BULL. AUSTRAL. MATH. SOC. 30C80
VOL. 30 (1984), 457-462. (30F99)

LENGTH AND AREA INEQUALITIES FOR THE DERIVATIVE OF A BOUNDED AND HOLOMORPHIC FUNCTION

SHINJI YAMASHITA

The Schwarz-Pick lemma,

$$\left(1-|z|^2\right)|f'(z)|/\left(1-|f(z)|^2\right) \le 1$$

for f analytic and bounded, $|f| < 1$, in the disk $|z| < 1$, is refined:

$$\left(1-|z|^2\right)|f'(z)|/\left(1-|f(z)|^2\right) \le \Phi(z, r) \le \Psi(z, r) \le 1 ,$$

where $\Phi(z, r)$ is a quantity determined by the non-Euclidean area of the image of

$$D(z, r) = \{w;\ |w| < 1,\ |w-z|/|1-\bar{z}w| < r\} , \quad 0 < r < 1 ,$$

and $\Psi(z, r)$ is that determined by the non-Euclidean length of the image of the boundary of $D(z, r)$. The multiplicities in both images by f are not counted.

1. Introduction

Let f be a function nonconstant, holomorphic, and bounded, $|f| < 1$, in the unit disk $D = \{|z| < 1\}$. Let

$$f^*(z) = |f'(z)|/\left(1-|f(z)|^2\right) , \quad z \in D ,$$

Received 19 July 1984.

and let

$$D(z, r) = \{w \in D;\ |w-z|/|1-\bar{z}w| < r\}\ ,\quad z \in D\ ,\quad 0 < r < 1\ ,$$

be the non-Euclidean disk of the non-Euclidean center z and the non-Euclidean radius $\tanh^{-1}r$. Let $\Delta(z, r) = f(D(z, r))$ be the image of $D(z, r)$, that is, the projection of the Riemannian image of $D(z, r)$ by f. Let $A(z, r, f)$ be the non-Euclidean area of $\Delta(z, r)$, and let $L(z, r, f)$ and $L^{\#}(z, r, f)$ be the non-Euclidean lengths of the boundary $\partial\Delta(z, r)$ and the exact outer boundary $\partial^{\#}\Delta(z, r)$ of $\Delta(z, r)$, respectively. To explain $\partial^{\#}\Delta(z, r)$ we let E be the unbounded complement of the closure of $\Delta(z, r)$ in the complex palne. Then $\partial^{\#}\Delta(z, r)$ is the boundary of $\Delta^{\#}(z, r) = D\backslash E \supset \Delta(z, r)$. Roughly, $\Delta^{\#}(z, r)$ is the "island" $\Delta(z, r)$ plus its reclaimed "lakes" and "bays". Apparently, $L^{\#}(z, r, f) \le L(z, r, f)$ by $\partial^{\#}\Delta(z, r) \subset \partial\Delta(z, r)$.

The Schwarz-Pick lemma

$$(1.1)\qquad \left(1-|z|^2\right)f^*(z) \le 1\ ,\quad z \in D\ ,$$

referred to in the abstract, will be refined in

THEOREM 1. *Let f be a function nonconstant, holomorphic, and bounded, $|f| < 1$, in D. Then, for each $z \in D$, and for each r, $0 < r < 1$,*

$$(1.2)\quad \left(1-|z|^2\right)f^*(z) \le \Phi(A(z, r, f)) \le \Psi\left(L^{\#}(z, r, f)\right) \le \Psi(L(z, r, f)) \le 1\ ,$$

where

$$\Phi(x) = x^{\frac{1}{2}}/\left\{r(x+\pi)^{\frac{1}{2}}\right\}\ ,$$

$$\Psi(x) = \left\{\left(x^2+\pi^2\right)^{\frac{1}{2}}-\pi\right\}/(rx)\ ,\quad 0 < x < +\infty\ .$$

For the sharpness, it is apparent that if f is a conformal homeomorphism from D onto D,

$$(1.3)\qquad f(w) = e^{i\alpha}(w-\beta)/(1-\bar{\beta}w)\ ,$$

where α is a real constant and $\beta \in D$, then all the equalities in (1.2) hold because the left- and the right-most are identical. Conversely, it will be shown that if the last equality in (1.2) holds, $\Psi(L) = 1$, for a certain pair z, r, then f is a conformal homeomorphism of (1.3).

2. Proofs of some parts of Theorem 1

Since Ψ is increasing, the third inequality in (1.2), $\Psi(L^{\#}) \le \Psi(L)$, is obvious by $L^{\#} \le L$. Furthermore, since

$$(2.1)\quad L(z, r, f) \le \int_{\partial D(z,r)} f^*(w)|dw| \le \int_{\partial D(z,r)} \left(1-|w|^2\right)^{-1}|dw| = 2\pi r/\left(1-r^2\right)$$

by (1.1), the fourth inequality $\Psi(L) \le 1$ in (1.2) immediately follows. The second quantity in (2.1) is the length of the Riemannian image of $\partial D(z, r)$.

To prove the second inequality $\Phi(A) \le \Psi\left(L^{\#}\right)$ in (1.2), we let $A^{\#}(z, r, f)$ be the area of the simply connected domain $\Delta^{\#}(z, r)$. Since the Gauss curvature of the non-Euclidean space D endowed with the metric in the differential form $\left(1-|w|^2\right)^{-1}|dw|$ is the constant -4, the isoperimetric inequality [3, Theorem 4.3, (4.25), p. 1206] reads

$$(2.2)\qquad \lambda^2 \ge 4\pi\sigma + 4\sigma^2 ,$$

where σ is the non-Euclidean area of a simply connected domain in D and λ is the non-Euclidean length of its boundary. Applying (2.2) to $\Delta^{\#}(z, r)$ we obtain

$$(2.3)\quad L^{\#}(z, r, f)^2 \ge 4\pi A^{\#}(z, r, f) + 4A^{\#}(z, r, f)^2 \ge 4\pi A(z, r, f) + 4A(z, r, f)^2 ,$$

from which we have $\Phi(A) \le \Psi\left(L^{\#}\right)$.

3. An area theorem

To complete the proof of Theorem 1 use is made of

THEOREM 2. *Let f be a function nonconstant, holomorphic, and bounded, $|f| < 1$, in D. Then the function $\Phi(A(0, r, f))$ of r, $0 < r < 1$, is nondecreasing.*

https://doi.org/10.1017/S0004972700002173 Published online by Cambridge University Press

Proof. The proof is based on a version of Dufresnoy's idea [2]. We first find a simply connected domain $G(r)$ in the disk $D(0, r)$, where f is univalent and

$$(3.1) \qquad A(r) \equiv A(0, r, f) = \iint_{G(r)} f^*(w)^2 du dv \quad (w = u + iv) .$$

The projections of all the branch points of the Riemannian image of $D(0, r)$ by f are a finite number of distinct points, $a_1, \ldots, a_n$ in $\Delta(r) = \Delta(0, r)$. First we find a finite number of analytic cross-cuts and analytic end-cuts of $\Delta(r)$ [1, p. 168], $\gamma_1, \ldots, \gamma_k$, such that

$\Delta_1(r) = \Delta(r) \setminus \bigcup_{j=1}^{k} \gamma_k$ is simply connected and $a_l \notin \bigcup_{j=1}^{k} \gamma_k$ for $1 \le l \le n$. Then find an analytic end-cut γ_0 of $\Delta_1(r)$ on which all the points a_l, $1 \le l \le n$, lie. See Figure 1. Let $G(r)$ be one of the preimages of $\Delta_1(r) \setminus \gamma_0$ by f. Since the area of $\Delta(r)$ is the same as that of $\Delta_1(r) \setminus \gamma_0$ we have (3.1).

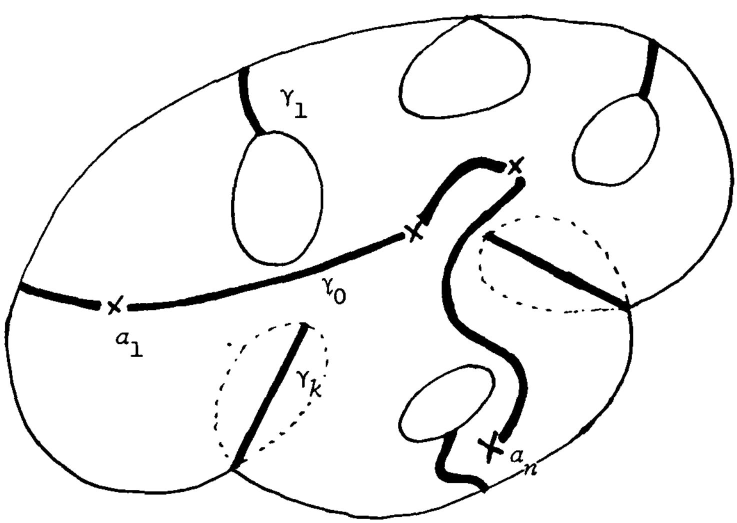

Figure 1

https://doi.org/10.1017/S0004972700002173 Published online by Cambridge University Press

It now follows from

$$A(r) = \int_0^r t\,dt \int_{\Gamma(r,t)} f^*\left(te^{i\theta}\right)^2 d\theta ,$$

where

$$\Gamma(r, t) = \left\{\theta;\ 0 \le \theta \le 2\pi,\ te^{i\theta} \in \overline{G(r)}\right\} ,$$

that

$$(3.2) \qquad dA(r)/dr = r \int_{\Gamma(r,r)} f^*\left(re^{i\theta}\right)^2 d\theta .$$

Furthermore, the length $L(r) = L(0, r, f)$ is given by

$$(3.3) \qquad L(r) = r \int_{\Gamma(r,r)} f^*\left(re^{i\theta}\right) d\theta .$$

It then follows from the Schwarz inequality

$$\left\{\int_{\Gamma(r,r)} f^*\left(re^{i\theta}\right) d\theta\right\}^2 \le \int_{\Gamma(r,r)} f^*\left(re^{i\theta}\right)^2 d\theta \int_{\Gamma(r,r)} d\theta ,$$

with

$$\int_{\Gamma(r,r)} d\theta \le 2\pi ,$$

(3.2), and (3.3), that

$$(3.4) \qquad L(r)^2 \le 2\pi r\,dA(r)/dr .$$

On the other hand, it follows from (2.3) for $z = 0$ that

$$(3.5) \qquad L(r)^2 \ge L^{\#}(0, r, f)^2 \ge 4\pi A(r) + 4A(r)^2 .$$

Combining (3.4) with (3.5) we have

$$(3.6) \qquad (2/r)dr \le A(r)^{-1}dA(r) - \left(A(r)+\pi\right)^{-1}dA(r) , \quad 0 < r < 1 .$$

On integrating (3.6) from r_1 to r_2, $0 < r_1 \le r_2 < 1$, we observe, after a short computation, that $\Phi\left(A\left(0, r_1, f\right)\right) \le \Phi\left(A\left(0, r_2, f\right)\right)$.

4. Completion of the proof of Theorem 1

For the proof of the first inequality in (1.2) we set

https://doi.org/10.1017/S0004972700002173 Published online by Cambridge University Press

$$g(w) = f\{(w+z)/(1+\bar{z}w)\} , \quad w \in D , \tag{4.1}$$

so that $g^*(0) = (1-|z|^2)f^*(z)$. Since

$$\lim_{\delta\to+0} \Phi(A(0, \delta, g)) = g^*(0) \text{ and } A(0, r, g) = A(z, r, f) ,$$

Theorem 2 now yields the desired conclusion.

It remains to show that if $\Psi(L) = 1$ in (1.2), then f must be of the form (1.3). We may suppose that $z = 0$; otherwise, we examine g of (4.1). Consider the part

$$P = \{re^{i\theta}; \theta \in \Gamma(r, r)\} = \overline{G(r)} \cap \partial D(0, r)$$

of $\partial D(0, r)$. Then

$$\int_P f^*(w)|dw| = L(0, r, f) = 2\pi r/(1-r^2) \geq \int_P (1-|w|^2)^{-1}|dw| .$$

This, combined with (1.1), shows that $f^*(w) = (1-|w|^2)^{-1}$ at each point of P . Thus f must be a conformal homeomorphism of D onto itself.

References

[1] E.F. Collingwood and A.J. Lohwater, *The theory of cluster sets* (Cambridge University Press, Cambridge, 1966).

[2] J. Dufresnoy, "Sur l'aire sphérique décrite par les valeurs d'une fonction méromorphe", *Bull. Sci. Math.* 65 (1941), 214-219.

[3] R. Osserman, "The isoperimetric inequality", *Bull. Amer. Math. Soc.* 84 (1978), 1182-1238.

Department of Mathematics,
Tokyo Metropolitan University,
Fukasawa,
Setagaya,
Tokyo 158,
Japan.