubic in the $(\lambda, \mu, \nu)$ plane which touches the quadratic family of parabolas

$$\sum \mu\nu \sin^2 A = 0$$

at the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, and which passes through the points $(0, \gamma^2, -\beta^2)$, $(-\gamma^2, 0, \alpha^2)$, and $(\beta^2, -\alpha^2, 0)$. That is, effectively through nine points; however these nine points do not suffice to determine the cubic, since they are common to all cubics of the pencil generated by varying $K$ in

$$\{\sum \mu\nu \sin^2 A\}\{\lambda\alpha^2+\mu\beta^2+\nu\gamma^2\}-K\lambda\mu\nu = 0.$$

## PART III

**The excluded types of net.** We have assumed up to this point that our net of conics

$$\lambda S_1+\mu S_2+\nu S_3 = 0$$

has the following properties:

1. The locus of the centres of its rectangular hyperbolas is a non-degenerate circle. From (9) this leads to the condition that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ h_1 & h_2 & h_3 \end{vmatrix} \neq 0. \tag{40}$$

2. There exists a unique circle in the net. From (2) and (4) this leads to the condition that

$$RANK\begin{pmatrix} a_1-b_1 & a_2-b_2 & a_3-b_3 \\ h_1 & h_2 & h_3 \end{pmatrix} = 2. \tag{41}$$

3. There exists a unique rectangular hyperbola whose asymptotes are parallel to the co-ordinate axes. {Cf. (32).} This leads to the condition that

$$RANK\begin{pmatrix} a_1 & a_2 & a_3 \\ h_1 & h_2 & h_3 \end{pmatrix} = 2. \tag{42}$$

4. There exists a unique rectangular hyperbola whose axes are parallel to the co-ordinate axes. {Cf. (33).} This leads to the condition that

$$RANK\begin{pmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ h_1 & h_2 & h_3 \end{pmatrix} = 2. \tag{43}$$

5. There exist at least three distinct degenerate parabolas in the net. This is equivalent to the condition that equation (14) have at least three distinct roots, or, also equivalently, that the three points of contact between the net-cubic (10) and the net-conic (12) be distinct.

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It may, of course, happen that the net-cubic (10) degenerates into the net-conic (12) and a straight line. In this case all the parabolas of the net are degenerate, and we can always choose three of them as base conics of the net.

It is immediately obvious that condition (43) is equivalent to the geometric condition that the three linearly independent conics $S_1$, $S_2$, $S_3$ intersect the line at infinity in three pairs of points which are *not* in involution. Also, that the condition (40) implies conditions (41), (42), and (43), but *not* conversely.

The conditions under which property 5 holds are less transparent, and therefore need some elaboration. Since condition (40) assures the existence of a unique circle and two unique rectangular hyperbolas in the net, as in properties 3 and 4, we can choose the base conics of the net as follows.

First, we choose the centre of the unique circle as the origin of our co-ordinate system. Then we can take

$$\left.\begin{aligned} S_1 &\equiv x^2-y^2+2g_1x+2f_1y+c_1\\ S_2 &\equiv \;2xy\; +2g_2x+2f_2y+c_2\\ S_3 &\equiv x^2+y^2+\;*\;+\;*\;+c_3\end{aligned}\right\} \tag{44}$$

A further simplification is obtained by a rotation about the origin through a suitable angle $\Phi$, followed by a new choice of $S_1$ and $S_2$. In this way we can, for instance, arrange to have $f_1=0$. We do this as follows. The rotation through an angle $\Phi$ about the origin corresponds to the transformation

$$\begin{aligned} x &\to x\cos\Phi-y\sin\Phi,\\ x &\to x\sin\Phi+y\cos\Phi. \end{aligned}$$

This leaves $S_3$ unchanged, but $S_1$ and $S_2$ become

$$\begin{aligned} S_1 \equiv{} & x^2\cos 2\Phi-2xy\sin 2\Phi-y^2\cos 2\Phi+2(g_1\cos\Phi+f_1\sin\Phi)x\\ & +2(-g_1\sin\Phi+f_1\cos\Phi)y+c_1,\\ S_2 \equiv{} & x^2\sin 2\Phi+2xy\cos 2\Phi-y^2\sin 2\Phi+2(g_2\cos\Phi+f_2\sin\Phi)x\\ & +2(-g_2\sin\Phi+f_2\cos\Phi)y+c_2. \end{aligned}$$

We now choose the following two distinct rectangular hyperbolas as our new base conics $\bar{S}_1$, $\bar{S}_2$.

$$\begin{aligned} \bar{S}_1 &= \quad S_1\cos 2\Phi+S_2\sin 2\Phi,\\ \bar{S}_2 &= -S_1\sin 2\Phi+S_2\cos 2\Phi. \end{aligned}$$

Then

$$\begin{aligned} \bar{S}_1 = {} & x^2-y^2+2\{(g_1\cos\Phi+f_1\sin\Phi)\cos 2\Phi+(g_2\cos\Phi+f_2\sin\Phi)\sin 2\Phi\}x\\ & +2(-g_1\sin\Phi+f_1\cos\Phi)\cos 2\Phi+(-g_2\sin\Phi+f_2\cos\Phi)\sin 2\Phi\}y\\ & +c_1\cos 2\Phi+c_2\sin 2\Phi,\\ \bar{S}_2 = {} & 2xy+2\{-(g_1\cos\Phi+f_1\sin\Phi)\sin 2\Phi+(g_2\cos\Phi+f_2\sin\Phi)\cos 2\Phi\}x\\ & +2\{-(-g_1\sin\Phi+f_1\cos\Phi)\sin 2\Phi+(-g_2\sin\Phi+f_2\cos\Phi)\cos 2\Phi\}y\\ & -c_1\sin 2\Phi+c_2\cos 2\Phi. \end{aligned}$$

