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Non-extendable Zero Sets of Harmonic and Holomorphic Functions

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Abstract. In this paper we study the zero sets of harmonic functions on open sets in $\mathbb{R}^N$ and holomorphic functions on open sets in $\mathbb{C}^N$. We show that the non-extendability of such zero sets is a generic phenomenon.

Recall that a subset Y of a topological space X is said to be *residual* (in X) if X is of second Baire category and $X \setminus Y$ is of first Baire category; *i.e.*, it can be written as a countable union of nowhere dense subsets of X. In particular, if X is of second category and Y is a dense G_δ subset of X, then Y is residual in X.

We will show that the zero sets of most (in the sense of Baire category) harmonic functions are not extendable near every boundary point. Also, we shall consider the analogous situation for holomorphic functions on domains in $\mathbb{C}^N$. These results relate well to the work of other authors on the genericity and non-extendability of universal functions. Bernal-González and Ordóñez Cabrera [1] among others have made contributions that touch on the results here. Many of these authors have also considered topics like "lineability" of properties considered in this paper. We might study this in the future.

1 The Harmonic Case

We begin with an example that motivated our results.

Example 1.1 Consider a bounded open set D in $\mathbb{R}^N$. We show that there exists a harmonic function u on D with the property that there is no harmonic function on any open set G containing $\overline{D}$, whose zeros on D are the same as those of u. Let F be a closed subset of D consisting of the union of pairwise disjoint closed segments $[a_k, b_k]$, $k = 1, 2, \dots$, in D, whose respective lengths tend to zero, form a locally finite family in D, and accumulate at each point of ∂D. We note that for this set $F \subset \Omega$, the hypotheses of [2, Theorem 3.19] are satisfied. Define a continuous function f on F by mapping each segment $[a_k, b_k]$ to the interval $[-1, +1] \subset \mathbb{R}$. Let u be a harmonic function on D, which approximates f within $1/2$ on F. Let p be an arbitrary point of ∂D and $\varepsilon > 0$. We may choose a segment $[a_k, b_k]$ within ε distance of p. There is a point $c \in [a_k, b_k]$ such that $u(c) = 0$. Thus, u is a non-constant harmonic function in D, whose zeros accumulate at every point of ∂D. Now, if G is an open set containing $\overline{D}$ and v is a harmonic function on G whose zeros on D coincide with those of u, then

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v vanishes at each point of ∂D. By the Maximum Principle, v must vanish identically in D. This contradicts the assumption that the zeros of v on D are the same as those of u.

In this section we will show that the non-extendability of zero sets of harmonic functions is a generic phenomenon. To formulate our results we introduce some notation. For a function g, defined on a set E, let $\|g\|_E := \sup\{|g(x)| : x \in E\}$ denote its supremum norm on E. Further, we denote by $Z(g|U)$ the set of zeros of the function g on a set $U \subset E$. Finally, for an open set $\Omega \subset \mathbb{R}^N$, we denote by $H(\Omega)$ the space of harmonic functions on Ω endowed with the topology of local uniform convergence.

Definition 1.2 Let Ω be a proper domain in $\mathbb{R}^N$ $(N \geq 2)$. We say that a function $u \in H(\Omega)$ belongs in the class $\mathcal{N}(\Omega)$ of *hypernull functions* on Ω if it satisfies the following property:

> For every $p \in \partial\Omega$, for every ball B_p centred at p, and for every component U of $\Omega \cap B_p$, there are no functions v_p real analytic in B_p and not identically 0 such that $Z(u|U) \subset Z(v_p|U)$.

Theorem 1.3 *Let Ω be a proper domain in $\mathbb{R}^N$. Then $\mathcal{N}(\Omega)$ is a dense G_δ subset of the space $H(\Omega)$.*

For the purposes of the proof we introduce the following notion. By an *horoball* in an open set Ω, we understand a pair (A, q), where A is an open ball in Ω such that $\{q\} = \partial\Omega \cap \partial A$.

The following remark is borrowed from a paper we are currently writing with Myrto Manolaki.

Remark 1.4 Let B_p be a ball centered at a point $p \in \mathbb{R}^N$ and let C be an open cone with vertex at p. Then there is a sequence x_n tending to p in $C \cap B_p$, such that if u is analytic in B_p and vanishes on this sequence, then $u = 0$.

Proof We may assume that $p = 0$, B_0 is the unit ball B and $C = r\theta : \theta \in U, 0 < r < 1$, where U is an open subset of the unit sphere. Let $\theta_1, \theta_2, \ldots,$ be a countable dense subset of U, and consider the countable set of radial segments $S_1, S_2, \ldots,$ emanating from 0, where $S_j = \{r\theta_j : 0 < r < 1/j\}$, $j = 1, 2, \ldots$. Now, let $\{x_n\}$ be a sequence tending to 0 on the union of the S_j, such that for each j, there is a subsequence on S_j. If u is analytic in B and vanishes on this sequence, then $u = 0$ on the segment $L_j \cap B$, where L_j is the line on which S_j lies, since the zeros have an accumulation point, namely 0. Since u is continuous and zero on a dense subset of the cone C, it is zero on all of C, but since C is an open subset of B, it follows that $u = 0$ on all of B. ∎

Lemma 1.5 *For each proper open subset $\Omega \subset \mathbb{R}^N$, there is a countable collection of horoballs (A_k, q_k) in Ω, such that for each $p \in \partial\Omega$, for each ball B_p centred at p and for each component U_p of $\Omega \cap B_p$, we have $A_k \subset U_p$ for some k and $q_k \in B_p$.*

Proof Let X be a countable dense subset of $\partial\Omega$. For every fixed $x \in X$ and $j \in \mathbb{N}$, let $\{U_{x,j,i} : i \in I_{x,j}\}$, where $I_{x,j} \subset \mathbb{N}$ is the set of connected components of $\Omega \cap B(x, 1/j)$. For each component $U_{x,j,i}$, choose a point $y \in \partial U_{x,j,i} \cap B(x, 1/j)$ and let V be a ball in $U_{x,j,i}$, whose closure is closer to y than to the boundary $S(x, 1/j)$ of $B(x, 1/j)$. Let S_y be the segment from the center of V to the point y. We displace the ball V by moving its center along the segment S_y until V first meets a boundary point q of $U_{x,j,i}$. By construction, $q \in \partial V \cap \partial\Omega$, but it may not be the only such point. We may choose an horoball (A, q), by taking A to be an open ball in V, such that $q = \partial A \cap \partial V$. With each x, j, i we have associated an horoball (A, q). This gives a countable family of horoballs $(A_{x,j,i}, q_{x,j,i})$, We may arrange these in a sequence (A_k, q_k), where the q_k may not be all distinct. We thus obtain a countable family of horoballs, whose boundary points q_k are dense on $\partial\Omega$.

Let $p \in \partial\Omega$, B_p be a ball centred at p and U_p a component of $\Omega \cap B_p$. Choose $q \in \partial U_p \cap B_p$ and $j \in \mathbb{N}$ such that the ball $B_{q,j}$ of centre q and radius $1/j$ is contained in B_p. Let $U_{q,j,i}$ be a component of $\Omega \cap B_{q,j}$ that meets U_p. By construction, one of the horoballs (A_k, q_k) corresponds to this $U_{q,j,i}$. Thus, $A_k \subset U_{q,j,i} \subset U_p$. This concludes the proof. ■

Finally, we will make use of the following fact.

Harmonic Hurwitz Theorem *Let Ω be a domain in $\mathbb{R}^N$ and suppose u_n is a sequence of zero-free harmonic functions in Ω converging locally uniformly in Ω to a function u. Then either u is a zero-free harmonic function or $u \equiv 0$.*

The proof of this fact is straightforward. We can assume that $u_n > 0$. Hence, $u \geq 0$. Suppose $u(z_0) = 0$. Then, u assumes its minimum and so $u \equiv 0$.

Now we have all the tools to prove the main theorem of this section.

Proof of Theorem 1.3

Step 1. First, we show that $\mathcal{N}(\Omega)$ is non-empty.

Suppose first that $\partial\Omega$ consists of finitely many points $p_1, \ldots, p_m$. Let $r_1 > r_2 > \cdots$ with $r_j \searrow 0$, and for each $k \in \{1, 2, \ldots, m\}$, denote by $B_{k,j}$ the ball of radius r_j centred at p_k. Choose r_1 so small that the closed balls $\overline{B}_{k,1}$ are disjoint and contained in Ω, except for their respective centres p_k. For each k, let K_k be an open cone, with vertex p_k. We may form the spherical caps $C_{k,j} := K_k \cap \partial B_{k,j}$. Let F be the relatively closed subset of Ω formed by the union of the caps $C_{k,j}$. We define a continuous function ϕ on F, by setting $\phi = (-1)^j$ on $C_{k,j}$, for each j. By [2, Theorem 3.19], there exists a harmonic function u on Ω such that $|u - \phi| < 1/2$ on F. On each ray $R \subset K_k$, the function u has a sequence of zeros converging to p_k. Suppose $\widetilde{u}$ is real analytic in an open ball B_k, centred at p_k. If the zeros of $\widetilde{u}$ contain those of u in $\Omega \cap B_k$, then $\widetilde{u} = 0$ on a sequence of points tending to p_k on $R \cap B_k$. Thus, $\widetilde{u} = 0$ on $R \cap B_k$. Since this is true for every $R \subset K_k$, we have that $\widetilde{u} = 0$ on $K_k \cap B_k$. Consequently, $\widetilde{u} = 0$ on B_k. This completes the proof in case $\partial\Omega$ is a finite set.

Now suppose that $\partial\Omega$ is infinite. We claim that to prove a function u belongs in $\mathcal{N}(\Omega)$ it is sufficient to show the following. For every q in a dense subset Q of $\partial\Omega$, for every ball B_q centred at q, and for every component U of $\Omega \cap B_q$, there are no functions v_q real analytic in B_q and not identically 0 such that

$$Z(u \,|\, U) \subset Z(v_p|U).$$

To see this, suppose that the above holds for a dense subset Q of $\partial\Omega$. Let $p \in \partial\Omega$. Suppose there is a ball B_p centred at p, a component U of $\Omega \cap B_p$, and a function v_p real analytic on B_p such that $Z(v_p|B_p) \supset Z(u|U)$. Choose $q \in Q \cap \partial U \cap B_p$ and a ball $B_q \subset B_p$ centred at q. Let $v_q := v_p|B_q$ and let U_q be any component of $U \cap B_q$. Then, v_q is a real analytic function on B_q for which $Z(v_q|(U_q \cap B_q)) \supset Z(u|(U_q \cap B_q))$. From the definition of Q, it follows that $v_q = 0$ on B_q and consequently that $v_p = 0$ on B_p. This establishes the claim.

By Lemma 1.5, we may choose a dense sequence of points $q_1, q_2, \dots$ on $\partial\Omega$ with the property that for each $p \in \partial\Omega$, for each ball B centred at p, and for each component U of $\Omega \cap U$, there is an open ball A_k contained in U for which $\partial A_k \cap \partial\Omega = \{q_k\}$.

In general, suppose we have an open ball A whose closure is contained in Ω except for one point $q \in \partial A \cap \partial\Omega$. Let B_q be a ball centred at q and of radius r less than that of A. Then the closed spherical cap $C = \overline{A} \cap \partial B_q$ is non empty. Let K be the cone with vertex q generated by the cap C. Let $r = r_1 > r_2 > \cdots$ with $r_j \searrow 0$. We may form the spherical caps $C_j = K \cap \partial B_j$, where B_j is the ball of radius r_j centred at q.

We apply this procedure for each q_k, to form a sequence $C_{k,j}$ of corresponding spherical caps converging to q_k, but we must do this carefully. First, we choose $C_{1,1}$. Next, we choose $C_{1,2}$ and $C_{2,1}$, making sure that $C_{2,1}$ is disjoint from $C_{1,1}$ and $C_{1,2}$. The general procedure is as follows. Consider the infinite matrix

$$\big\{ (k,j) : k = 1,2,\dots,\ j = 1,2,\dots \big\}.$$

Let $D_\ell := \{(m, \ell - m + 1) : m = 1, 2. \dots, \ell\}$ be the entries of this matrix along the ℓ-th anti-diagonal, that is along the segment starting at $(1,\ell)$ and running southwest in a straight line to $(\ell, 1)$. We choose the caps successively in $D_1, D_2, \dots, D_n, \dots$. At each step, for an entry (k, j) in some D_n, we choose a cap $C_{k,j}$ sufficiently close to q_k that it is disjoint from the previously constructed caps.

The union of these caps is a relatively closed subset E of Ω that satisfies the hypotheses of [2, Theorem 3.19]. Define a continuous function ϕ on E by setting $\phi = (-1)^j$ on $C_{k,j}$ for each k and j. There exists a harmonic function u on Ω, such that $|u - \phi| < 1/2$ on E. Fix q_k, a ball B centred at q_k and a component U of $\Omega \cap B$. There is an horoball (A_k, q_k) with $A_k \subset U$. The argument used for the case that $\partial\Omega$ was finite shows that u has the desired property for each q_k. Since these points are dense in $\partial\Omega$, this concludes the proof of Step 1.

Step 2. Next, we show that the family of functions $u \in \mathcal{N}(\Omega)$ is dense in $H(\Omega)$.

Let h be a function in $H(\Omega)$, let K be a compact subset of Ω, and let ε be a positive number. Denote by $\widehat{K}$ the Ω-hull of K, that is, the union of K and all bounded components of $\mathbb{R}^N \setminus K$ that are relatively compact in Ω. Then $\widehat{K}$ is also a compact subset of Ω. In Step 1, we can drop finitely many caps, and so we can assume that all caps are disjoint from $\widehat{K}$. Since $\widehat{K} \cup E$ satisfies the hypotheses of [2, Theorem 3.19], instead of merely approximating ϕ on E, we can simultaneously approximate h on $\widehat{K}$. We thus obtain a function $u \in H(\Omega)$, which, not only has the desired behaviour on E, but also approximates h within ε on $\widehat{K}$, and *a fortiori* on K. This concludes the proof of Step 2.

Step 3. We now prove that the family X_p of functions in $H(\Omega)$ that fail to have the property of non-extendability at a particular boundary point p is of first Baire category.

Fix a compact ball $K \subset \Omega$. If $u \in X_p$, then certainly $u \not\equiv 0$ so

$$\max\{|u(x)| : x \in K\} > 0.$$

Also, there exists a ball B_p centred at p, a component U of $\Omega \cap B_p$ and a function v_p real analytic and not identically 0 on B_p such that $Z(u|U) \subset Z(v_p|B_p)$. We can consider $\mathbb{R}^N$ as the real part of $\mathbb{C}^N = \mathbb{R}^N + i\mathbb{R}^N$. In a neighborhood of $p = p + i0$, the function v_p extends to a holomorphic function $\widetilde{v}_p$. By choosing B_p smaller, we can assume that $\widetilde{v}_p$ is bounded on the ball $\widetilde{B}_p$, centered at $p + i0$ in $\mathbb{C}^N = \mathbb{R}^N + i\mathbb{R}^N$ and having the same radius as B_p. Multiplying by a small positive number, we may assume that $|\widetilde{v}_p| \le 1$. Since $v_p \not\equiv 0$, it follows that $\|v_p\|_S > 0$, where S is the sphere centred at p whose radius is half that of B_p. Let j and k be positive integers. Denote by B_j the ball centred at p and of radius $1/j$ and $S(j)$ the sphere of centre p and radius half that of B_j.

Let $U_{i,j}$, $i = 1, 2, \dots$, be the components of $\Omega \cap B_j$ and denote by $H_{i,j}(u)$ the family of functions v analytic in the ball B_j, which are respectively restrictions of holomorphic functions $\widetilde{v}$ bounded by 1 on $\widetilde{B}_j$, for which $Z(u|U_{i,j}) \subset Z(v|B_j)$. Let

$$X_{i,j,k} = \left\{ u \in H(\Omega) : \|u\|_K \ge 1/k, \text{ there is } v \in H_{i,j}(u), \|v\|_{S(j)} \ge 1/k \right\}.$$

Then

$$X_p \subset \bigcup_{i,j,k} X_{i,j,k},$$

and we will show that each $X_{i,j,k}$ is closed and nowhere dense in $H(\Omega)$.

To show that $X_{i,j,k}$ is closed, suppose $u_1, u_2, \dots$ are in $X_{i,j,k}$ and $u_n \to u$ locally uniformly on Ω. Then $\|u\|_K \ge 1/k$ and so $u \not\equiv 0$. For each u_n let v_n be a function associated with u_n by the definition of $X_{i,j,k}$; that is,

$$v_n \in H_{i,j}(u_n) \quad \text{and} \quad \|v_n\|_{S(j)} \ge 1/k.$$

Also, let $\widetilde{v}_n$ denote the holomorphic extension of v_n on the complex ball $\widetilde{B}_j$. Since $\{\widetilde{v}_n\}$ is a normal family, we may assume that $v_n \to v$, where v is an analytic function on B_j, which is the restriction of a function $\widetilde{v}$ holomorphic on $\widetilde{B}_j$.

Clearly $|v| \le 1$ and $\max|v| \ge 1/k$ on $S(j)$. Thus, $v \not\equiv 0$ on $U_{i,j}$. Let $q \in U_{i,j}$ be a point where $v(q) \ne 0$. Let B_q be a closed ball centred at q in $U_{i,j}$ on which v is zero free. Then, v_n is zero free on B_q for large n, and consequently u_n is also zero free on B_q. By the harmonic version of Hurwitz Theorem, either $u \equiv 0$ on B_q or u is zero-free on B_q. Since $u \not\equiv 0$ on Ω, it follows that u is zero-free on B_q. In particular, $u(q) \ne 0$. We have shown that

$$Z(u|U_{i,j}) \subset Z(v|U_{i,j}).$$

Thus, $v \in H_{i,j}(u)$ and $X_{i,j,k}$ is closed.

Next, we claim that the sets $X_{i,j,k}$ are nowhere dense. Indeed, suppose we are given a function $h \in H(\Omega)$, a compact set $K \subset \Omega$, and $\varepsilon > 0$. In Step 2, we showed that there exists a function $u \in H(\Omega)$ having the property that for each $p \in \partial\Omega$, for each ball B_p centred at p, and for each component U of $\Omega \cap B$, there is no non-constant analytic function v_p on B_p whose zeros contain those of u on U. Hence, $u \notin X_p$. In constructing this function u we were able moreover to ensure that $|u - h| < \varepsilon$ on K. Thus, $H(\Omega) \setminus X_p$ is dense, and consequently each $X_{j,k}$ is a nowhere dense set. It follows that X_p is of first Baire category in $H(\Omega)$. Since $H(\Omega)$ is a complete metric space, the subset X_p is residual. This concludes the proof of Step 3.

Since a countable union of sets of category one is still of category one, for every countable subset Q of $\partial\Omega$, the family of functions in $H(\Omega)$, that fail to satisfy the conclusion of the theorem for some $q \in Q$, is still of first category. Let Q be a dense countable subset of $\partial\Omega$. Then the family X of functions in $H(\Omega)$ that satisfy the conclusion of the theorem with respect to each point in Q is residual, that is its complement is of first category. But, as in Step 1, if a function u has the desired property for a dense subset of the boundary, then it has it for every point of the boundary. This concludes the proof. ∎

2 The Holomorphic Case

For a complex manifold X and an open set $U \subset X$, denote by $\mathcal{O}(U)$ the family of functions holomorphic on U. For a compact set $K \subset X$, we denote $\mathcal{O}(K)$ the family of functions f on K, which are holomorphic on some open neighbourhood U of K (depending on f) .

The following lemma on simultaneous approximation and interpolation is a particular case of [3, Theorem 3.1].

Lemma 2.1 *Let X be a Stein manifold and let $K \subset X$ be a compact set that is holomorphically convex. If $B = \{b_i\}_{i=1}^{\infty}$ is a discrete sequence of points in X with $B \subset X \setminus K$, and if $\{w_i\}_{i=1}^{\infty}$ is a sequence in $\mathbb{C}$, then for every $f \in \mathcal{O}(K)$ and every $\varepsilon > 0$, there exists a function $g \in \mathcal{O}(X)$ such that*

(i) *$|g(x) - f(x)| < \varepsilon$ for all $x \in K$, and*
(ii) *$g(b_i) = w_i$ for all $i \in \mathbb{N}$.*

We also need the following Hurwitz type lemma.

Lemma 2.2 *If a sequence g_n of zero-free holomorphic functions on a domain $\Omega \subset \mathbb{C}^N$ converges locally uniformly to a function g, then g is either zero-free or identically zero.*

Proof Let g_n be a a sequence of zero-free holomorphic functions on Ω that converges locally uniformly on Ω to a function g and suppose that $g(p) = 0$ for some $p \in \Omega$. If B is a ball in Ω centred at p, then we can apply the one-variable Hurwitz theorem, for every complex line ℓ through p, to conclude that the function g is identically zero on $\ell \cap B$. Thus, $g \equiv 0$ on B and consequently on Ω. ∎

To state our result for the case of $\mathbb{C}^N$, we first generalize the notion of hypernull functions in the natural way.

Definition 2.3 A holomorphic function f on a domain Ω of $\mathbb{C}^N$ is called *hypernull on* Ω if it has the property that, for every $p \in \partial\Omega$ and for every ball B_p in $\mathbb{C}^N$ centred at p, if g_p is a function holomorphic in B_p and, for some component U_p of $\Omega \cap B_p$, we have $Z(f|U_p) \subset Z(g_p|U_p)$, then $g_p = 0$.

Remark 2.4 We have shown that for each domain $\Omega \subset \mathbb{R}^N$ there exist nonconstant harmonic hypernull functions on Ω, and so harmonic functions on Ω that cannot be extended harmonically to any larger domain. This is no longer true for holomorphic functions on domains of $\mathbb{C}^N$ for $N \geq 2$, as a consequence of the Hartogs Lemma. For this reason, in the statement of the following theorem, it is essential to restrict our attention to domains of holomorphy.

Theorem 2.5 *Let Ω be a domain of holomorphy in $\mathbb{C}^N$. Then the set of hypernull holomorphic functions on Ω is a dense G_δ subset of the space $\mathcal{O}(\Omega)$ of holomorphic functions on Ω, endowed with the topology of local uniform convergence.*

Proof If $\Omega = \mathbb{C}^N$, the theorem is trivial, since there is nothing to prove. Suppose $\Omega \neq \mathbb{C}^N$. The proof is similar to that of the harmonic case, except that we will replace caps by points.

By Lemma 1.5 there is a countable collection of balls $A_k \subset \Omega$, such that for each $p \in \partial\Omega$, for each ball B_p centred at p and for each component U_p of $\Omega \cap B_p$, we have that $A_k \subset U_p$, for some k, and there is a point q_k in $B_p \cap \partial\Omega \cap \partial A_k$.

We claim that a function $f \in \mathcal{O}(\Omega)$ satisfies the required conclusion at each point $p \in \partial\Omega$ if it does so at each q_k. To see this, let f satisfy the required property at each q_k and let p be an arbitrary point of $\partial\Omega$. Suppose we have a ball B_p centred at p and g_p a function holomorphic in B_p and, for some component U_p of $\Omega \cap B_p$, we have $Z(f|U_p) \subset Z(g_p)$. Choose $A_k \subset U_p$ with $q_k \in \partial\Omega \cap \partial A_k$. Let B_k be a ball centred at q_k and contained in B_p and let U_k be the component of $\Omega \cap B_k$ which meets A_k. Then $U_k \subset U_p$ and so $Z(f|U_k) \subset Z(f|U_p) \subset Z(g_p)$. Since g_p is holomorphic in B_k, it follows that $g_p = 0$ on B_k. Consequently, $g_p = 0$ on B_p, which confirms the claim.

For each k, it follows from the definition of A_k and q_k that we may construct a sequence $b_{k,j}$ as in Remark 1.4 that converges to q_k in Ω as $j \to \infty$. By a diagonal process, we may construct a sequence b_ℓ of distinct points in Ω that is eventually outside of every compact subset of Ω and that, for every q_k, contains such a subsequence that tends to q_k.

Given a function $h \in \mathcal{O}(\Omega)$, an $\mathcal{O}(\Omega)$-convex compact set $K \subset \Omega$ and $\varepsilon > 0$, we may assume that the sequence b_ℓ is disjoint from K. There is a function $f \in \mathcal{O}(\Omega)$ such that $|f - h| < \varepsilon$ on K and $f(b_\ell) = 0$, for each ℓ. This follows from Lemma 2.1, since domains of holomorphy are Stein manifolds. The function f has the properties required in the theorem. We have shown that the functions satisfying the required properties form a dense subfamily of $\mathcal{O}(\Omega)$.

We shall now show that most functions in $\mathcal{O}(\Omega)$ have the required properties. Namely, we shall show that the exceptional functions form a family of first Baire category and since the space $\mathcal{O}(\Omega)$ is of second Baire category, functions satisfying the conclusion will be generic in the sense of Baire category. Since we have shown that the family of functions that fail to satisfy the property is the same as the family of functions which, for some k, fail to satisfy the property at q_k, and since a countable union of first category sets is still of first category, it suffices to fix a boundary point p, and show that the family X_p of functions that fail to have the property of the conjecture, for this particular boundary point p is of first Baire category. Fix a compact ball $K \subset \Omega$. If $f \in X_p$, then certainly $f \not\equiv 0$, so $\max_K |f| > 0$. Also, there exists a ball B_p centred at p and a function g_p holomorphic and not identically 0 on B_p such that $Z(f|U_p) \subset Z(g_p)$ for some component U_p of $\Omega \cap B_p$. Since the same is true for every smaller ball, we can assume that g_p is bounded on the ball B_p. Multiplying by a small positive number, we can assume that $|g_p| \le 1$. Since $g_p \not\equiv 0$, it follows that $\max_S |g_p| > 0$, where S is the sphere centred at p whose radius is half that of B_p.

Denote by B_j the ball centred at p and of radius $1/j$ and $S(j)$ the sphere of centre p and radius half that of B_j. Let U_i, $i = 1, 2, \dots$, be the components of $\Omega \cap B_j$. Denote by $\mathcal{O}_{i,j}(f)$ the family of functions g holomorphic in the ball B_j, bounded by 1 on B_j, for which $Z(f|U_i) \subset Z(g)$. Set

$$X_{i,j,k} = \{f \in \mathcal{O}(\Omega) : \max_K |f| \ge 1/k, \exists g \in \mathcal{O}_{i,j}(u), \max_{S(j)} |g| \ge 1/k\}.$$

Then

$$X_p \subset \bigcup_{i,j,k} X_{i,j,k}$$

and we will show that each $X_{i,j,k}$ is closed and nowhere dense in $\mathcal{O}(\Omega)$.

To show that $X_{i,j,k}$ is closed, suppose $f_1, f_2, \dots$ are in $X_{i,j,k}$ and $f_n \to f$. Then $\max_K |f| \ge 1/k$ and so $f \not\equiv 0$. For each f_n, let g_n be a function associated to f_n by the definition of $X_{i,j,k}$. