Proceedings of the Royal Society of Edinburgh, **155**, 307–344, *2025*
DOI:10.1017/prm.2023.95

Asymptotic formulas of the eigenvalues for the linearization of the scalar field equation

Yasuhito Miyamoto
Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan
(miyamoto@ms.u-tokyo.ac.jp)

Haruki Takemura
Department of Integrated Science, College of Arts and Science, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan
(takemura-haruki@g.ecc.u-tokyo.ac.jp)

Tohru Wakasa
Department of Basic Sciences, Kyushu Institute of Technology, Sensuicho, Tobata-ku, Kitakyushu, Fukuoka 804-8550, Japan
(wakasa@mns.kyutech.ac.jp)

(Received 2 April 2022; accepted 16 August 2023)

We establish asymptotic formulas for all the eigenvalues of the linearization problem of the Neumann problem for the scalar field equation in a finite interval

$$\begin{cases} \varepsilon^2 u_{xx} - u + u^3 = 0, & 0 < x < 1, \\ u_x(0) = u_x(1) = 0. \end{cases}$$

In the previous paper of the third author [T. Wakasa and S. Yotsutani, J. Differ. Equ. **258** (2015), 3960–4006] asymptotic formulas for the Allen–Cahn case $\varepsilon^2 u_{xx} + u - u^3 = 0$ were established. In this paper, we apply the method developed in the previous paper to our case. We show that all the eigenvalues can be classified into three categories, i.e., near -3 eigenvalues, near 0 eigenvalues and the other eigenvalues. We see that the number of the near -3 eigenvalues (resp. the near 0 eigenvalues) is equal to the number of the interior and boundary peaks (resp. the interior peaks) of a solution for the nonlinear problem. The main technical tools are various asymptotic formulas for complete elliptic integrals.

Keywords: reaction–diffusion equation; linearized eigenvalue problem; representation of eigenfunctions; elliptic integrals

2020 *Mathematics Subject Classification:* Primary: 34A05, 34B05, secondary 34B15, 35K57

1. Introduction and main results

We are concerned with the Neumann problem of a reaction–diffusion equation

$$\begin{cases}\varepsilon^2 u_{xx}(x) + f(u(x)) = 0, & 0 < x < 1,\\ u_x(0) = u_x(1) = 0\end{cases} \tag{1.1}$$

and its linearized eigenvalue problem associated with a positive solution $u(x)$

$$\begin{cases}\varepsilon^2 \varphi_{xx}(x) + f_u(u(x))\varphi(x) = -\lambda\varphi(x), & 0 < x < 1,\\ \varphi_x(0) = \varphi_x(1) = 0,\end{cases} \tag{1.2}$$

where $\varepsilon > 0$ is a parameter and f is smooth enough. We denote $F(u)$ by

$$F(u) = \int_0^u f(s)\,\mathrm{d}s.$$

Precise information of eigenvalues is crucial not only in a stability analysis of a stationary solution of the associated parabolic problem but also in a study of dynamics. The goal of the present paper is to obtain a rather explicit expression of all the eigenvalues of (1.2) in the case

$$f(u) = -u + u^3.$$

The equation $\varepsilon^2 u_{xx} - u + u^p = 0$, $p > 1$, is called the scalar field equation which appears in the study of a standing wave of a nonlinear Schrödinger equation [**2**], of elastic curves in differential geometry [**6**] and of the Gierer–Meinhardt model in mathematical biology [**5**]. This equation has attracted much attention for these three decades. A complete bifurcation diagram of the positive solutions for (1.1) with $f(u) = -u + u^p$, $p > 1$, was obtained in [**7**, **16**]. See Fig. 1. Let $p > 1$ and $\varepsilon_n := \sqrt{p-1}/n\pi$. For each $n \geqslant 1$, (1.1) has exactly two n-mode solutions $u^{\pm}_{n,\varepsilon}$ if $0 < \varepsilon < \varepsilon_n$. Here,

$$u^+_{n,\varepsilon}(0) = \min_{0\leqslant x\leqslant 1} u^+_{n,\varepsilon}(x), \quad u^-_{n,\varepsilon}(0) = \max_{0\leqslant x\leqslant 1} u^-_{n,\varepsilon}(x).$$

In particular, if $n = 2k + 1$, then both $u^+_{n,\varepsilon}$ and $u^-_{u,\varepsilon}$ have k interior peaks and 1 boundary peak. On the other hand, if $n = 2k$, then $u^+_{n,\varepsilon}$ (resp. $u^-_{n,\varepsilon}$) has k interior peaks and no boundary peak (resp. $k - 1$ interior peaks and 2 boundary peaks). Each peak becomes sharp as $\varepsilon \to 0$.

Hereafter, we consider the case $f(u) = -u + u^3$. Then

$$\varepsilon_n = \frac{\sqrt{2}}{n\pi}.$$

All the solutions of (1.1) can be written in terms of elliptic functions:

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

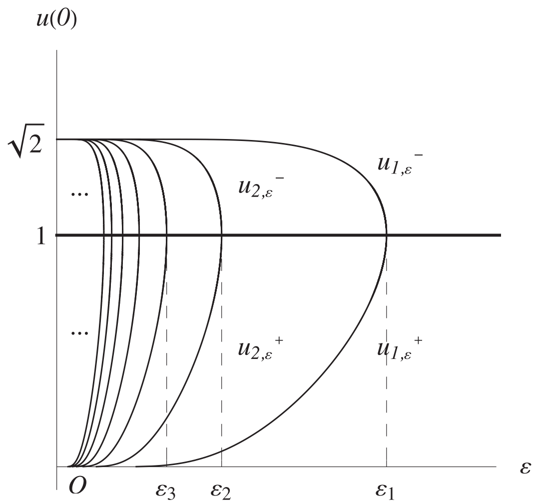

Figure 1. The complete bifurcation diagram for (1.1) with $f(u) = -u + u^3$.

PROPOSITION 1.1. *Let $n \geqslant 1$ be fixed and let $0 < \varepsilon < \varepsilon_n$. Let $k_\varepsilon = k_{n,\varepsilon} \in (0,1)$ be a solution of*

$$\sqrt{2-k^2}K(k) = \frac{1}{n\varepsilon}. \tag{1.3}$$

Note that k_ε is uniquely determined. Then $u^+_{n,\varepsilon}$ and $u^-_{n,\varepsilon}$ can be written as follows:

$$u^\pm_{n,\varepsilon}(x) = \sqrt{\frac{2}{2-k_\varepsilon^2}}\mathrm{DN}^\pm_n(x, k_\varepsilon), \tag{1.4}$$

where

$$\mathrm{DN}^+_n(x, k_\varepsilon) := \mathrm{dn}(K(k_\varepsilon)(1+nx), k_\varepsilon), \quad \mathrm{DN}^-_n(x, k_\varepsilon) := \mathrm{dn}(nK(k_\varepsilon)x, k_\varepsilon).$$

The proof of proposition 1.1 is given in §2.

In this paper, we frequently use Jacobi's elliptic functions $\mathrm{sn}(x,k)$ and $\mathrm{dn}(x,k)$ and complete elliptic integrals $K(k)$, $E(k)$ and $\Pi(\nu,k)$. The definitions and various properties are summarized in Appendix A of the present paper.

The linearized eigenvalue problem (1.2) associated with $u^\pm_{n,\varepsilon}$ is as follows:

$$\begin{cases} \varepsilon^2\varphi_{xx}(x) + f_u(u^\pm_{n,\varepsilon}(x))\varphi(x) = -\lambda\varphi(x), & 0 < x < 1, \\ \varphi_x(0) = \varphi_x(1) = 0. \end{cases} \tag{LP$_\pm$}$$

Hereafter, $\lambda^\pm_{j,\varepsilon}$, $j \geqslant 0$ denotes the $j+1$-th eigenvalue of (LP$_\pm$) and $\varphi^\pm_{j,\varepsilon}$ denotes an associated eigenfunction. It is well known that each eigenvalue $\lambda^\pm_{j,\varepsilon}$ is simple.

The following three exact eigenvalues of (LP$_\pm$) were obtained in the previous paper of the third author [11]:

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

Proposition 1.2. *Let $n \geqslant 1$,*

$$\mathrm{SN}_n^+(x,k) := \mathrm{sn}(K(k)(1+nx),k), \quad \mathrm{SN}_n^-(x,k) := \mathrm{sn}(nK(k)x,k),$$
$$\mathrm{CN}_n^+(x,k) := \mathrm{cn}(K(k)(1+nx),k), \quad \mathrm{CN}_n^-(x,k) := \mathrm{cn}(nK(k)x,k).$$

The following hold:

(i) *The problem* (LP$_+$) *has the following two pairs of eigenvalues and eigenfunctions*:

$$\lambda_{0,\varepsilon}^+ = -1 - \frac{2\sqrt{1-k_\varepsilon^2+k_\varepsilon^4}}{2-k_\varepsilon^2}, \quad \varphi_{0,\varepsilon}^+(x) = 1$$
$$- \left(1+k_\varepsilon^2 - \sqrt{1-k_\varepsilon^2+k_\varepsilon^4}\right)\mathrm{SN}_n^+(x,k_\varepsilon)^2,$$
$$\lambda_{n,\varepsilon}^+ = -1 + \frac{2\sqrt{1-k_\varepsilon^2+k_\varepsilon^4}}{2-k_\varepsilon^2}, \quad \varphi_{n,\varepsilon}^+(x) = -1$$
$$+ \left(1+k_\varepsilon^2 + \sqrt{1-k_\varepsilon^2+k_\varepsilon^4}\right)\mathrm{SN}_n^+(x,k_\varepsilon)^2.$$

Moreover, if n is even, then (LP$_+$) has one more pair:

$$\lambda_{n/2,\varepsilon}^+ = -\frac{3(1-k_\varepsilon^2)}{2-k_\varepsilon^2}, \quad \varphi_{n/2,\varepsilon}^+(x) = \mathrm{SN}_n^+(x,k_\varepsilon)\mathrm{DN}_n^+(x,k_\varepsilon).$$

(ii) *The problem* (LP$_-$) *has the following pairs of eigenvalues and eigenfunctions*:

$$\lambda_{0,\varepsilon}^- = -1 - \frac{2\sqrt{1-k_\varepsilon^2+k_\varepsilon^4}}{2-k_\varepsilon^2}, \quad \varphi_{0,\varepsilon}^-(x) = 1$$
$$- \left(1+k_\varepsilon^2 - \sqrt{1-k_\varepsilon^2+k_\varepsilon^4}\right)\mathrm{SN}_n^-(x,k_\varepsilon)^2,$$
$$\lambda_{n,\varepsilon}^- = -1 + \frac{2\sqrt{1-k_\varepsilon^2+k_\varepsilon^4}}{2-k_\varepsilon^2}, \quad \varphi_{n,\varepsilon}^-(x) = -1$$
$$+ \left(1+k_\varepsilon^2 + \sqrt{1-k_\varepsilon^2+k_\varepsilon^4}\right)\mathrm{SN}_n^-(x,k_\varepsilon)^2.$$

Moreover, if n is even, then (LP$_-$) has one more pair:

$$\lambda_{n/2,\varepsilon}^- = -\frac{3}{2-k_\varepsilon^2}, \quad \varphi_{n/2,\varepsilon}^-(x) = \mathrm{CN}_n^-(x,k_\varepsilon)\mathrm{DN}_n^-(x,k_\varepsilon).$$

Singularly perturbed problems have attracted great attention, because of a wide variety of applications. It follows from (1.3) that a singularly perturbed problem corresponds to the case where k is close to 1. The main result of the paper is the following asymptotic formula of the other eigenvalues as $k \to 1$:

Theorem 1.3. *Let $n \geqslant 1$ and $j > 0$ be fixed. Let k_ε be as in proposition 1.1. The following hold for* (LP$_\pm$):

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

(i) *For* $0 < j < n/2$,

$$\lambda_{j,\varepsilon}^{\pm} = -3 + 3(1-k_\varepsilon^2) + \frac{3}{4}\left(\sin^2\frac{j\pi}{n} - 5\right)(1-k_\varepsilon^2)^2 + o\left((1-k_\varepsilon^2)^2\right) \ \textit{as}\ k_\varepsilon \to 1.$$

(ii) *For* $n/2 < j < n$,

$$\lambda_{j,\varepsilon}^{\pm} = -3\left(\sin^2\frac{j\pi}{n}\right)(1-k_\varepsilon^2) + o(1-k_\varepsilon^2) \ \textit{as}\ k_\varepsilon \to 1.$$

(iii) *For* $j > n$,

$$\lambda_{j,\varepsilon}^{\pm} = 1 + \frac{(j-n)^2\pi^2}{n^2}\frac{1}{K(k_\varepsilon)^2} + o\left(\frac{1}{K(k_\varepsilon)^2}\right) \ \textit{as}\ k_\varepsilon \to 1,$$

where $K(k_\varepsilon) \to \infty$ *as* $k_\varepsilon \to 1$.

Combining proposition 1.2 and theorem 1.3, we can relate locations of eigenvalues with the number of the peaks of $u_{n,\varepsilon}^{\pm}$ as follows: All the eigenvalues can be classified into three categories, i.e.,

$$\begin{cases} \text{(a) near } -3 \text{ eigenvalues which converge to } -3 \text{ as } k_\varepsilon \to 1, \\ \text{(b) near } 0 \text{ eigenvalues which converge to } 0 \text{ as } k_\varepsilon \to 1, \\ \text{(c) the other eigenvalues which converge to } 1 \text{ as } k_\varepsilon \to 1. \end{cases} \tag{1.5}$$

The number of the near -3 eigenvalues is equal to the number of the interior and boundary peaks of $u_{n,\varepsilon}^{\pm}$. The number of the near 0 eigenvalues is equal to the number of the interior peaks of $u_{n,\varepsilon}^{\pm}$.

Example 1.4.

(i) The solution $u_{10,\varepsilon}^{+}$ has 5 interior peaks and no boundary peak. Hence, (LP_+) has 5 eigenvalues near -3 and 5 eigenvalues near 0 when $\varepsilon > 0$ is small. On the other hand, $u_{10,\varepsilon}^{-}$ has 4 interior peaks and 2 boundary peaks. Hence (LP_-) has 6 eigenvalues near -3 and 4 eigenvalues near 0 when $\varepsilon > 0$ is small.

(ii) Each of $u_{11,\varepsilon}^{+}$ and $u_{11,\varepsilon}^{-}$ has 5 interior peaks and 1 boundary peak. Hence, ($LP_\pm$) has 6 eigenvalues near -3 and 5 eigenvalues near 0 when $\varepsilon > 0$ is small.

In figures 2–4 profiles of eigenfunctions on ($LP_\pm$) with $u_{10,\varepsilon}^{\pm}$ and profiles of eigenfunctions on (LP_+) with $u_{11,\varepsilon}^{+}$ are shown. In each figure (a-0), (a-1) and (a-2) belong to category (a) in ($LP_\pm$), and so on.

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

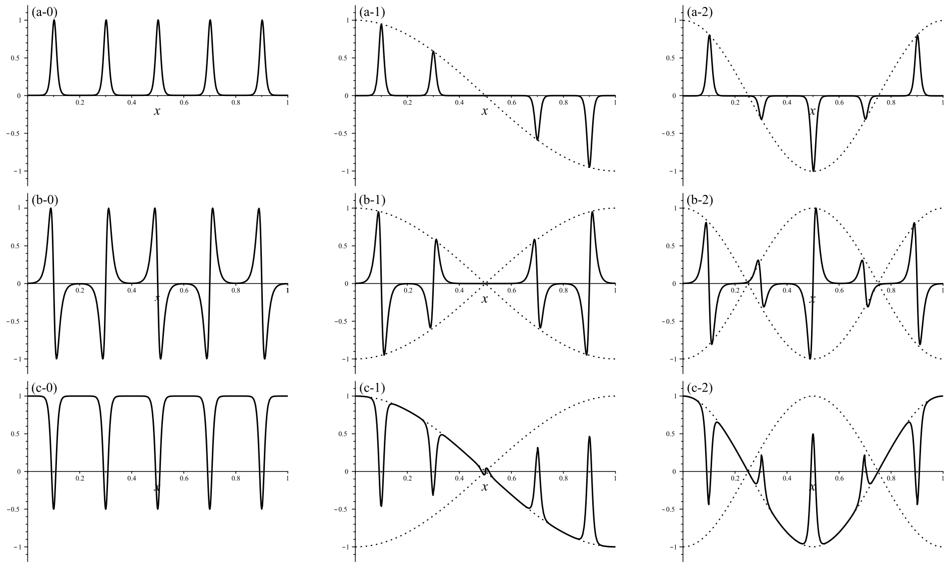

Figure 2. Profiles of eigenfunctions for (LP_+) with $u^+_{10,\varepsilon}(x)$; (a-0) φ^+_0, (a-1) φ^+_1, (a-2) φ^+_2, (b-0) φ^+_5, (b-1) φ^+_6, (b-2) φ^+_7, (c-0) φ^+_{10}, (c-1) φ^+_{11} and (c-2) φ^+_{12} .

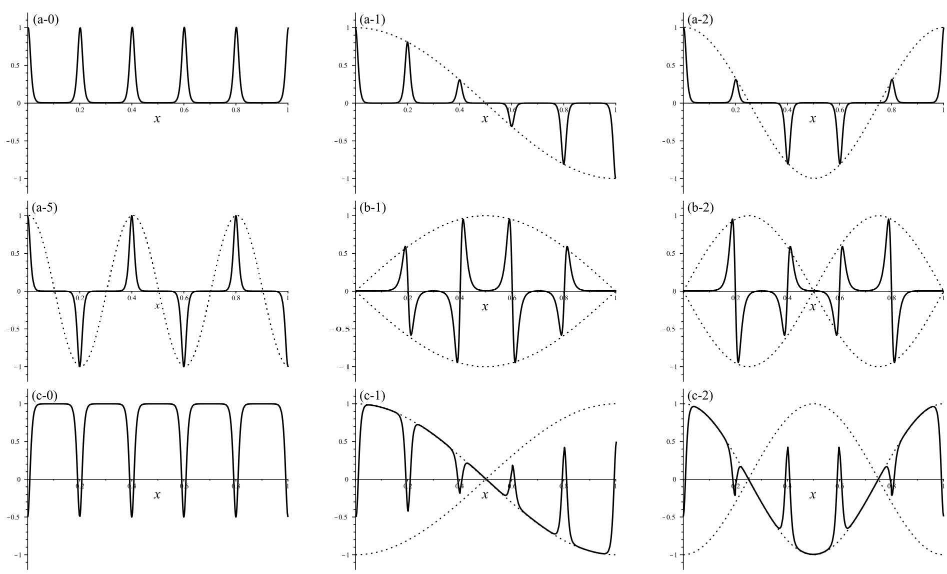

Figure 3. Profiles of eigenfunctions for (LP_-) with $u^-_{10,\varepsilon}(x)$; (a-0) φ^-_0, (a-1) φ^-_1, (a-2) φ^-_2, (a-5) φ^-_5, (b-1) φ^-_6, (b-2) φ^-_7, (c-0) φ^-_{10}, (c-1) φ^-_{11} and (c-2) φ^-_{12}.

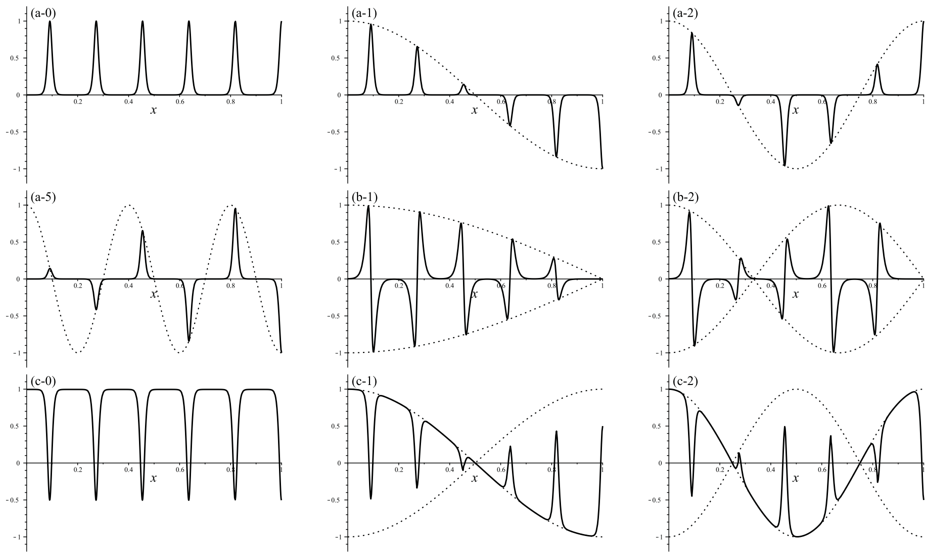

Figure 4. Profiles of eigenfunctions for (LP_+) with $u^+_{11,\varepsilon}(x)$; (a-0) φ^+_0, (a-1) φ^+_1, (a-2) φ^+_2, (a-5) φ^+_5, (b-1) φ^+_6, (b-2) φ^+_7, (c-0) φ^+_{11}, (c-1) φ^+_{12} and (c-2) φ^+_{13}.

Let us consider the stretched problem

$$\begin{cases} \tilde{u}_{yy} - \tilde{u} + \tilde{u}^3 = 0, & -\frac{x_{\ell,\varepsilon}}{\varepsilon} < y < \frac{1-x_{\ell,\varepsilon}}{\varepsilon}, \\ \tilde{u}_y\left(-\frac{x_{\ell,\varepsilon}}{\varepsilon}\right) = \tilde{u}_y\left(\frac{1-x_{\ell,\varepsilon}}{\varepsilon}\right) = 0 \end{cases}$$

and the limit problem

$$\begin{cases} \bar{u}_{yy} - \bar{u} + \bar{u}^3 = 0, & -\infty < y < \infty, \\ \bar{u} \in L^2(\mathbb{R}). \end{cases} \tag{1.6}$$

Here $\tilde{u}(y) := u(x)$, $y := (x - x_{\ell,\varepsilon})/\varepsilon$ and $x_{\ell,\varepsilon} \in (0, 1)$ is a position of an ℓ-th interior peak. Then $\bar{u}(y) = \frac{\sqrt{2}}{\cosh(y)}$ is a solution of (1.6) which has a one-peak at $y = 0$. The associated eigenvalue problem is as follows:

$$\begin{cases} \bar{\varphi}_{yy} + (-1 + 3\bar{u}^2)\bar{\varphi} = -\lambda\bar{\varphi}, & -\infty < y < \infty, \\ \bar{\varphi} \in H^1(\mathbb{R}). \end{cases} \tag{1.7}$$

It is known that the spectral set consists of two eigenvalues -3 and 0 and the continuous spectrum $[1, \infty)$. The near -3 eigenvalues of ($\mathrm{LP}_\pm$) come from the -3 eigenvalue of (1.7). Figures 2–4 indicate that an eigenfunction in category (a) can be approximated by a linear combination of translations of a compressed first eigenfunction of (1.7). The near 0 eigenvalues of ($\mathrm{LP}_\pm$) come from a translation invariance of the one-peak solution $\bar{u}(y)$, which is indicated by a shape of an eigenfunction of category (b), e.g., (b-0), (b-1) and (b-2) of figure 2. Specifically, an eigenfunction is close to $cu'(x)$, $c \in \mathbb{R}$, near each peak of $u(x)$. The other eigenvalues converge to 1 which is an end point of $[1, \infty)$. The spectrum 1 of (1.7) is

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

not an eigenvalue, since a corresponding eigenfunction is not in $H^1(\mathbb{R})$. Indeed, a graph of an eigenfunction of category (c) in figures 2–4 is close to $\cos(m\pi x)$ for *most* $x \in (0,1)$, and hence its stretched function is not in $H^1(\mathbb{R})$ or 0.

REMARK 1.5. If n is odd, then the eigenvalue sets of (LP$_+$) and (LP$_-$) are the same, i.e., $\lambda^+_{j,\varepsilon} = \lambda^-_{j,\varepsilon}$ for all $j \geqslant 0$, since $u^-_{n,\varepsilon}(x) = u^+_{n,\varepsilon}(1-x)$. However, if n is even, then $\lambda^+_{j,\varepsilon} = \lambda^-_{j,\varepsilon}$ for $j \neq n/2$, but $\lambda^+_{n/2,\varepsilon} \neq \lambda^-_{n/2,\varepsilon}$. In the even case the number of all the peaks of $u^+_{n,\varepsilon}$ is different from that of $u^-_{n,\varepsilon}$.

Corollaries 1.6 and 1.7 are asymptotic formulas with respect to $\mathrm{e}^{-2/n\varepsilon}$.

COROLLARY 1.6. *Let $n \geqslant 1$ be fixed. The following hold for (LP$_\pm$):*

(i) $\lambda^\pm_{0,\varepsilon} = -3 + 48\,\mathrm{e}^{-\frac{2}{n\varepsilon}} + o(\mathrm{e}^{-\frac{2}{n\varepsilon}})$ *as* $\varepsilon \to 0$.

(ii) *If n is even, then as $\varepsilon \to 0$,*

$$\lambda^+_{n/2,\varepsilon} = -48\,\mathrm{e}^{-\frac{2}{n\varepsilon}} + o(\mathrm{e}^{-\frac{2}{n\varepsilon}}), \quad \lambda^-_{n/2,\varepsilon} = -3 + 48\,\mathrm{e}^{-\frac{2}{n\varepsilon}} + o(\mathrm{e}^{-\frac{2}{n\varepsilon}}).$$

(iii) $\lambda^\pm_{n,\varepsilon} = 1 - 48\,\mathrm{e}^{-\frac{2}{n\varepsilon}} + o(\mathrm{e}^{-\frac{2}{n\varepsilon}})$ *as* $\varepsilon \to 0$.

COROLLARY 1.7. *Let $n \geqslant 1$ and $j \geqslant 0$ be fixed. The following hold for (*LP$_\pm$*):*

(i) *For* $0 < j < n/2$,

$$\lambda^\pm_{j,\varepsilon} = -3 + \left\{1 + 4\left(\sin^2\frac{j\pi}{n}\right)\mathrm{e}^{-\frac{2}{n\varepsilon}} + o(\mathrm{e}^{-\frac{2}{n\varepsilon}})\right\}(\lambda^\pm_{0,\varepsilon} + 3) \ \text{as } \varepsilon \to 0.$$

(ii) *For* $n/2 < j < n$,

$$\lambda^\pm_{j,\varepsilon} = -48\left(\sin^2\frac{j\pi}{n}\right)\mathrm{e}^{-\frac{2}{n\varepsilon}} + o(\mathrm{e}^{-\frac{2}{n\varepsilon}}) \ \text{as } \ \varepsilon \to 0.$$

(iii) *For* $j > n$,

$$\lambda^\pm_{j,\varepsilon} = 1 + (j-n)^2\pi^2\varepsilon^2 + o(\varepsilon^2) \ \text{as } \varepsilon \to 0.$$

It follows from corollary 1.7 (i) that $\lambda^\pm_{j,\varepsilon}$, $0 < j < n/2$, can be also written as follows:

$$\lambda^\pm_{j,\varepsilon} = -3 + 48\,\mathrm{e}^{-\frac{2}{n\varepsilon}} + o(\mathrm{e}^{-\frac{2}{n\varepsilon}}) \text{ as } \ \varepsilon \to 0.$$

However a j-dependence does not appear in the first two terms.

Let us recall known results. It is difficult to obtain exact expressions of eigenvalues of an elliptic differential operator even if a domain is a finite interval. Only few examples are known for elliptic differential operators with variable coefficients. A model case is the linearization problem for the Allen–Cahn equation (1.1) with $f(u) = u - u^3$. The problem (1.1) with $f(u) = u - u^3$ has n-mode solutions $u^\pm_{n,\varepsilon}$ for small $\varepsilon > 0$ which has n transition layers in the interval. In [4] the so-called very slow dynamics of a transition layer solution of an associated parabolic problem were

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

studied. Then the first n eigenvalues of (LP$_\pm$), $\{\lambda^{\pm}_{j,\varepsilon}\}_{j=0}^{n-1}$, play a crucial role. It was shown in [**4**, Corollary 4.2] that for $0 \leqslant j < n$,

$$\lambda^{\pm}_{j,\varepsilon} = O(\mathrm{e}^{-d/\varepsilon}) \text{ with some } d > 0. \tag{1.8}$$

These near 0 eigenvalues come from translation invariance of each transition layer. Later exact expressions of three special eigenvalues $\lambda^{\pm}_{j,\varepsilon}$, $j = 0, n, 2n$, were obtained in [**9**]. Using these three exact eigenvalues, one can obtain

$$\begin{aligned}
\lambda^{\pm}_{0,\varepsilon} &= -96\,\mathrm{e}^{-\frac{\sqrt{2}}{n\varepsilon}} + o(\mathrm{e}^{-\frac{\sqrt{2}}{n\varepsilon}}) \text{ as } \varepsilon \to 0,\\
\lambda^{\pm}_{n,\varepsilon} &= \frac{3}{2} - 12\,\mathrm{e}^{-\frac{1}{\sqrt{2}n\varepsilon}} + o(\mathrm{e}^{-\frac{1}{\sqrt{2}n\varepsilon}}) \text{ as } \varepsilon \to 0,\\
\lambda^{\pm}_{2n,\varepsilon} &= 2 + 96\,\mathrm{e}^{-\frac{\sqrt{2}}{n\varepsilon}} + o(\mathrm{e}^{-\frac{\sqrt{2}}{n\varepsilon}}) \text{ as } \varepsilon \to 0.
\end{aligned}$$

Note that these three eigenvalues correspond to $\lambda^{\pm}_{j,\varepsilon}$, $j = 0, n/2, n$, in our case $f(u) = -u + u^3$. Then an exact representation formula of eigenvalues for general f, which is lemma 2.2 in our case, was obtained in [**12**]. The authors of [**12**] applied it to the case $f(u) = \sin u$ and obtained an asymptotic formula for every eigenvalue. After that in [**14**] asymptotic formulas of all the eigenvalues for the Allen–Cahn case $f(u) = u - u^3$ were established. Specifically,

$$\begin{aligned}
&\text{for } 0 < j < n,\ \lambda^{\pm}_{j,\varepsilon} = -96\left(\cos^2\frac{j\pi}{2n}\right)\mathrm{e}^{-\frac{\sqrt{2}}{n\varepsilon}} + o(\mathrm{e}^{-\frac{\sqrt{2}}{n\varepsilon}}) \text{ as } \varepsilon \to 0,\\
&\text{for } n < j < 2n,\ \lambda^{\pm}_{j,\varepsilon} = \frac{3}{2} - 12\left(\cos\frac{(j-n)\pi}{n}\right)\mathrm{e}^{-\frac{1}{\sqrt{2}n\varepsilon}} + o(\mathrm{e}^{-\frac{1}{\sqrt{2}n\varepsilon}}) \text{ as } \varepsilon \to 0,\\
&\text{for } j > 2n, \lambda^{\pm}_{j,\varepsilon} = 2 + (j-2n)^2\pi^2\varepsilon^2 + o(\varepsilon^2) \text{ as } \varepsilon \to 0.
\end{aligned}$$

These formulas show that (1.8) holds for $0 \leqslant j < n$.