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Since the angle of rotation $\Phi$ is still arbitrary we can choose it in such a way that the coefficient of $y$ in $\bar{S}_1$ be zero. That is

$$(-g_1 \sin \Phi + f_1 \cos \Phi)\cos 2\Phi + (-g_2 \sin \Phi + f_2 \cos \Phi)\sin 2\Phi = 0.$$

Note that there always exists a real angle $\Phi$ which satisfies this equation, since it is of degree three in $t = \tan \Phi$. In fact it is

$$(-g_1 t + f_1)(1 - t^2) + 2t(-g_2 t + f_2) = 0.$$

We have thus shown that an orthogonal cartesian co-ordinate system $(x, y)$ in our original plane exists such that, if condition (40) is satisfied, then the three base conics of the net can be chosen to be

$$\left.\begin{aligned} S_1 &\equiv x^2 - y^2 + 2g_1 x + \ * \ + c_1, \\ S_2 &\equiv \ \ 2xy \ \ + 2g_2 x + 2f_2 y + c_2, \\ S_3 &\equiv x^2 + y^2 + \ * \ + \ * \ + c_3. \end{aligned}\right\} \tag{45}$$

This representation is particularly appropriate since the co-ordinates $(\lambda, \mu, \nu)$ of a point-element in the net-geometry corresponding to the conic

$$\lambda S_1 + \mu S_2 + \nu S_3 = 0$$

can be interpreted as homogeneous orthogonal cartesian co-ordinates. Thus $S_3$ corresponds to the origin (0, 0, 1); the pencil of rectangular hyperbolas corresponds to the line at infinity, $\nu = 0$; and the family of parabolas corresponds to the unit circle {Cf. (12)}.

$$\lambda^2 + \mu^2 - \nu^2 = 0. \tag{46}$$

Hence the parabolas which are also rectangular hyperbolas correspond to the circular points $(1, i, 0)$ and $(1, -i, 0)$.

The curve which corresponds to the degenerate conics of the net is the cubic {Cf. (10)}

$$\begin{vmatrix} \lambda + \nu & \mu & \lambda g_1 + \mu g_2 \\ \mu & -\lambda + \nu & \mu f_2 \\ \lambda g_1 + \mu g_2 & \mu f_2 & \lambda c_1 + \mu c_2 + \nu c_3 \end{vmatrix} = 0. \tag{47}$$

As has been shown earlier, the cubic (47) is tangent to the circle (46) in three points, $P_1, P_2, P_3$, say. The rotation through an angle $\Phi$ in the $(x, y)$ plane, as outlined previously, corresponds to a rotation in the $(\lambda, \mu, \nu)$ plane such that one of these three points, say $P_1$, be the point (1, 0, 1). This is easily checked by putting these values in (47). The corresponding cubic equation (14) in $p$ then reduces to the quadratic equation

$$(g_1 - 2f_2)p^2 + 2g_2 p - g_1 = 0. \tag{48}$$

Hence (14) has the roots $p_1, p_2$ of (48), and the third root is $p_3 = \infty$.

We are now in a position to interpret the conditions for property 5 (that there exist at least three degenerate parabolas in the net). In order that this should happen it is necessary and sufficient that the equation (48) have neither the root $p_3=\infty$, nor that its two roots $p_1, p_2$ co-incide. That is, both

$$g_1-2f_2 \neq 0, \tag{49}$$

and

$$g_1(g_1-2f_2)+g_2^2 \neq 0. \tag{50}$$

We thus have:

THEOREM. *In order that a net of conics in the euclidean plane satisfying condition* (40) *should have three degenerate parabolas* (*at least*) *it is necessary and sufficient that, when reduced to the canonical form* (45), *conditions* (49) *and* (50) *be satisfied.*

Thus the net whose base conics are

$$\left.\begin{aligned} S_1 &\equiv x^2-y^2+2g_1x+ * +c_1,\\ S_2 &\equiv \quad 2xy \ +2g_2x+g_1y+c_2,\\ S_3 &\equiv x^2+y^2+ \ * \ + * +c_3.\end{aligned}\right\} \quad g_1 \neq 0, g_2 \neq 0,$$

has only two degenerate parabolas. Also, the net whose basic conics are

$$\left.\begin{aligned} S_1 &\equiv x^2-y^2+2g_1x+ \ * \ +c_1\\ S_2 &\equiv \quad 2xy \ + \ * \ +g_1y+c_2\\ S_3 &\equiv x^2+y^2+ \ * \ + \ * \ +c_3\end{aligned}\right\} \quad g_1 \neq 0,$$

has only one degenerate parabola.

**Classification of nets.** The representation (45) of a net of conics in the euclidean plane (to which all nets satisfying condition (40) can be reduced) can be made the basis of a classification of such nets. This is because there will clearly be as many, and only as many, types of such nets as there are distinct relative positions of the cubic (47) and the circle (46). One would also have to take into consideration the reality and multiplicity of the common points of (46) and (47); these matters have not been dealt with in this paper.

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UNIVERSITY OF SASKATCHEWAN,
SASKATOON, SASKATCHEWAN.

TRENT UNIVERSITY,
PETERBOROUGH, ONTARIO.