Detailed studies of all the eigenfunctions were made for the case $f(u) = \sin u$ in [**13**] and for the case $f(u) = u - u^3$ in [**15**]. The authors of [**8**] obtained exact eigenvalues and eigenfunctions of a one-dimensional Gel'fand problem $f(u) = e^u$, using the method developed in [**12**]. In this paper, we apply the method developed in [**14**] to the scalar field equation (1.1) with $f(u) = -u + u^3$ to establish asymptotic formulas. This method seems to work for other nonlinearities and other boundary conditions. They may be future works.

The paper consists of five sections. In §2 we prove proposition 1.1. We obtain asymptotic formulas for three special eigenvalues $\lambda^{\pm}_{0,\varepsilon}$, $\lambda^{\pm}_{n/2,\varepsilon}$ and $\lambda^{\pm}_{n,\varepsilon}$ (corollary 1.6). We recall an exact representation formulas for the case $f(u) = -u + u^3$ in lemma 2.2. In particular, the other eigenvalues are given by a unique solution of

$$\mathcal{A}_0(k, \lambda_j) = \frac{j\pi}{n} \text{ for } j \neq 0, n/2, n,$$

where $\mathcal{A}_0$ is given in lemma 2.2. Let $\mu := (2 - k^2)\lambda$. We use μ instead of λ, since various formulas becomes simple. We define $\mathcal{A}(k, \mu) := A_0(k, \lambda)$. We see in lemma 2.3 that the characteristic function $\mathcal{A}(k, \mu)$ can be written in terms of the complete

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

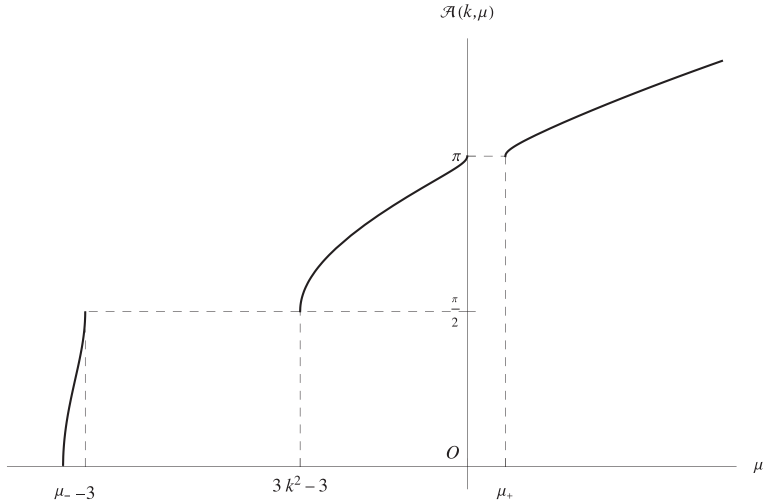

Figure 5. A graph of $\mathcal{A}(k,\mu)$ with $k = 3/4$. $\mathcal{A}(k,\mu)$ is defined on Σ and monotone increasing in μ.

elliptic integral of the third kind. In § 3 we study the shape of the graph of $\mathcal{A}(k,\mu)$, since the graph of $\mathcal{A}(k,\mu)$ is directly related to the $j+1$-th eigenvalue by the equation

$$\mathcal{A}(k,\mu) = \frac{j\pi}{n} \text{ for } j \neq 0, n/2, n. \tag{1.9}$$

Specifically, we show that $\mathcal{A}(k,\mu)$ is increasing in μ and that for each j, (1.9) has a unique solution. See figure 5 in § 2 for a graph of $\mathcal{A}(k,\mu)$. In § 4 we prove theorem 1.3 and corollary 1.7, using results proved in § 3. In Appendix A we recall definitions and basic properties of Jacobi's elliptic functions $\mathrm{sn}(x,k)$, $\mathrm{cn}(x,k)$ and $\mathrm{dn}(x,k)$. Then we recall basic properties of the complete elliptic integrals $K(k)$, $E(k)$ and $\Pi(\nu,k)$. We also recall various asymptotic formulas for $K(k)$ and $\Pi(\nu,k)$ which are used in proofs of lemmas in § 3 and theorem 1.3.

2. Preliminaries

2.1. Exact solutions of the nonlinear problem

Proof of proposition 1.1. Let $y = \mathrm{dn}(x,k)$. Then, $y(x)$ satisfies $y'' - (2-k^2)y + 2y^3 = 0$. We can find a solution, assuming $u(x) = c_0 y(c_1 x)$ and fixing c_0 and c_1. However, we take a different approach here.

We apply a change of variables $\tau = \sqrt{1-k^2s^2}$ to (A.1). We have

$$x = \int_0^{\mathrm{sn}(x,k)} \frac{\mathrm{d}s}{\sqrt{(1-s^2)(1-k^2s^2)}} = \int_{\mathrm{dn}(x,k)}^1 \frac{\mathrm{d}\tau}{\sqrt{(1-\tau^2)(k^2-1+\tau^2)}}, \tag{2.1}$$

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

where (A.2) is used. Let $\mathrm{dn}^{-1}(y,k)$ denote the inverse function of $y=\mathrm{dn}(x,k)$. Substituting $x=\mathrm{dn}^{-1}(y,k)$ into (2.1), we obtain

$$\mathrm{dn}^{-1}(y,k)=\int_y^1 \frac{\mathrm{d}\tau}{\sqrt{(1-\tau^2)(k^2-1+\tau^2)}}. \tag{2.2}$$

We consider only the case $u^-_{n,\varepsilon}(x)$, since the case $u^+_{n,\varepsilon}(x)$ is similar. For simplicity, u stands for $u^-_{n,\varepsilon}$. By (1.1) we have

$$\frac{\varepsilon^2 u'^2}{2}-\frac{u^2}{2}+\frac{u^4}{4}=-\frac{\alpha^2}{2}+\frac{\alpha^4}{4},$$

where

$$\alpha := u(0)=\max_{0\leqslant x\leqslant 1} u(x)(>1).$$

Since $2\varepsilon^2u'^2=(\alpha^2-u^2)(\alpha^2-2+u^2)$, we have

$$\frac{x}{\sqrt{2}\varepsilon}=\int_0^x \frac{-u'\,\mathrm{d}x}{\sqrt{(\alpha^2-u^2)(\alpha^2-2+u^2)}},$$

where we use the fact that $u'(x)<0$ for $0<x<1/n$. Let k be defined by the relation

$$\alpha=\sqrt{\frac{2}{2-k^2}}.$$

Using a change of variables $\tau := u/\alpha$, we have

$$\begin{aligned}\frac{x}{\sqrt{2}\varepsilon}&=\int_{u(x)}^{\alpha}\frac{\mathrm{d}u}{\sqrt{(\alpha^2-u^2)(\alpha^2-2+u^2)}}=\frac{1}{\alpha}\int_{\frac{u(x)}{\alpha}}^{1}\frac{\mathrm{d}\tau}{\sqrt{(1-\tau^2)\left(1-\frac{2}{\alpha^2}+\tau^2\right)}}\\&=\sqrt{\frac{2-k^2}{2}}\int_{\sqrt{\frac{2-k^2}{2}}u(x)}^{1}\frac{\mathrm{d}\tau}{\sqrt{(1-\tau^2)(k^2-1+\tau^2)}}\\&=\sqrt{\frac{2-k^2}{2}}\mathrm{dn}^{-1}\left(\sqrt{\frac{2-k^2}{2}}u(x),k\right),\end{aligned}$$

where (2.2) is used in the last equality. Thus,

$$u(x)=\sqrt{\frac{2}{2-k^2}}\mathrm{dn}\left(\frac{x}{\varepsilon\sqrt{2-k^2}},k\right). \tag{2.3}$$

Since $K(k)$ is a half-period of $\mathrm{dn}(x,k)$ and $1/n$ is a half-period of $u(x)$, we have

$$\frac{1}{n\varepsilon\sqrt{2-k^2}}=K(k),$$

and hence (1.3) is obtained. Substituting $\varepsilon=1/\sqrt{2-k^2}nK(k)$ into (2.3), we have (1.4). □

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

2.2. Three special eigenvalues

Proposition 1.2 says that there are three (resp. two) exact eigenvalues of (LP$_\pm$) if n is even (resp. odd). These three or two eigenvalues are special and play a key role in this paper. Proposition 1.2 can be proved by direct calculation. We define $\lambda_\pm(k)$ by

$$\lambda_+ := -1 + \frac{2\sqrt{1-k^2+k^4}}{2-k^2}, \quad \lambda_- := -1 - \frac{2\sqrt{1-k^2+k^4}}{2-k^2}. \tag{2.4}$$

Then it is obvious that $\lambda^\pm_{0,\varepsilon} = \lambda_-$ and $\lambda^\pm_{n,\varepsilon} = \lambda_+$.

LEMMA 2.1. *Let $k_{n,\varepsilon}$ be the unique solution of (1.3). Then,*

$$1 - k^2_{n,\varepsilon} = 16\,\mathrm{e}^{-\frac{2}{n\varepsilon}} + o(\mathrm{e}^{-\frac{2}{n\varepsilon}}) \ \text{as} \ \ \varepsilon \to 0.$$

Proof. By lemma A.3 we have

$$(1-k^2)K(k) + \frac{1}{2}(1-k^2)\log(1-k^2) - 2(1-k^2)\log 2 = o(1) \ \text{as} \ \ k \to 1,$$

and hence $(1-k^2)K(k) \to 0$ as $k \to 1$. By (1.3) we have

$$-\frac{2}{\sqrt{2-k^2}n\varepsilon} = -\frac{2}{n\varepsilon} + \frac{2(1-k^2)K(k)}{\sqrt{2-k^2}+1} = -\frac{2}{n\varepsilon} + o(1) \ \text{as} \ \ \varepsilon \to 0.$$

Using $K(k) = 1/\sqrt{2-k^2}n\varepsilon$, we have

$$\begin{aligned} 1 - k^2 = \exp\left(-2K(k) + 4\log 2 + o(1)\right) &= 16\exp\left(\frac{-2}{\sqrt{2-k^2}n\varepsilon} + o(1)\right) \\ &= 16\exp\left(\frac{-2}{n\varepsilon} + o(1)\right) \\ &= 16\exp\left(-\frac{2}{n\varepsilon}\right)(1+o(1)) \ \text{ as } \varepsilon \to 0. \end{aligned}$$

We obtain the desired result. □

Using lemma 2.1, we obtain asymptotic expansions of three special eigenvalues.

Proof of corollary 1.6. First we prove (i) and (iii). By lemma 2.1 we have

$$\frac{2\sqrt{1-k^2+k^4}}{2-k^2} = 2 - 3(1-k^2) + o(1-k^2) = 2 - 48\,\mathrm{e}^{-\frac{2}{n\varepsilon}} + o(\mathrm{e}^{-\frac{2}{n\varepsilon}}) \text{ as } \varepsilon \to 0.$$

Then

$$\begin{aligned} \lambda^\pm_{0,\varepsilon} &= \lambda_- = -1 - \frac{2\sqrt{1-k^2+k^4}}{2-k^2} = -3 + 48\,\mathrm{e}^{-\frac{2}{n\varepsilon}} + o\left(\mathrm{e}^{-\frac{2}{n\varepsilon}}\right) \ \text{ as } \varepsilon \to 0, \\ \lambda^\pm_{n,\varepsilon} &= \lambda_+ = -1 + \frac{2\sqrt{1-k^2+k^4}}{2-k^2} = 1 - 48\,\mathrm{e}^{-\frac{2}{n\varepsilon}} + o\left(\mathrm{e}^{-\frac{2}{n\varepsilon}}\right) \ \text{ as } \varepsilon \to 0. \end{aligned}$$

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

Next we prove (ii). If n is even, then

$$\lambda^+_{n/2,\varepsilon} = -\frac{3(1-k^2)}{2-k^2} = -3(1-k^2) + o(1-k^2) = -48\,\mathrm{e}^{-\frac{2}{n\varepsilon}} + o(\mathrm{e}^{-\frac{2}{n\varepsilon}}) \text{ as } \varepsilon \to 0,$$

$$\lambda^-_{n/2,\varepsilon} = -\frac{3}{2-k^2} = -3 + 3(1-k^2) + o(1-k^2) = -3 + 48\,\mathrm{e}^{-\frac{2}{n\varepsilon}} + o(\mathrm{e}^{-\frac{2}{n\varepsilon}}) \text{ as } \varepsilon \to 0. \qquad \square$$

2.3. Exact representation of the other eigenvalues

Before we consider the scalar field equation, let us briefly explain a theory for a general nonlinear term developed in Wakasa–Yotsutani [**12**].

Let $u(x)$ be a solution of (1.1), and let α denote the maximum value of $u(x)$. Substituting $\varphi(x) = \sqrt{\psi(x)}$ into the equation in (1.2), we see that $\psi(x)$ satisfies

$$\varepsilon^2(2\psi\psi'' - \psi'^2) + 4(f'(u) + \lambda)\psi^2 = 0.$$

Let us consider the following function:

$$\Psi(x) := \varepsilon^2(2\psi\psi'' - \psi'^2) + 4(f'(u) + \lambda)\psi^2. \tag{2.5}$$

We have

$$\Psi'(x) = 2\psi\left\{\varepsilon^2\psi''' + 4(f'(u) + \lambda)\psi' + 2f''(u)u'\psi\right\}.$$

We look for a solution of the equation

$$\varepsilon^2\psi''' + 4(f'(u) + \lambda)\psi' + 2f''(u)u'\psi = 0 \tag{2.6}$$

of the form $\psi(x) = h(u(x))$, where $h(\,\cdot\,)$ is an unknown positive function. Substituting $\psi(x) = h(u(x))$ into (2.6), we obtain a key equation

$$2(F(\alpha) - F(u))h'''(u) - 3f(u)h''(u) + (3f'(u) + 4\lambda)h'(u) + 2f''(u)h(u) = 0, \tag{2.7}$$

where we use the following relations:

$$\varepsilon^2 u'^2 = 2(F(\alpha) - F(u)) \text{ and } \varepsilon^2 u'' = -f(u).$$

We assume that an exact expression $h(u)$ of a positive solution of (2.7) can be obtained. Then, $h(u)$ is also a solution of (2.6). Thus, $\Psi'(x) \equiv 0$ and Ψ is constant. Substituting $h(u)$ into (2.5), we see that there exists $\rho \in \mathbb{R}$ such that

$$\Psi(x) = (F(\alpha) - F(u))(4hh'' - 2h'^2) - 2f(u)hh' + 4(f'(u) + \lambda)h^2 =: 4\rho. \tag{2.8}$$

Now, we construct an eigenfunction of the form

$$\varphi(x) = \sqrt{h(u(x))}W(\theta(x)), \tag{2.9}$$

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

using a solution $h(u)$ of (2.7). Here, $W(\,\cdot\,)$ and $\theta(x)$ are defined later. Using (2.8), we have

$$\varepsilon^2\frac{\mathrm{d}^2\sqrt{h(u(x))}}{\mathrm{d}x^2}+f'(u)\sqrt{h(u(x))}+\lambda\sqrt{h(u(x))}$$
$$=\frac{1}{4h^{3/2}}\left\{(F(\alpha)-F(u))(4hh''-2h'^2)-2f(u)hh'+4(f'(u)+\lambda)h^2\right\}=\frac{\rho}{h^{3/2}}. \tag{2.10}$$

Substituting (2.9) into (1.2), we have

$$0=\varepsilon^2\left\{\frac{\mathrm{d}^2\sqrt{h}}{\mathrm{d}x^2}W+2\frac{\mathrm{d}\sqrt{h}}{\mathrm{d}x}W'\theta'+\sqrt{h}\left(W''\theta'^2+W'\theta''\right)\right\}+f'(u)\sqrt{h}W+\lambda\sqrt{h}W$$
$$=\varepsilon^2\sqrt{h}\theta'^2\left(W''+\frac{2\frac{\mathrm{d}\sqrt{h}}{\mathrm{d}x}\theta'+\sqrt{h}\theta''}{\sqrt{h}\theta'^2}W'\right)+\left(\varepsilon^2\frac{\mathrm{d}^2\sqrt{h}}{\mathrm{d}x^2}+f'(u)\sqrt{h}+\lambda\sqrt{h}\right)W$$
$$=\varepsilon^2\sqrt{h}\theta'^2\left(W''+\frac{1}{h\theta'^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(h\theta'\right)W'+\frac{\rho}{\varepsilon^2h^2\theta'^2}W\right), \tag{2.11}$$

where (2.10) is used in the last equality. We assume that $\rho>0$. We define $\theta(x)$ by a solution of

$$h(u(x))\theta'(x)=\frac{\sqrt{\rho}}{\varepsilon}. \tag{2.12}$$

Specifically, the following function is a solution of (2.12):

$$\theta(x)=\frac{1}{\varepsilon}\int_0^x\frac{\sqrt{\rho}\mathrm{d}\xi}{h(u(\xi))}+\theta_0.$$

Moreover, it follows from (2.11) that $W''(\theta)+W(\theta)=0$, and hence $W(\theta)=C\cos(\theta+\theta_1)$. By (2.9) we obtain

$$\varphi(x)=C\sqrt{h(u(x))}\cos\left(\frac{1}{\varepsilon}\int_0^x\frac{\sqrt{\rho_0}\,\mathrm{d}\xi}{h(u(\xi))}+x_0\right), \tag{2.13}$$

where $x_0:=\theta_0+\theta_1$. In other words, a general solution of $\varepsilon^2\varphi''+f'(u)\varphi+\lambda\varphi=0$ can be written in terms of a solution $u(x)$ of the nonlinear problem (1.1). Readers can find more details of (2.7) and (2.13) in [**8**, **12**, **14**].

In our case the maximum value of $u^{\pm}_{n,\varepsilon}(x)$, which is denoted by α, is given by

$$\alpha:=\sqrt{\frac{2}{2-k^2}} \tag{2.14}$$

and (2.7) becomes

$$2\left(-\frac{\alpha^2}{2}+\frac{\alpha^4}{4}+\frac{u^2}{2}-\frac{u^4}{4}\right)h'''+3(u-u^3)h''+\left\{3(-1+3u^2)+4\lambda\right\}h'+12uh=0. \tag{2.15}$$

There seems to be no solution formula of (2.7) for general nonlinearities. In the case (2.15) every coefficient is a polynomial of u. We substitute $h(u)=u^p$ into (2.15).

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

Then it does not vanish, and the top term becomes

$$-\frac{1}{2}(p-4)(p+1)(p+6)u^{p+1}. \tag{2.16}$$

This suggests that a bi-quadratic polynomial is a candidate of a solution, since (2.16) vanishes for $p = 4$. Actually, h defined by (2.19) is a solution of (2.15). We evaluate (2.8) at a maximum point of u. Then,

$$2\rho_0 = -f(\alpha)h(\alpha)h'(\alpha) + 2(f'(\alpha) + \lambda)h(\alpha)^2,$$

which yields (2.18). Using $h(u)$ and ρ_0, we can construct a general solution of $\varepsilon^2\varphi'' + f(u)\varphi + \lambda\varphi = 0$. We take the Neumann boundary condition into account. Let $x_0 = 0$ in (2.13). Since $1/n$ is a half-period of $u^{\pm}_{n,\varepsilon}(x)$, the following holds:

$$\frac{n}{\varepsilon}\int_0^{1/n} \frac{\sqrt{\rho_0}\,\mathrm{d}\xi}{h(u^{\pm}_{n,\varepsilon}(\xi))} = j\pi \text{ for } j \in \{1, 2, \ldots\}.$$

Eigenvalues can be found by solving the above equation.

Finding a solution of (2.7) is key in this theory. A solution for (2.7) was found for

$$f(u) = \sin u \text{ in [12]}, \quad f(u) = u - u^3 \text{ in [14]}, \quad f(u) = \mathrm{e}^{\pm u} \text{ in [8] and } f(u) = \sinh u \text{ in [1]}.$$

We summarize these results in the following lemma:

Lemma 2.2. *Let $j \neq 0, n$ and $j \neq n/2$ if n is even. Let k_ε be the unique solution of (1.3). We define $\mathcal{A}_0$ by*

$$\mathcal{A}_0(k, \lambda) := \frac{1}{\varepsilon}\int_0^{1/n} \frac{\sqrt{\rho_0(\lambda, k)}\mathrm{d}\xi}{|h(u^{\pm}_{n,\varepsilon}(\xi), \lambda, k)|} \tag{2.17}$$

for $\lambda \in \{\lambda \in \mathbb{R};\ \rho_0(\lambda, k) > 0\}$, where

$$\rho_0(\lambda, k) := \frac{16}{81}\lambda\left(\lambda + \frac{3}{2-k^2}\right)\left(\lambda + \frac{3-3k^2}{2-k^2}\right)(\lambda - \lambda_+)(\lambda - \lambda_-), \tag{2.18}$$

$$h(u, \lambda, k) = u^4 - 2\left(\frac{\lambda}{3} + 1\right)u^2 + \frac{4}{9}\lambda^2 + \frac{4}{3}\lambda + \frac{4(1-k^2)}{(2-k^2)^2}. \tag{2.19}$$

(i) *Let α be defined by (2.14). If $\rho_0(\lambda, k) > 0$, then either $h(u, \lambda, k) > 0$ or $h(u, \lambda, k) < 0$ for $u \in \left[\sqrt{2-\alpha^2}, \alpha\right]$.*

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

(ii) *If the equation for* λ

$$\mathcal{A}_0(k_\varepsilon, \lambda) = \frac{j\pi}{n} \tag{2.20}$$

has a solution $\lambda \in \{\lambda \in \mathbb{R} |\ \rho_0(\lambda, k) > 0\}$*, then* λ *is a* $j+1$*-th eigenvalue of* ($\mathrm{LP}_\pm$) *and*

$$\varphi^{\pm}_{j,\varepsilon}(x) = \sqrt{|h(u^{\pm}_{n,\varepsilon}(x), \lambda, k_\varepsilon)|} \cos\left(\frac{1}{\varepsilon}\int_0^x \frac{\sqrt{\rho_0(\lambda, k_\varepsilon)}\,\mathrm{d}\xi}{|h(u^{\pm}_{n,\varepsilon}(\xi), \lambda, k_\varepsilon)|}\right)$$

is an associated eigenfunction.

Proof. Here, we directly prove the lemma.

(i) By direct calculation we can check that $h(u)$ satisfies

$$(F(\alpha) - F(u))(2hh'' - h'^2) - f(u)hh' + 2(f'(u) + \lambda)h^2 = 2\rho_0. \tag{2.21}$$

Suppose the contrary, i.e., there exists $\bar{u} \in [\sqrt{2-\alpha^2}, \alpha]$ such that $h(\bar{u}) = 0$. Since $F(\alpha) - F(\bar{u}) \geqslant 0$, it follows from (2.21) that

$$0 \geqslant -(F(\alpha) - F(\bar{u}))h'^2 = 2\rho_0 > 0,$$

which is a contradiction. This contradiction leads to the assertion (i).

(ii) Without loss of generality, we can assume that $h(u^{\pm}_{n,\varepsilon}(x)) > 0$ for $x \in [0, 1]$. Let

$$\varphi(x) := \sqrt{h(u^{\pm}_{n,\varepsilon}(x))} \cos\left(\frac{1}{\varepsilon}\int_0^x \frac{\sqrt{\rho_0}\,\mathrm{d}\xi}{h(u^{\pm}_{n,\varepsilon}(\xi))}\right).$$

For simplicity, we write $u^{\pm}_{n,\varepsilon}$ and $h(u^{\pm}_{n,\varepsilon})$ as u and h, respectively. Then,

$$\begin{aligned}
&\varepsilon^2\varphi_{xx} + f'(u)\varphi + \lambda\varphi \\
&= \frac{1}{2h^{3/2}}\left\{\frac{\varepsilon^2 u'^2}{2}(2hh'' - h'^2) + \varepsilon^2 u'' hh' - 2\rho_0 + 2f'(u)h^2 + 2\lambda h^2\right\} \\
&\quad \cos\left(\frac{1}{\varepsilon}\int_0^x \frac{\sqrt{\rho_0}\,\mathrm{d}\xi}{h(u(\xi))}\right) \\
&= \frac{1}{2h^{3/2}}\left\{(F(\alpha) - F(u))(2hh'' - h'^2) - f(u)hh' - 2\rho_0 + 2f'(u)h^2 + 2\lambda h^2\right\} \\
&\quad \cos\left(\frac{1}{\varepsilon}\int_0^x \frac{\sqrt{\rho_0}\,\mathrm{d}\xi}{h(u(\xi))}\right) = 0.
\end{aligned}$$

By (2.20) we see that $\varphi(x)$ has exactly j zero(s) in $0 \leqslant x \leqslant 1$. It follows from Sturm–Liouville theory that $\varphi(x)$ is a $j+1$-th eigenfunction and λ is the associated eigenvalue. □

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

Since $\sqrt{2-\alpha^2} \leqslant u^{\pm}_{n,\varepsilon}(x) \leqslant \alpha$ for $0 \leqslant x \leqslant 1$ (proposition 1.1), either $h(u^{\pm}_{n,\varepsilon}(x)) > 0$ or $h(u^{\pm}_{n,\varepsilon}(x)) < 0$ holds in the case $\rho_0(\lambda, k) > 0$. Hence, $\mathcal{A}_0(k, \lambda)$ is well defined. It follows from lemma 2.2 that the $j+1$-th eigenvalue is determined by (2.20) and that the other eigenvalues are on $\{\rho_0 > 0\}$. Hence, it is important to study the function $\mathcal{A}_0(k, \lambda)$.

Let

$$\mu := (2-k^2)\lambda.$$

We sometimes use μ instead of λ, since various formulas become simple. We define

$$\mu_+ := (2-k^2)\lambda_+ = -2 + k^2 + 2\sqrt{1-k^2+k^4}, \quad \mu_- := (2-k^2)\lambda_- = -2 + k^2 - 2\sqrt{1-k^2+k^4}. \tag{2.22}$$

We will show that the other eigenvalues are on $\{\lambda \in \mathbb{R} |\ \rho_0(\lambda, k) > 0\}$. The set $\{(k, \lambda) |\ k \in (0,1), \rho_0(\lambda, k) > 0\}$ corresponds to

$$\Sigma := \left\{ (k,\mu) |\ k \in (0,1),\ \rho_0\left(\frac{\mu}{2-k^2}, k\right) > 0 \right\}. \tag{2.23}$$

The set Σ is split into three components, i.e., $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, where

$$\begin{aligned} \Sigma_0 &:= \{(k,\mu) |\ k \in (0,1),\ \mu \in (\mu_-, -3)\}, \\ \Sigma_1 &:= \{(k,\mu) |\ k \in (0,1),\ \mu \in (-3+3k^2, 0)\}, \\ \Sigma_2 &:= \{(k,\mu) |\ k \in (0,1),\ \mu \in (\mu_+, \infty)\}. \end{aligned} \tag{2.24}$$

Let

$$\mathcal{A}(k,\mu) := \mathcal{A}_0(k,\lambda), \tag{2.25}$$

where $\mathcal{A}_0$ is defined by (2.17). Then the characteristic equation (2.20) becomes

$$\mathcal{A}(k,\mu) = \frac{j\pi}{n}.$$

Figure 5 shows the graph of $\mathcal{A}(k, \cdot)$. In § 3 we rigorously study the graph of $\mathcal{A}(k, \cdot)$. Specifically, end points of $\mathcal{A}(k, \cdot)$ are obtained in lemma 3.1 and the monotone increase of $\mathcal{A}(k,\mu)$ in μ is proved in lemma 3.4.

We would like to obtain a more simple expression of $\mathcal{A}(k,\mu)$. The following lemma indicates that $\mathcal{A}(k,\mu)$ can be decomposed into two complete elliptic integrals of the third kind.

Lemma 2.3. *Suppose that the assumptions of lemma 2.2 hold. Then the following hold*:

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

(i) *If* $(k,\mu)\in\Sigma_0\cup\Sigma_1$*, then*

$$\mathcal{A}(k,\mu)=|\mathrm{sgn}(\nu_+)\mathcal{M}(\nu_+,k)-\mathrm{sgn}(\nu_-)\mathcal{M}(\nu_-,k)|, \tag{2.26}$$

where $\mathrm{sgn}(\,\cdot\,)$ *denotes the sign function,*

$$\mathcal{M}(\nu,k):=\sqrt{\frac{(1+\nu)(k^2+\nu)}{\nu}}\Pi(\nu,k),\quad \Pi(\nu,k) := \int_0^1\frac{\mathrm{d}s}{(1+\nu s^2)\sqrt{(1-s^2)(1-k^2s^2)}}, \tag{2.27}$$

$$\nu_\pm(k,\mu):=\frac{3k^2}{2}\frac{\mu-3k^2\pm\sqrt{-3\mu^2+6(k^2-2)\mu+9k^4}}{\mu(\mu+3-3k^2)}. \tag{2.28}$$

(ii) *If* $(k,\mu)\in\Sigma_2$*, then*

$$\mathcal{A}(k,\mu)=\frac{1}{9k^4}\sqrt{\mu(\mu+3)(\mu+3-3k^2)(\mu-\mu_+)(\mu-\mu_-)}\tilde{\Pi}(a,b,k),$$

where

$$\tilde{\Pi}(a,b,k):=\int_0^1\frac{\mathrm{d}s}{\sqrt{(1-s^2)(1-k^2s^2)\{a+(b-s^2)^2\}}}, \tag{2.29}$$

$$a(k,\mu):=\frac{1}{12k^2}(\mu-\mu_+)(\mu-\mu_-),\quad b(k,\mu):=\frac{3k^2-\mu}{6k^2}. \tag{2.30}$$

Proof. (i) We consider the case $h(u^\pm_{n,\varepsilon}(x))>0$ for $x\in[0,1]$. In both cases $u^+_{n,\varepsilon}$ and $u^-_{n,\varepsilon}$ by lemma 2.2 we see that

$$\mathcal{A}_0(k,\lambda)=\frac{\sqrt{\rho_0}}{\varepsilon}\int_0^{1/n}\frac{\mathrm{d}\xi}{h(u^-_{n,\varepsilon}(\xi))},$$

because

$$\int_0^{1/n}\frac{\mathrm{d}\xi}{h(u^+_{n,\varepsilon}(\xi))}=\int_0^{1/n}\frac{\mathrm{d}\xi}{h(u^-_{n,\varepsilon}(\xi))}.$$

Using the change of variables $w=u^-_{n,\varepsilon}(\xi)$, we have

$$\mathcal{A}_0(k,\lambda)=\frac{\sqrt{\rho_0}}{\varepsilon}\int_{\sqrt{2-\alpha^2}}^{\alpha}\frac{\mathrm{d}w}{\left|\frac{\mathrm{d}u^-_{n,\varepsilon}}{\mathrm{d}x}(\xi)\right|h(u^-_{n,\varepsilon}(\xi))}=\sqrt{\frac{\rho_0}{2}}\int_{\sqrt{2-\alpha^2}}^{\alpha}\frac{\mathrm{d}w}{\sqrt{F(\alpha)-F(w)}h(w)},$$

where we use

$$\varepsilon^2\left(\frac{\mathrm{d}u^-_{n,\varepsilon}}{\mathrm{d}x}\right)^2=2(F(\alpha)-F(u^-_{n,\varepsilon})).$$

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

In order to bring the integral closer to the expression of the complete elliptic integral of the third kind, we let $s = w/\alpha$. Since

$$2(F(\alpha) - F(\alpha s)) = \frac{\alpha^4}{2}(1 - s^2)(s^2 - 1 + k^2),$$

we have

$$\mathcal{A}_0(k,\lambda) = \frac{\sqrt{2\rho_0}}{\alpha}\int_{\sqrt{1-k^2}}^{1} \frac{\mathrm{d}s}{\sqrt{(1-s^2)(s^2-1+k^2)}h(\alpha s)}. \tag{2.31}$$

We define $\sigma_\pm$ by

$$\sigma_\pm := \frac{\alpha^2}{\frac{\lambda}{3} + 1 \pm \sqrt{\frac{3k^4 - (2-k^2)^2(\lambda^2+2\lambda)}{3(2-k^2)^2}}}. \tag{2.32}$$

Then

$$\frac{1}{h(\alpha s)} = \frac{\sqrt{3}}{2\alpha^2\sqrt{-(\lambda-\lambda_+)(\lambda-\lambda_-)}}\left(\frac{\sigma_-}{1-\sigma_- s^2} - \frac{\sigma_+}{1-\sigma_+ s^2}\right). \tag{2.33}$$

Note that $-(\lambda - \lambda_+)(\lambda - \lambda_-) > 0$ for $(k, \mu) \in \Sigma_0 \cup \Sigma_1$. Using (2.33), we have

$$\mathcal{A}_0(k,\lambda) = \frac{\sqrt{6\rho_0(k,\lambda)}}{2\alpha^3\sqrt{-(\lambda-\lambda_+)(\lambda-\lambda_-)}}\int_{\sqrt{1-k^2}}^{1}\frac{1}{\sqrt{(1-s^2)(s^2-1+k^2)}} \left(\frac{\sigma_-}{1-\sigma_- s^2} - \frac{\sigma_+}{1-\sigma_+ s^2}\right)\mathrm{d}s.$$

Using (2.18), which is the definition of ρ_0, and

$$\left(\lambda + \frac{3}{2-k^2}\right)\left(\lambda + \frac{3-3k^2}{2-k^2}\right) = \frac{4\alpha^4}{9\sigma_+\sigma_-},$$

we have

$$\mathcal{A}_0(k,\lambda) = \sqrt{\frac{(2-k^2)(-\lambda)}{3\sigma_+\sigma_-}}\left(\sigma_-\Pi_0(\sigma_-,k) - \sigma_+\Pi_0(\sigma_+,k)\right),$$

where

$$\Pi_0(\sigma,k) := \int_{\sqrt{1-k^2}}^{1}\frac{\mathrm{d}s}{(1-\sigma s^2)\sqrt{(1-s^2)(s^2-1+k^2)}}.$$

Using the change of variables $s := \sqrt{1-k^2\tau^2}$, we have

$$\Pi_0(\sigma,k) = \int_0^1 \frac{\mathrm{d}\tau}{(1-\sigma+\sigma k^2\tau^2)\sqrt{(1-\tau^2)(1-k^2\tau^2)}} = \frac{1}{1-\sigma}\Pi\left(\frac{k^2\sigma}{1-\sigma},k\right).$$

Therefore,

$$\mathcal{A}_0(k,\lambda) = \sqrt{\frac{(2-k^2)(-\lambda)}{3\sigma_+\sigma_-}}\left(\frac{\sigma_-}{1-\sigma_-}\Pi\left(\frac{k^2\sigma_-}{1-\sigma_-},k\right) - \frac{\sigma_+}{1-\sigma_+}\Pi\left(\frac{k^2\sigma_+}{1-\sigma_+},k\right)\right).$$

Since $\lambda = \mu/(2-k^2)$, we can check that

$$\nu_+ = \frac{k^2\sigma_-}{1-\sigma_-} \quad \text{and} \quad \nu_- = \frac{k^2\sigma_+}{1-\sigma_+}. \tag{2.34}$$

Here $\sigma_\pm$ are given by (2.32) and $\nu_\pm$ are given by (2.28). Using

$$\sigma_+ = \frac{\nu_-}{\nu_- + k^2} \quad \text{and } \sigma_- = \frac{\nu_+}{\nu_+ + k^2},$$

by lemma 2.4 below we have

$$\begin{aligned}
\mathcal{A}(k,\mu) &= \sqrt{\frac{-\mu}{3}}\sqrt{\frac{(\nu_+ + k^2)(\nu_- + k^2)}{\nu_+\nu_-}}\left(\frac{\nu_+}{k^2}\Pi(\nu_+,k) - \frac{\nu_-}{k^2}\Pi(\nu_-,k)\right) \\
&= \frac{1}{k^2}\sqrt{\frac{-\mu}{3}}\left(\sqrt{\left|\frac{\nu_+(\nu_+ + k^2)(\nu_- + k^2)}{\nu_-}\right|}\operatorname{sgn}(\nu_+)\Pi(\nu_+,k)\right. \\
&\quad \left. - \sqrt{\left|\frac{\nu_-(\nu_+ + k^2)(\nu_- + k^2)}{\nu_+}\right|}\operatorname{sgn}(\nu_-)\Pi(\nu_-,k)\right) \\
&= \operatorname{sgn}(\nu_+)\mathcal{M}(\nu_+,k) - \operatorname{sgn}(\nu_-)\mathcal{M}(\nu_-,k).
\end{aligned} \tag{2.35}$$

Here $\mathcal{M}$ and Π are defined by (2.27). When $h(u^-_{n,\varepsilon}(x)) < 0$ for $x \in [0,1]$, we have

$$\mathcal{A}_0(k,\lambda) = \frac{\sqrt{\rho_0}}{\varepsilon}\int_0^{1/n} \frac{\mathrm{d}\xi}{-h(u^-_{n,\varepsilon}(\xi))} = -\left(\operatorname{sgn}(\nu_+)\mathcal{M}(\nu_+,k) - \operatorname{sgn}(\nu_-)\mathcal{M}(\nu_-,k)\right) > 0.$$

We obtain (2.26).

(ii) We can check that if $(k,\mu) \in \Sigma_2$, then $h(u^\pm_{n,\varepsilon}(x)) > 0$ for $x \in [0,1]$. By the same way as in (i) we have (2.31). Using the change of variables $s := \sqrt{1-k^2\tau^2}$, we have

$$\mathcal{A}(k,\mu) = \frac{\sqrt{2\rho_0}}{\alpha^5 k^4}\int_0^1 \frac{\mathrm{d}\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}\,\{a + (b-\tau^2)^2\}},$$

where

$$\begin{aligned}
a &= \frac{1}{\alpha^4 k^4}\left(\frac{\lambda^2 + 2\lambda}{3} - \frac{k^4}{(2-k^2)^2}\right) = \frac{1}{12k^4}(\mu-\mu_+)(\mu-\mu_-), \\
b &= \frac{1}{\alpha^2 k^2}\left(\alpha^2 - 1 - \frac{\lambda}{3}\right) = \frac{3k^2-\mu}{6k^2}.
\end{aligned}$$

In this case the polynomial $a + (b-\tau^2)^2$ cannot be factored into two real quadratic polynomials. We have

$$\mathcal{A}(k,\mu) = \frac{1}{9k^4}\sqrt{\mu(\mu+3)(\mu+3-3k^2)(\mu-\mu_+)(\mu-\mu_-)}\,\tilde{\Pi}(a,b,k).$$

The proof of (ii) is complete. □

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

The following equality, which is somewhat nontrivial, is left in the proof of lemma 2.3.

LEMMA 2.4. *Let* $\sigma_\pm$ *and* $\nu_\pm$ *be defined by* (2.32) *and* (2.28), *respectively. If* $(k,\mu)\in\Sigma_0\cup\Sigma_1$, *then*

$$\frac{1}{k^2}\sqrt{\frac{-\mu}{3}}\sqrt{\left|\frac{\nu_\pm(\nu_+ + k^2)(\nu_- + k^2)}{\nu_\mp}\right|}=\sqrt{\frac{(1+\nu_\pm)(k^2+\nu_\pm)}{\nu_\pm}}. \tag{2.36}$$

Proof. By (2.28) we can check that

$$\nu_+ + \nu_- = \frac{3k^2(\mu-3k^2)}{\mu(\mu+3-2k^2)},\quad \nu_+\nu_- = \frac{9k^4}{\mu(\mu+3-3k^2)}. \tag{2.37}$$

Using (2.37), we have

$$\begin{aligned}-\frac{\mu}{3k^4}\frac{(\nu_+ + k^2)(\nu_- + k^2)}{\nu_\pm\nu_\mp} &= -\frac{\mu}{3k^4}\frac{\nu_+\nu_- + k^2(\nu_+ + \nu_-)+k^4}{\nu_+\nu_-}\\ &= -\frac{1}{27k^4}\mu(\mu+3)(\mu+3-3k^2).\end{aligned} \tag{2.38}$$

On the other hand, we consider the right-hand side of (2.36). Let $\gamma_\pm := \alpha^2/\sigma_\pm$. Then $\gamma_\pm$ satisfies

$$\gamma_\pm^2 - 2\left(\frac{\lambda}{3}+1\right)\gamma_\pm + \frac{4}{9}\lambda^2 + \frac{4}{3}\lambda + \frac{4(1-k^2)}{(2-k^2)^2} = 0. \tag{2.39}$$

Since

$$\frac{1}{\nu_\pm} = \frac{1}{k^2}\left(\frac{1}{\sigma_\mp}-1\right) = \frac{1}{k^2}\left(\frac{2-k^2}{2}\gamma_\mp - 1\right),$$

we have

$$\begin{aligned}\frac{(1+\nu_\pm)(k^2+\nu_\pm)}{\nu_\pm^3} &= \frac{1}{\nu_\pm}\left(1+\frac{1}{\nu_\pm}\right)\left(1+\frac{k^2}{\nu_\pm}\right)\\ &= \frac{1}{k^2}\left(\frac{2-k^2}{2}\gamma_\mp - 1\right)\left(1+\frac{2-k^2}{2k^2}\gamma_\mp - \frac{1}{k^2}\right)\\ &\quad\left(1+\frac{2-k^2}{2}\gamma_\mp - 1\right)\\ &= \frac{(2-k^2)^3}{8k^4}\left(\gamma_\mp^2 - 2\gamma_\mp + \frac{4(1-k^2)}{(2-k^2)^2}\right)\gamma_\mp.\end{aligned}$$

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

By (2.39) we have

$$\begin{aligned}\frac{(1+\nu_\pm)(k^2+\nu_\pm)}{\nu_\pm^3} &= \frac{(2-k^2)^3}{8k^4}\left(\frac{2}{3}\lambda\gamma_\mp - \frac{4}{9}\lambda^2 - \frac{4}{3}\lambda\right)\gamma_\mp \\ &= \frac{(2-k^2)^3}{12k^4}\lambda\left\{\gamma_\mp^2 - 2\left(\frac{\lambda}{3}+1\right)\gamma_\mp\right\} \\ &= -\frac{(2-k^2)^3}{12k^4}\lambda\left(\frac{4}{9}\lambda^2 + \frac{4}{3}\lambda + \frac{4(1-k^2)}{(2-k^2)^2}\right) \\ &= -\frac{1}{27k^4}\mu(\mu+3)(\mu+3-3k^2). \qquad (2.40)\end{aligned}$$

It follows from (2.40) and (2.38) that

$$-\frac{\mu}{3k^4}\frac{(\nu_+ + k^2)(\nu_- + k^2)}{\nu_\pm\nu_\mp} = \frac{(1+\nu_\pm)(k^2+\nu_\pm)}{\nu_\pm^3}.$$

Multiplying both sides by $\nu_\pm^2$, we have

$$-\frac{\mu}{3k^4}\frac{\nu_\pm(\nu_+ + k^2)(\nu_- + k^2)}{\nu_\mp} = \frac{(1+\nu_\pm)(k^2+\nu_\pm)}{\nu_\pm}.$$

The assertion holds. □

Now the proof of lemma 2.3 is complete.

3. Fundamental properties of $\mathcal{A}$

Let $\mathcal{A}(k,\mu)$ be defined by (2.25), and let Σ be defined by (2.23).

LEMMA 3.1. *Let $k \in (0,1)$ be fixed and $(k,\mu) \in \Sigma$, and let $\mu_\pm$ be defined by (2.22). Then the following hold*:

(i) $\mathcal{A}(k,\mu) \to 0$ *as* $\mu \to \mu_-$.

(ii) $\mathcal{A}(k,\mu) \to \pi/2$ *as* $\mu \to -3$.

(iii) $\mathcal{A}(k,\mu) \to \pi/2$ *as* $\mu \to -3+3k^2$.

(iv) $\mathcal{A}(k,\mu) \to \pi$ *as* $\mu \to 0$.

(v) $\mathcal{A}(k,\mu) \to \pi$ *as* $\mu \to \mu_+$.

(vi) $\mathcal{A}(k,\mu) \to \infty$ *as* $\mu \to \infty$.

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

Proof. (i) In this case we can check that $h(u^{\pm}_{n,\varepsilon}(x)) > 0$ for $x \in [0, 1]$. Let $\nu_{\pm}$ be defined by (2.28). We see that there exists ν_* such that

$$\lim_{\mu \to \mu_-} \nu_+ = \nu_*, \quad \lim_{\mu \to \mu_-} \nu_- = \nu_*.$$

By direct calculation we can check that $-1 < \nu_* < 0$. Let $\mathcal{M}$ be defined by (2.27). Since

$$\lim_{\nu_+ \to \nu_*} \mathcal{M}(\nu_+, k) = \mathcal{M}(\nu_*, k), \quad \lim_{\nu_- \to \nu_*} \mathcal{M}(\nu_-, k) = \mathcal{M}(\nu_*, k),$$

we have

$$\lim_{\mu \to \mu_-} \mathcal{A}(k, \mu) = 0.$$

(ii) In this case we can check that $h(u^{\pm}_{n,\varepsilon}(x)) > 0$ for $x \in [0, 1]$. We see that $\nu_+ \to -k^2$ $(\mu \to -3)$ and $\nu_- \to -1$ $(\mu \to -3)$. By lemma A.9 we have

$$\lim_{\nu_+ \to -k^2} \mathrm{sgn}(\nu_+)\mathcal{M}(\nu_+, k) = 0, \quad \lim_{\nu_- \to -1} \mathrm{sgn}(\nu_-)\mathcal{M}(\nu_-, k) = -\frac{\pi}{2}.$$

Then,

$$\lim_{\mu \to -3} \mathcal{A}(k, \mu) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

(iii) In this case we can check that $h(u^{\pm}_{n,\varepsilon}(x)) < 0$ for $x \in [0, 1]$. We see that $\nu_+ \to -k^2$ $(\mu \to -3 + 3k^2)$ and $\nu_- \to \infty$ $(\mu \to -3 + 3k^2)$. By lemma A.9 we have

$$\lim_{\nu_+ \to -k^2} \mathrm{sgn}(\nu_+)\mathcal{M}(\nu_+, k) = 0, \quad \lim_{\nu_- \to \infty} \mathrm{sgn}(\nu_-)\mathcal{M}(\nu_-, k) = \frac{\pi}{2}.$$

Then,

$$\lim_{\mu \to -3+3k^2} \mathcal{A}(k, \mu) = -\left(0 - \frac{\pi}{2}\right) = \frac{\pi}{2}.$$

(iv) In this case we can check that $h(u^{\pm}_{n,\varepsilon}(x)) < 0$ for $x \in [0, 1]$. We see that $\nu_+ \to -1$ $(\mu \to 0)$ and $\nu_- \to \infty$ $(\mu \to 0)$. By lemma A.9 we have

$$\lim_{\nu_+ \to -1} \mathrm{sgn}(\nu_+)\mathcal{M}(\nu_+, k) = -\frac{\pi}{2}, \quad \lim_{\nu_- \to \infty} \mathrm{sgn}(\nu_-)\mathcal{M}(\nu_-, k) = \frac{\pi}{2}.$$

Then,

$$\lim_{\mu \to 0} \mathcal{A}(k, \mu) = -\left(-\frac{\pi}{2} - \frac{\pi}{2}\right) = \pi.$$

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

(v) In this case we can check that $h(u^{\pm}_{n,\varepsilon}(x)) > 0$ for $x \in [0,1]$. Let a, b be defined by (2.30). By lemmas 2.3 and A.7 we have

$$\begin{aligned}\mathcal{A}(k,\mu) &= \frac{1}{9k^4}\sqrt{\mu(\mu+3)(\mu+3-3k^2)(\mu-\mu_+)(\mu-\mu_-)}\tilde{\Pi}(a,b,k)\\ &= \frac{2}{3\sqrt{3}k^2}\sqrt{\mu(\mu+3)(\mu+3-3k^2)}\sqrt{a}\tilde{\Pi}(a,b,k)\\ &\to \frac{2}{3\sqrt{3}k^2}\sqrt{\mu_+(\mu_+{+}3)(\mu_+{+}3-3k^2)}\\ &\qquad \frac{\pi}{2\sqrt{b_0(1-b_0)(1-k^2b_0)}} \quad \text{as} \quad \mu \to \mu_+.\end{aligned}$$

Here, $b_0 := (3k^2 - \mu_+)/6k^2$. Since

$$\frac{2}{3\sqrt{3}k^2}\sqrt{\mu_+(\mu_+{+}3)(\mu_+{+}3-3k^2)} = \frac{2}{3\sqrt{3}k^2}\sqrt{(1-k^2+k^4)\mu_+{+}3k^4(2-k^2)},$$

$$\frac{\pi}{2\sqrt{b_0(1-b_0)(1-k^2b_0)}} = \frac{3\sqrt{3}\pi k^2}{2\sqrt{(1-k^2+k^4)\mu_+{+}3k^4(2-k^2)}},$$

we have

$$\lim_{\mu\to\mu_+}\mathcal{A}(k,\mu) = \pi.$$

(vi) Let $\lambda_\pm$ be defined by (2.4). We see that $\lambda_+ : (0,1) \to \mathbb{R}$ is increasing in k,

$$\lim_{k\to 0}\lambda_+(k) = 0, \quad \lim_{k\to 1}\lambda_+(k) = 1.$$

Hence, $0 < \lambda_+ < 1$ for $k \in (0,1)$.
Let $a(k,\mu)$, $b(k,\mu)$ be defined by (2.30). Since $\lambda > \lambda_+ > 0$, we have see $\mu > \mu_+ > 0$ and

$$(b(k,\mu)-1)^2 - b(k,\mu)^2 = \frac{\mu}{3k^2} > 0.$$

Hence,

$$0 < a(k,\mu) + b(k,\mu)^2 < a(k,\mu) + (b(k,\mu)-1)^2.$$

Using the inequality

$$\tilde{\Pi}(a,b,k) \geqslant \min\left\{\frac{1}{a+b^2}, \frac{1}{a+(b-1)^2}\right\}K(k)$$

for any $a > 0$, $b \in (0,1)$, $k \in (0,1)$, we have

$$\begin{aligned}\mathcal{A}(k,\mu) &= \frac{1}{9k^4}\sqrt{\mu(\mu+3)(\mu+3-3k^2)(\mu-\mu_+)(\mu-\mu_-)}\tilde{\Pi}(a(k,\mu),b(k,\mu),k)\\ &\geqslant \frac{1}{9k^4}\sqrt{\mu(\mu+3)(\mu+3-3k^2)(\mu-\mu_+)(\mu-\mu_-)}\\ &\quad \frac{K(k)}{a(k,\mu)+(b(k,\mu)-1)^2}\\ &= \sqrt{\frac{(\mu+3-3k^2)(\mu-\mu_+)(\mu-\mu_-)}{\mu(\mu+3)}}K(k)\\ &\geqslant c\sqrt{\mu-\mu_+}K(k),\end{aligned}$$

because there exists $c > 0$ such that $\sqrt{\frac{(\mu+3-3k^2)(\mu-\mu_-)}{\mu(\mu+3)}} > c$ for $\mu > \mu_+$. Since

$$\mathcal{A}(k,\mu) \geqslant c\sqrt{\mu-\mu_+}K(k) \to \infty \quad \text{as} \quad \mu \to \infty,$$

the assertion (vi) holds. □

Let $\nu_\pm$ be defined by (2.28). We need two lemmas to prove a monotonicity of $\mathcal{A}(k,\mu)$ in μ.

Lemma 3.2. *Let $\nu_\pm$ be defined by (2.28). Then the following hold*:

$$\frac{\partial}{\partial\mu}\left[\left\{\frac{k^2}{2}\left(\frac{1}{\nu_+}+\frac{1}{\nu_-}\right)+1\right\}\left(-\frac{1}{\nu_+}+\frac{1}{\nu_-}\right)\right] = \frac{-\{\mu^2+3(2-k^2)\mu+6(1-k^2)\}}{3\sqrt{3}k^2\sqrt{-\mu^2+(2k^2-4)\mu+3k^4}},$$

$$\frac{\partial}{\partial\mu}\left(-\frac{1}{\nu_+}+\frac{1}{\nu_-}\right) = \frac{-(\mu-k^2+2)}{\sqrt{3}k^2\sqrt{-\mu^2+(2k^2-4)\mu+3k^4}}.$$

Proof. By (2.28) we can check that

$$\frac{1}{\nu_+}+\frac{1}{\nu_-} = \frac{\mu-3k^2}{3k^2}, \quad -\frac{1}{\nu_+}+\frac{1}{\nu_-} = \frac{\sqrt{-\mu^2+(2k^2-4)\mu+3k^4}}{\sqrt{3}k^2}.$$

Then by direct calculation we obtain the conclusion. We omit details. □

We define

$$\mathcal{B}(k,\mu) := \left\{\mu^2+3(2-k^2)\mu+6(1-k^2)\right\}K(k) - 3(\mu-k^2+2)E(k), \tag{3.1}$$

since a sign of $\mathcal{B}(k,\mu)$ is important in the study of $\partial\mathcal{A}/\partial\mu$.

Lemma 3.3. *The following holds*:

$$\mathcal{B}(k,0) < 0 \quad \textit{for} \quad k \in (0,1).$$

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

Proof. It is obvious that $\mathcal{B}(0,0)=0$. By lemma A.1 (i) and (ii) we have

$$\begin{aligned}\frac{\partial \mathcal{B}}{\partial k}(k,0) &= -12kK(k)+6(1-k^2)\frac{E(k)-(1-k^2)K(k)}{k(1-k^2)}\\ &\quad + 6kE(k)+3(k^2-2)\frac{E(k)-K(k)}{k}\\ &= 9k(E(k)-K(k))<0\end{aligned}$$

for $k\in(0,1)$. Thus, the assertion holds. □

LEMMA 3.4. *Let Σ_i, $i=0,1,2$, be defined by (2.24). For each $i=0,1,2$,*

$$\frac{\partial \mathcal{A}}{\partial \mu}(k,\mu)>0 \quad \textit{for} \quad (k,\mu)\in\Sigma_i. \tag{3.2}$$

Proof. We consider the case $(k,\mu)\in\Sigma_0$. Let $\nu_\pm$ be defined by (2.28). By (2.37) and (2.40) we can see that

$$-1<\nu_-<\nu_+<-k^2,\ \frac{(1+\nu_-)(k^2+\nu_-)}{\nu_-^2}<0,\ \frac{(1+\nu_+)(k^2+\nu_+)}{\nu_+^2}<0. \tag{3.3}$$

Then, $\mathcal{A}(k,\mu)=-\mathcal{M}(\nu_+,k)+\mathcal{M}(\nu_-,k)$. By lemma A.1(iv) we have

$$\begin{aligned}\frac{\partial \mathcal{A}}{\partial \mu} &= -\frac{\partial \mathcal{M}}{\partial \nu_+}\frac{\partial \nu_+}{\partial \mu}+\frac{\partial \mathcal{M}}{\partial \nu_-}\frac{\partial \nu_-}{\partial \mu}\\ &= -\frac{1}{2}\sqrt{\frac{(1+\nu_+)(k^2+\nu_+)}{\nu_+}}\left(\frac{-K(k)}{\nu_+(1+\nu_+)}+\frac{E(k)}{(1+\nu_+)(k^2+\nu_+)}\right)\frac{\partial \nu_+}{\partial \mu}\\ &\quad +\frac{1}{2}\sqrt{\frac{(1+\nu_-)(k^2+\nu_-)}{\nu_-}}\left(\frac{-K(k)}{\nu_-(1+\nu_-)}+\frac{E(k)}{(1+\nu_-)(k^2+\nu_-)}\right)\frac{\partial \nu_-}{\partial \mu}.\end{aligned}$$

By (3.3) we have

$$\begin{aligned}\frac{\partial \mathcal{A}}{\partial \mu} &= -\frac{1}{2}\sqrt{\frac{\nu_+^3}{(1+\nu_+)(k^2+\nu_+)}}\left(\frac{k^2+\nu_+}{\nu_+^3}K(k)-\frac{1}{\nu_+^2}E(k)\right)\frac{\partial \nu_+}{\partial \mu}\\ &\quad +\frac{1}{2}\sqrt{\frac{\nu_-^3}{(1+\nu_-)(k^2+\nu_-)}}\left(\frac{k^2+\nu_-}{\nu_-^3}K(k)-\frac{1}{\nu_-^2}E(k)\right)\frac{\partial \nu_-}{\partial \mu}.\end{aligned}$$

By (2.40) we have

$$\frac{(1+\nu_\pm)(k^2+\nu_\pm)}{\nu_\pm^3}=-\frac{1}{27k^4}\mu(\mu+3)(\mu+3-3k^2)=:\mathcal{R}.$$

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

Then,

$$\begin{aligned}\frac{\partial \mathcal{A}}{\partial \mu} &= \frac{1}{2\sqrt{\mathcal{R}}}\left\{\left(-\frac{k^2+\nu_+}{\nu_+^3}\frac{\partial \nu_+}{\partial \mu} + \frac{k^2+\nu_-}{\nu_-^3}\frac{\partial \nu_-}{\partial \mu}\right)K(k)\right.\\ &\quad \left. + \left(\frac{1}{\nu_+^2}\frac{\partial \nu_+}{\partial \mu} - \frac{1}{\nu_-^2}\frac{\partial \nu_-}{\partial \mu}\right)E(k)\right\}\\ &= \frac{1}{2\sqrt{\mathcal{R}}}\left[\frac{\partial}{\partial \mu}\left\{\left(-\frac{k^2}{2}\left(\frac{1}{\nu_+}+\frac{1}{\nu_-}\right)-1\right)\left(-\frac{1}{\nu_+}+\frac{1}{\nu_-}\right)\right\}K(k)\right.\\ &\quad \left. + \frac{\partial}{\partial \mu}\left(-\frac{1}{\nu_+}+\frac{1}{\nu_-}\right)E(k)\right].\end{aligned}$$

By lemma 3.2 we have

$$\frac{\partial \mathcal{A}}{\partial \mu} = \frac{\mathcal{B}(k,\mu)}{6\sqrt{3}k^2\sqrt{\mathcal{R}}\sqrt{-\mu^2+(2k^2-4)\mu+3k^4}},$$

where $\mathcal{B}(k,\mu)$ is defined by (3.1). Since $(k,\mu) \in \Sigma_0$, we see that $-3(\mu - k^2 + 2) > 0$. By lemma A.2 we have

$$\begin{aligned}\mathcal{B}(k,\mu) &= \left\{\mu^2 + 3(2-k^2)\mu + 6(1-k^2)\right\}K(k) - 3(\mu-k^2+2)E(k)\\ &> \left\{\mu^2 + 3(2-k^2)\mu + 6(1-k^2) - 3(\mu-k^2+2)(1-k^2)\right\}K(k)\\ &= \left\{\mu^2 + 3\mu + 3k^2(1-k^2)\right\}K(k) > 0.\end{aligned}$$

Thus, (3.2) holds for $(k,\mu) \in \Sigma_0$.

We consider the case $(k,\mu) \in \Sigma_1$. By (2.37) and (2.40) we can see that

$$\nu_+ < 0 < \nu_-, \ \frac{(1+\nu_-)(k^2+\nu_-)}{\nu_-^2} > 0, \ \frac{(1+\nu_+)(k^2+\nu_+)}{\nu_+^2} < 0.$$

Then, $\mathcal{A}(k,\mu) = \mathcal{M}(\nu_-,k) + \mathcal{M}(\nu_+,k)$. By a similar calculation as in (i) we have

$$\begin{aligned}\frac{\partial \mathcal{A}}{\partial \mu} &= \frac{1}{2\sqrt{\mathcal{R}}}\left[\frac{\partial}{\partial \mu}\left\{\left(\frac{k^2}{2}\left(\frac{1}{\nu_+}+\frac{1}{\nu_-}\right)+1\right)\left(-\frac{1}{\nu_+}+\frac{1}{\nu_-}\right)\right\}K(k)\right.\\ &\quad \left. + \frac{\partial}{\partial \mu}\left(\frac{1}{\nu_+}-\frac{1}{\nu_-}\right)E(k)\right]\\ &= \frac{-\mathcal{B}(k,\mu)}{6\sqrt{3}k^2\sqrt{\mathcal{R}}\sqrt{-\mu^2+(2k^2-4)\mu+3k^4}}.\end{aligned}$$

We show that $\mathcal{B}(k,\mu) < 0$ for $(k,\mu) \in \Sigma_1$. Since $\mathcal{B}(k,\mu)$ is convex in μ, it is enough to show that $\mathcal{B}(k,-3+3k^2) < 0$ and $\mathcal{B}(k,0) < 0$. By lemma A.2 we have

$$\begin{aligned}\mathcal{B}(k,-3+3k^2) &= -3(1-k^2)K(k) + 3(1-2k^2)E(k)\\ &\leqslant -3\frac{1-k^2}{1-\frac{k^2}{2}}E(k) + 3(1-2k^2)E(k) = -\frac{3k^2(3-2k^2)}{2-k^2}E(k) < 0\end{aligned}$$

for $k \in (0,1)$. By lemma 3.3 we see that $\mathcal{B}(k,0) < 0$ for $k \in (0,1)$. Thus, (3.2) holds for $(k,\mu) \in \Sigma_1$.

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

We consider the case $(k,\mu)\in\Sigma_2$. In this case by (2.35) we see that $\mathcal{A}(k,\mu)=\mathcal{M}(\nu_+,k)-\mathcal{M}(\nu_-,k)$, where we consider $\mathcal{M}(\nu_+,k)$ and $\mathcal{M}(\nu_-,k)$ as complex-valued functions. Then by a similar calculation as in (i) we obtain

$$\frac{\partial\mathcal{A}}{\partial\mu}=\frac{-\mathcal{B}(k,\mu)}{6\sqrt{3}k^2\sqrt{\mathcal{R}}\sqrt{-\mu^2+(2k^2-4)\mu+3k^4}},$$

Since $\mathcal{R}<0$ and $-\mu^2+(2k^2-4)\mu+3k^4<0$, we have

$$\frac{\partial\mathcal{A}}{\partial\mu}=\frac{\mathcal{B}(k,\mu)}{6\sqrt{3}k^2\sqrt{|\mathcal{R}|}\sqrt{|-\mu^2+(2k^2-4)\mu+3k^4|}}.$$

Thus, we show that $\mathcal{B}(k,\mu)>0$ for $\mu>\mu_+$. Since $\mu-k^2+2>0$ for $\mu>\mu_+$, by lemma A.2 we have

$$\begin{aligned}\mathcal{B}(k,\mu)&\geqslant\left\{\mu^2+3(2-k^2)\mu+6(1-k^2)\right\}K(k)-3(\mu-k^2+2)\left(1-\frac{1}{2}k^2\right)K(k)\\&=\left\{\mu^2+3\left(1-\frac{1}{2}k^2\right)\mu-\frac{3}{2}k^4\right\}K(k)\\&>\left\{\left(k^2-2+2\sqrt{1-k^2+k^4}\right)^2\right.\\&\quad\left.+3\left(1-\frac{1}{2}k^2\right)\left(k^2-2+2\sqrt{1-k^2+k^4}\right)-\frac{3}{2}k^4\right\}K(k)\\&=2\sqrt{1-k^2+k^4}\left\{\sqrt{1-k^2+k^4}-\left(1-\frac{1}{2}k^2\right)\right\}K(k)>0.\end{aligned}$$

Thus, (3.2) holds for $(k,\mu)\in\Sigma_2$. □

4. Asymptotic formulas

In this section let $\lambda_j(k)$, $j\geqslant 0$, denote the $j+1$-th eigenvalue of (LP$_\pm$) and let $\mu_j(k):=(2-k^2)\lambda_j(k)$.

We know the following:

- $\mathcal{A}(k,\mu)$ is defined on $\Sigma_0\cup\Sigma_1\cup\Sigma_2$ (lemma 2.2),
- $\mathcal{A}(k,\mu)$ is increasing in μ (lemma 3.4),
- the range of $\mathcal{A}(k,\cdot\,)$ is $\mathbb{R}_+\setminus\{\pi/2,\pi\}$, where $\mathbb{R}_+:=\{x|\ x>0\}$ (lemma 3.1).

The equation

$$\mathcal{A}(k,\mu_j)=\frac{j\pi}{n},\ j\neq 0,n/2,n$$

has a unique solution μ_j in Σ_0 for $0<j<n/2$, in Σ_1 for $n/2<j<n$ and in Σ_2 for $j>n$. Then, $\lambda_j=\mu_j/(2-k^2)$ is the $j+1$-th eigenvalue for $j\neq 0,n/2,n$. In this section we obtain an asymptotic expansion of λ_j as $k\to 1$ for $j\neq 0,n/2,n$.

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

Proof of theorem 1.3(i). Let $\lambda_\pm$ be defined by (2.4). We consider the case $\lambda_- < \lambda_j < -3/(2-k^2)$, which corresponds to the case $0 < j < n/2$. Because of the definition of λ_- we have

$$\frac{2}{(2-k^2)(2-k^2+\sqrt{1-k^2+k^4})} = \frac{\frac{\lambda_-}{3}+1}{1-k^2} \leqslant \frac{\frac{\lambda_j}{3}+1}{1-k^2} \leqslant \frac{\frac{-3}{3(2-k^2)}+1}{1-k^2} = \frac{1}{2-k^2},$$

and hence

$$\lim_{k\to 1} \frac{\frac{\lambda_j}{3}+1}{1-k^2} = 1. \tag{4.1}$$

We define $r_j(k)$ by

$$r_j := \frac{\lambda_j+3}{\lambda_-+3} - 1.$$

It follows from corollary 1.6(i) that $\lambda_- + 3 > 0$ for k close to 1. Since $r_j = (\lambda_j - \lambda_-)/(\lambda_- + 3) > 0$, We have

$$\begin{aligned} 0 \leqslant \frac{\lambda_j+3}{\lambda_-+3} - 1 &\leqslant \frac{\frac{-3}{2-k^2}+3}{-1-\frac{2\sqrt{1-k^2+k^4}}{2-k^2}+3} - 1 \\ &= \frac{1-k^2}{2(k^2+\sqrt{1-k^2+k^4})} = \left(\frac{1}{4}+o(1)\right)(1-k^2) \quad \text{as} \quad k\to 1. \end{aligned}$$

Therefore,

$$0 \leqslant \frac{r_j}{1-k^2} \leqslant \frac{1}{4} + o(1) \quad \text{as} \quad k \to 1.$$

Then there exists $r_j^* \in [0, 1/4]$ such that

$$\lim_{m\to\infty} \frac{r_j(k_m)}{1-k_m^2} = r_j^*, \text{ i.e., } r_j = (r_j^* + o(1))(1-k_m^2) \quad \text{as} \quad m \to \infty$$

with a suitable monotonically increasing sequence $\{k_m\}_{m=1}^\infty$ satisfying $k_m \to 1$ as $m \to \infty$.

Let $\sigma_\pm(k_m)$ be defined by (2.32) with k_m and $\lambda_j(k_m)$. Let $\nu_\pm(k_m)$ be defined by (2.28) with k_m and $\mu_j(k_m)$. Next we calculate the limit $\lim_{m\to\infty}(1+\nu_\pm)/(1-k_m^2)$. By (2.4) we see that

$$\lambda_-+3 = 3(1-k^2) - \frac{15}{4}(1-k^2)^2 + o((1-k^2)^2) \quad \text{as} \quad k \to 1. \tag{4.2}$$

Using (4.2), we have

$$\begin{aligned} \lambda_j - \lambda_- &= r_j(\lambda_-+3) = (r_j^*+o(1))(1-k_m^2)\cdot(3+o(1))(1-k_m^2) \\ &= (3r_j^*+o(1))(1-k_m^2)^2. \end{aligned} \tag{4.3}$$

Since

$$3k^4 - (2-k^2)^2(\lambda^2+2\lambda) = -(2-k^2)^2(\lambda-\lambda_+)(\lambda-\lambda_-),$$

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

by (4.1) and (4.3) we have

$$\begin{aligned}\frac{1}{(1-k_m^2)\sigma_-} &= \frac{2-k_m^2}{2}\left(\frac{\frac{\lambda_j}{3}+1}{1-k_m^2} - \frac{1}{1-k_m^2}\sqrt{\frac{-(2-k_m^2)^2(\lambda_j-\lambda_+)(\lambda_j-\lambda_-)}{3(2-k_m^2)^2}}\right) \\ &= \frac{2-k_m^2}{2}\left(\frac{\frac{\lambda_j}{3}+1}{1-k_m^2} - \frac{1}{1-k_m^2}\right. \\ &\qquad \left.\sqrt{\frac{-(1+o(1))^2(-4+o(1))3(r_j^*+o(1))(1-k_m^2)^2}{3}}\right) \\ &= \frac{1-2\sqrt{r_j^*}}{2} + o(1) \quad \text{as} \quad m\to\infty. \end{aligned} \tag{4.4}$$

Moreover,

$$\frac{1}{\sigma_-} = \left(\frac{1-2\sqrt{r_j^*}}{2} + o(1)\right)(1-k_m^2) \to 0 \quad \text{as} \quad m\to\infty. \tag{4.5}$$

Thus, by (4.5), (4.4) and (2.34) we have

$$\nu_+^* := \lim_{m\to\infty}\frac{1+\nu_+}{1-k_m^2} = \lim_{m\to\infty}\frac{\frac{1}{(1-k_m^2)\sigma_-}-1}{\frac{1}{\sigma_-}-1} = \frac{1+2\sqrt{r_j^*}}{2}. \tag{4.6}$$

Similarly we have

$$\nu_-^* := \lim_{m\to\infty}\frac{1+\nu_-}{1-k_m^2} = \lim_{m\to\infty}\frac{\frac{1}{(1-k_m^2)\sigma_+}-1}{\frac{1}{\sigma_+}-1} = \frac{1-2\sqrt{r_j^*}}{2}. \tag{4.7}$$

By lemma A.5 with (4.6) and (4.7) we have

$$\begin{aligned}\lim_{m\to\infty}\mathcal{A}(k_m,\mu_j(k_m)) &= \lim_{m\to\infty}|\mathrm{sgn}(\nu_+)\mathcal{M}(\nu_+,k_m) - \mathrm{sgn}(\nu_-)\mathcal{M}(\nu_-,k_m)| \\ &= \left|(-1)\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{\nu_+^*}{1-\nu_+^*}}\right)\right. \\ &\qquad \left. -(-1)\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{\nu_-^*}{1-\nu_-^*}}\right)\right| \\ &= \left|\tan^{-1}\sqrt{\frac{1+2\sqrt{r_j^*}}{1-2\sqrt{r_j^*}}} - \tan^{-1}\sqrt{\frac{1-2\sqrt{r_j^*}}{1+2\sqrt{r_j^*}}}\right| \\ &= 2\tan^{-1}\sqrt{\frac{1+2\sqrt{r_j^*}}{1-2\sqrt{r_j^*}}} - \frac{\pi}{2}.\end{aligned}$$

Solving the equation

$$2\tan^{-1}\sqrt{\frac{1+2\sqrt{r_j^*}}{1-2\sqrt{r_j^*}}} - \frac{\pi}{2} = \frac{j\pi}{n}$$

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

with respect to r_j^*, we have $r_j^* = \frac{1}{4}\sin^2(\frac{j\pi}{n})$, which implies

$$\lim_{k\to 1}\frac{r_j(k)}{1-k^2} = \frac{1}{4}\sin^2\left(\frac{j\pi}{n}\right).$$

By (4.2) we have

$$\begin{aligned}\lambda_j &= -3 + (\lambda_-{+}3) + r_j(\lambda_-{+}3)\\ &= -3 + 3(1-k^2) + \frac{3}{4}\left(\sin^2\left(\frac{j\pi}{n}\right) - 5\right)(1-k^2)^2 + o((1-k^2)^2) \quad \text{as} \quad k\to 1.\end{aligned} \tag{4.8}$$

□

Proof of theorem 1.3(ii). We consider the case $(-3+3k^2)/(2-k^2) < \lambda_j < 0$, which corresponds to the case $n/2 < j < n$. We define $r_j(k)$ by

$$r_j := \frac{\lambda_j}{1-k^2}.$$

It is obvious that $\lambda_j \to 0$ as $k\to 1$. Since

$$\frac{1}{1-k^2}\frac{-3+3k^2}{2-k^2} < \frac{\lambda_j}{1-k^2} < 0,$$

there exists $r_j^* \in [-3, 0]$ such that

$$\lim_{m\to\infty} r_j(k_m) = r_j^*, \text{ i.e.}, \lambda_j = (r_j^* + o(1))(1-k_m^2) \quad \text{as} \quad m\to\infty$$

with a suitable monotonically increasing sequence $\{k_m\}_{m=1}^\infty$ satisfying $k_m \to 1$ as $m\to\infty$.

Let $\sigma_\pm(k_m)$ be defined by (2.32) with k_m and $\lambda_j(k_m)$. Let $\nu_\pm(k_m)$ be defined by (2.28) with k_m and $\mu_j(k_m)$. We calculate the limit $\lim_{m\to\infty}(1+\nu_+)/(1-k_m^2)$. Since

$$D_m := \frac{3k_m^4 - (2-k_m^2)^2(\lambda_j^2 + 2\lambda_j)}{3(2-k_m^2)^2} \to 1 \quad \text{as} \quad m\to\infty,$$

we have

$$\begin{aligned}\frac{1}{(1-k_m^2)\sigma_-} &= \frac{1}{1-k_m^2}\frac{2-k_m^2}{2}\left(\frac{\lambda_j}{3} + 1 - \sqrt{D_m}\right)\\ &= \frac{2-k_m^2}{6}\frac{\lambda_j}{1-k_m^2}\\ &\quad + \frac{2-k_m^2}{2(1+\sqrt{D_m})}\left(\frac{4}{(2-k_m^2)^2} + \frac{2}{3}\frac{\lambda_j}{1-k_m^2} + \frac{1}{3}\frac{\lambda_j}{1-k_m^2}\lambda_j\right)\\ &\to 1 + \frac{r_j^*}{3} \quad \text{as} \quad m\to\infty.\end{aligned} \tag{4.9}$$

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

Moreover,

$$\frac{1}{\sigma_-} = \left(1 + \frac{r_j^*}{3} + o(1)\right)(1 - k_m^2) \to 0 \quad \text{as} \quad m \to \infty. \tag{4.10}$$

Hence, by (4.10), (4.9) and (2.34) we have

$$\nu_+^* := \lim_{m\to\infty} \frac{1+\nu_+}{1-k_m^2} = \lim_{m\to\infty} \frac{\frac{1}{(1-k_m^2)\sigma_-} - 1}{\frac{1}{\sigma_-} - 1} = -\frac{r_j^*}{3}. \tag{4.11}$$

Since

$$\frac{1}{\sigma_+} = \frac{2-k_m^2}{2}\left(\frac{\lambda_j}{3} + 1 + \sqrt{D_m}\right) \to 1 \quad \text{as} \quad m \to \infty$$

and $\sigma_+ < 1$, we have

$$\nu_- = \frac{k_m^2\sigma_+}{1-\sigma_+} \to \infty \quad \text{as} \quad m \to \infty.$$

Since

$$0 \leqslant \frac{1}{\nu_-} = \frac{2-k_m^2}{2k_m^2} \frac{\frac{2\lambda_j}{3}\left(\frac{2\lambda_j}{3} + \frac{2-2k_m^2}{2-k_m^2}\right)}{\frac{\lambda_j}{3} - \frac{k_m^2}{2-k_m^2} - \sqrt{D_m}} \leqslant C_0(1-k_m^2),$$

by lemma A.3 we have

$$0 \leqslant \frac{K(k_m)}{\sqrt{\nu_-+1}} \leqslant \frac{K(k_m)}{\sqrt{\nu_-}} \leqslant \sqrt{C_0(1-k_m^2)}\left(\log\frac{1}{\sqrt{1-k_m^2}} + 2\log 2 + o(1)\right)$$
$$\to 0 \quad \text{as} \quad m \to \infty.$$

Let $\mathcal{A}$ and $\mathcal{M}$ be given in lemma 2.3. By lemma A.5 with (4.11) and lemma A.6 we have

$$\begin{aligned}
\lim_{m\to\infty} \mathcal{A}(k_m, \mu_j(k_m)) &= \lim_{m\to\infty} |\mathrm{sgn}(\nu_+)\mathcal{M}(\nu_+, k_m) - \mathrm{sgn}(\nu_-)\mathcal{M}(\nu_-, k_m)| \\
&= \lim_{m\to\infty} \Bigg|(-1)\mathcal{M}(\nu_+, k_m) \\
&\qquad - \sqrt{\frac{k_m^2+\nu_-}{\nu_-}}\left(J(\nu_-, k_m) + \frac{K(k_m)}{\sqrt{\nu_-+1}}\right)\Bigg| \\
&= \left|(-1)\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{\nu_+^*}{1-\nu_+^*}}\right) - \frac{\pi}{2}\right| \\
&= \pi - \tan^{-1}\sqrt{\frac{-r_j^*}{3+r_j^*}}.
\end{aligned}$$

Solving the equation

$$\pi - \tan^{-1}\sqrt{\frac{-r_j^*}{3+r_j^*}} = \frac{j\pi}{n}$$

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

with respect to r_j^*, we have $r_j^* = -3\sin^2(\frac{j\pi}{n})$, which implies

$$\lim_{k\to 1} r_j(k) = -3\sin^2\left(\frac{j\pi}{n}\right).$$

Since $\lambda_j/(1-k^2) = r_j^* + o(1)$, we have

$$\lambda_j = -3\sin^2\left(\frac{j\pi}{n}\right)(1-k^2) + o(1-k^2) \quad \text{as} \quad k \to 1. \qquad \square$$

Proof of theorem 1.3(iii). Let $\lambda_\pm$ be defined by (2.4). We consider the case $\lambda_j > \lambda_+$, which corresponds to the case $j > n$. Let $\mathcal{A}$ be given in lemma 2.3 and let $\mu_\pm$ be defined by (2.22). By the same argument as in the proof of lemma 3.1(vi) we have

$$\frac{j\pi}{n} = \mathcal{A}(k,\mu) \geqslant c\sqrt{\mu_j - \mu_+}K(k).$$

This implies that

$$\lim_{k\to 1}(\mu_j(k) - \mu_+(k)) = 0,$$

and in particular $\lim_{k\to 1}\mu_j(k) = 1$,

$$\lim_{k\to 1} a(k,\mu_j(k)) = 0, \quad \lim_{k\to 1} b(k,\mu_j(k)) = \frac{1}{3},$$

where $a(k,\mu_j)$ and $b(k,\mu_j)$ are defined by (2.30). Using

$$a(k,\mu_j) + (b(k,\mu_j)-1)^2 = \frac{1}{9k^4}\mu_j(\mu_j+3),$$

by lemma A.8 we have

$$\begin{aligned}
\sqrt{\mu_j-\mu_+}K(k) &= \left(\frac{9k^4}{\sqrt{\mu_j(\mu_j+3)(\mu_j+3-3k^2)(\mu_j-\mu_-)}}\frac{j\pi}{n}\right.\\
&\quad \left. - \frac{\sqrt{\mu_j-\mu_+}\tilde{J}(a(k,\mu_j),b(k,\mu_j),k)}{\sqrt{a(k,\mu_j)}}\right)\left\{a(k,\mu_j) + (1-b(k,\mu_j))^2\right\}\\
&= \sqrt{\frac{\mu_j(\mu_j+3)}{(\mu_j+3-3k^2)(\mu_j-\mu_-)}}\frac{j\pi}{n}\\
&\quad - \frac{2\mu_j(\mu_j+3)\tilde{J}(a(k,\mu_j),b(k,\mu_j),k)}{3\sqrt{3}k^2\sqrt{\mu_j-\mu_-}}\\
&\to \frac{j\pi}{n} - \pi \quad \text{as} \quad k \to 1.
\end{aligned}$$

Here we used

$$\lim_{k\to 1}\sqrt{\frac{\mu_j(\mu_j+3)}{(\mu_j+3-3k^2)(\mu_j-\mu_-)}} = \sqrt{\frac{1(1+3)}{(1+3-3)(1-(-3))}} = 1,$$

$$\lim_{k\to 1}\frac{2\mu_j(\mu_j+3)\tilde{J}(a(k,\mu_j),b(k,\mu_j),k)}{3\sqrt{3}k^2\sqrt{\mu_j-\mu_-}}=\frac{2\cdot 1(1+3)\frac{\pi}{2\sqrt{\frac{1}{3}}(1-\frac{1}{3})}}{3\sqrt{3}\cdot 1^2\cdot\sqrt{1-(-3)}}=\pi.$$

By lemmas 2.1 and (1.3) we see that $1-k^2=16\,\mathrm{e}^{-2/n\varepsilon}(1+o(1))$ and $K(k)=1/(1+o(1))n\varepsilon$. Hence,

$$1-k^2=o\left(\frac{1}{K(k)^2}\right).$$

Using this relation, we have

$$\begin{aligned}\lambda_j&=\lambda_+ +\frac{1}{2-k^2}\frac{(j-n)^2\pi^2}{n^2}\frac{1}{K(k)^2}+o\left(\frac{1}{K(k)^2}\right)\\&=1-3(1-k^2)+o(1-k^2)+\frac{(j-n)^2\pi^2}{n^2}\frac{1}{K(k)^2}(1+o(1))+o\left(\frac{1}{K(k)^2}\right)\\&=1+\frac{(j-n)^2\pi^2}{n^2}\frac{1}{K(k)^2}+o\left(\frac{1}{K(k)^2}\right).\end{aligned}$$

□

Proof of corollary 1.7. (i) By (4.8) and lemma 2.1 we have

$$\begin{aligned}\lambda_j&=-3+(1+r_j)(\lambda_- +3)\\&=-3+\left\{1+\frac{1}{4}\left(\sin^2\frac{j\pi}{n}\right)(1-k^2)+o(1-k^2)\right\}(\lambda_- +3)\\&=-3+\left\{1+4\left(\sin^2\frac{j\pi}{n}\right)\mathrm{e}^{-\frac{2}{n\varepsilon}}+o(\mathrm{e}^{-\frac{2}{n\varepsilon}})\right\}(\lambda_{0,\varepsilon}^{\pm}+3).\end{aligned}$$

(ii) By theorem 1.3(ii) and lemma 2.1 we have

$$\lambda_j=-3\left(\sin^2\frac{j\pi}{n}\right)(1-k^2)+o(1-k^2)=-48\left(\sin^2\frac{j\pi}{n}\right)\mathrm{e}^{-\frac{2}{n\varepsilon}}+o(\mathrm{e}^{-\frac{2}{n\varepsilon}}).$$

(iii) By theorem 1.3(iii), corollary 1.6(iii) and (1.3) we have

$$\begin{aligned}\lambda_j&=\lambda_+ +\frac{(j-n)^2\pi^2}{(2-k^2)n^2K(k)^2}+o\left(\frac{1}{K(k)^2}\right)=\lambda_+ +(j-n)^2\pi^2\varepsilon^2+o(\varepsilon^2)\\&=1+(j-n)^2\pi^2\varepsilon^2+o(\varepsilon^2).\end{aligned}$$

□

Acknowledgements

The authors are grateful to referees for helpful comments which improve the presentation of the paper.

Y. M. was supported by JSPS KAKENHI Grant Numbers 19H01797, 19H05599.

T. W. was supported by JSPS KAKENHI Grant Numbers 18K03374, 21H00993.

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

Appendix A. Elliptic integrals and functions

A.1. Elliptic functions

Let $k \in (0,1)$. We denote the complete elliptic integrals of the first kind by

$$K(k) := \int_0^1 \frac{\mathrm{d}s}{\sqrt{(1-s^2)(1-k^2 s^2)}}.$$

Jacobi's elliptic function $\mathrm{sn}(x,k)$ is an odd, periodic and analytic function with the period $4K(k)$ as a function for the real domain, and is defined locally by

$$x = \int_0^{\mathrm{sn}(x,k)} \frac{\mathrm{d}s}{\sqrt{(1-s^2)(1-k^2 s^2)}} \tag{A.1}$$

for $x \in [0, K(k)]$. The function $\mathrm{cn}(x,k)$ is an even and $4K(k)$-periodic function defined locally by

$$\mathrm{cn}(x,k) := \sqrt{1-\mathrm{sn}^2(x,k)},$$

for $x \in [0, K(k)]$ and $\mathrm{dn}(x,k)$ is an even and $2K(k)$-periodic function defined by

$$\mathrm{dn}(x,k) := \sqrt{1-k^2\mathrm{sn}^2(x,k)}. \tag{A.2}$$

In particular,

$$\mathrm{sn}^2(x,k) + \mathrm{cn}^2(x,k) = 1, \quad k^2\mathrm{sn}^2(x,k) + \mathrm{dn}^2(x,k) = 1$$

for $x \in \mathbb{R}$ and $k \in (0,1)$.

A.2. Complete elliptic integrals

Let $k \in [0,1)$ and $\nu \in \mathbb{C} \setminus (-\infty,-1]$. The complete elliptic integrals of the second and third kind are defined by

$$E(k) := \int_0^1 \sqrt{\frac{1-k^2s^2}{1-s^2}}\,\mathrm{d}s, \quad \Pi(\nu,k) := \int_0^1 \frac{\mathrm{d}s}{(1+\nu s^2)\sqrt{(1-s^2)(1-k^2s^2)}},$$

respectively. The function $K(k)$ is monotonically increasing in k,

$$K(0) = \frac{\pi}{2}, \quad \lim_{k\to 1} K(k) = \infty$$

and E is monotonically decreasing in k,

$$E(0) = \frac{\pi}{2}, \quad \lim_{k\to 1} E(k) = 1.$$

In [**10**] the following modified complete integral of the third kind was introduced

$$\mathcal{M}(\nu,k) := \sqrt{\frac{(1+\nu)(k^2+\nu)}{\nu}}\,\Pi(\nu,k)$$

for $k \in (0,1)$ and $\nu \in \mathbb{C} \setminus ((-\infty,-1] \cup [-k^2,0])$. The function $\mathcal{M}$ appears in (2.27).

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

We give standard formulas for $K(k)$, $E(k)$ and $\Pi(\nu, k)$ in lemmas A.1–A.3 without proofs. See [3] for details.

LEMMA A.1. *Let $k \in (0,1)$ and $\nu \neq 0, -1, -k^2$. Then,*

(i) $\dfrac{\mathrm{d}E}{\mathrm{d}k}(k) = \dfrac{E(k) - K(k)}{k}$.

(ii) $\dfrac{\mathrm{d}K}{\mathrm{d}k}(k) = \dfrac{E(k) - (1-k^2)K(k)}{k(1-k^2)}$.

(iii) $\dfrac{\partial \Pi}{\partial k}(\nu, k) = \dfrac{k(E(k) - (1-k-2)\Pi(\nu,k))}{(k^2+\nu)(1-k^2)}$.

(iv) $\dfrac{\partial \Pi}{\partial \nu}(\nu, k) = -\dfrac{K(k)}{2\nu(1+\nu)} + \dfrac{E(k)}{2(1+\nu)(k^2+\nu)} + \dfrac{(k^2-\nu^2)\Pi(\nu,k)}{2\nu(1+\nu)(k^2+\nu)}$.

LEMMA A.2. *Let $k \in (0,1)$. Then*

$$(1-k^2)K(k) < E(k) < \left(1 - \frac{1}{2}k^2\right) K(k).$$

LEMMA A.3. *Let $k \in (0,1)$. Then*

$$\lim_{k \to 1} \left(K(k) - \log \frac{1}{\sqrt{1-k^2}} - 2\log 2 \right) = 0.$$

Lemmas A.4–A.6 are formulas for $\Pi(\nu, k)$. Proofs can be found in [14].

LEMMA A.4. *Let $k \in (0,1)$ and $\nu > -1$. Then,*

(i) $\lim_{\nu \to -1} \sqrt{1+\nu}\,\Pi(\nu, k) = \dfrac{\pi}{2\sqrt{1-k^2}}$.

(ii) $\lim_{\nu \to \infty} \sqrt{1+\nu}\,\Pi(\nu, k) = \dfrac{\pi}{2}$.

LEMMA A.5. *Let $k \in (0,1)$. Suppose that ν is a continuous function on $(0,1)$ with $-1 < \nu(k) < -k^2$ for $k \in (0,1)$. Assume that there exists $\nu^* \in [0,1]$ such that*

$$\lim_{k \to 1} \frac{1+\nu(k)}{1-k^2} = \nu^*.$$

Then, for each $\nu^* \in [0,1]$,

$$\lim_{k \to 1} \sqrt{-(1+\nu(k))(k^2+\nu(k))}\,\Pi(\nu(k), k) = \frac{\pi}{2} - \tan^{-1} \sqrt{\frac{\nu^*}{1-\nu^*}}$$

and

$$\lim_{k \to 1} \mathcal{M}(\nu(k), k) = \frac{\pi}{2} - \tan^{-1} \sqrt{\frac{\nu^*}{1-\nu^*}}.$$

https://doi.org/10.1017/prm.2023.95 Published online by Cambridge University Press

LEMMA A.6. *Let* $J(\nu, k) := \sqrt{1+\nu}\Pi(\nu, k) - \frac{1}{\sqrt{1+\nu}}K(k)$. *Then,*

$$\lim_{\nu\to\infty, k\to 1} J(\nu, k) = \frac{\pi}{2}.$$

In [**14**] a kind of a complete elliptic integral $\tilde{\Pi}(a, b, k)$ defined by (2.29) was introduced. Lemmas A.7 and A.8 are formulas for $\tilde{\Pi}$ and proofs can be found in [**14**].

LEMMA A.7. *Suppose that* $a > 0$ *and* $b, b_0 \in (0, 1)$. *Then for each* $k \in (0, 1)$,

$$\lim_{a\to 0, b\to b_0} \sqrt{a}\tilde{\Pi}(a, b, k) = \frac{\pi}{2\sqrt{b_0(1-b_0)(1-k^2 b_0)}}.$$

LEMMA A.8. *Suppose that* $a > 0,\ b,\ b_0 \in (0, 1)$ *and* $k \in (0, 1)$. *Let*

$$\tilde{J}(a, b, k) := \sqrt{a}\tilde{\Pi}(a, b, k) - \frac{\sqrt{a}}{a + (b-1)^2}K(k).$$

Then,

$$\lim_{a\to 0, b\to b_0, k\to 1} \tilde{J}(a, b, k) = \frac{\pi}{2\sqrt{b_0}(1-b_0)}.$$

Lemma A.9 is a formula for $\mathcal{M}(\nu, k)$.

LEMMA A.9. *Let* $k \in (0, 1)$ *and* $\nu \in (-1, -k^2) \cup (0, \infty)$. *Then,*

(i) $\displaystyle\lim_{\nu\to -1} \mathcal{M}(\nu, k) = \frac{\pi}{2}$.

(ii) $\displaystyle\lim_{\nu\to -k^2} \mathcal{M}(\nu, k) = 0$.

(iii) $\displaystyle\lim_{\nu\to\infty} \mathcal{M}(\nu, k) = \frac{\pi}{2}$.

Lemma A.9(i) (resp. (iii)) follows from Lemma A.4(i) (resp. (ii)). Lemma A.9(ii) is trivial.

References

1 S. Aizawa, Y. Miyamoto and T. Wakasa. Asymptotic formulas of the eigenvalues for the linearization of a one-dimensional sinh-Poisson equation. *J. Elliptic Parabol. Equ*

2 H. Berestycki and P. L. Lions. Nonlinear scalar field equations. I. Existence of a ground state. *Arch. Ration. Mech. Anal.* **82** (1983), 313–345.

3 P. Byrd and M. Friedman. *Handbook of elliptic integrals for engineers and scientists*, 2nd edn, revised. Die Grundlehren der Mathematischen Wissenschaften, vol. 67 (New York-Heidelberg: Springer-Verlag, 1971).

4 J. Carr and R. Pego. Metastable patterns in solutions of $u_t = \epsilon^2 u_x x - f(u)$. *Commun. Pure Appl. Math.* **42** (1989), 523–576.

5 A. Gierer and H. Meinhardt. A theory of biological pattern formation. *Kybernetik (Berlin)* **12** (1972), 30–39.

6 J. Langer and D. Singer. The total squared curvature of closed curves. *J. Differ. Geom.* **20** (1984), 1–22.

7 Y. Miyamoto and K. Yagasaki. Monotonicity of the first eigenvalue and the global bifurcation diagram for the branch of interior peak solutions. *J. Differ. Equ.* **254** (2013), 342–367.

8 Y. Miyamoto and T. Wakasa. Exact eigenvalues and eigenfunctions for a one-dimensional Gel'fand problem. *J. Math. Phys.* **60** (2019), 021506.

9 T. Wakasa. Exact eigenvalues and eigenfunctions associated with linearization for Chafee-Infante problem. *Funkcial. Ekvac.* **49** (2006), 321–336.

10 T. Wakasa. Note on parameter dependence of eigenvalues for a linearized eigenvalue problem. *Bull. Kyushu Inst. Technol. Pure Appl. Math. No.* **64** (2017), 1–12.

11 T. Wakasa. Eigenvalue problems associated with 1-dimensional scalar field equations (Succession and innovation of studies on ODEs in real domains). *Surikaisekikenkyusho Kokyūroku* **2080** (2018), 94–105.

12 T. Wakasa and S. Yotsutani. Representation formulas for some 1-dimensional linearized eigenvalue problems. *Commun. Pure Appl. Anal.* **7** (2008), 745–763.

13 T. Wakasa and S. Yotsutani. Asymptotic profiles of eigenfunctions for some 1-dimensional linearized eigenvalue problems. *Commun. Pure Appl. Anal.* **9** (2010), 539–561.

14 T. Wakasa and S. Yotsutani. Limiting classification on linearized eigenvalue problems for 1-dimensional Allen–Cahn equation I-asymptotic formulas of eigenvalues. *J. Differ. Equ.* **258** (2015), 3960–4006.

15 T. Wakasa and S. Yotsutani. Limiting classification on linearized eigenvalue problems for 1-dimensional Allen–Cahn equation II-Asymptotic profiles of eigenfunctions. *J. Differ. Equ.* **261** (2016), 5465–5498.

16 K. Yagasaki. Monotonicity of the period function for $u'' - u + u^p = 0$ with $p \in \mathbb{R}$ and $p > 1$. *J. Differ. Equ.* **255** (2013), 1988–2001.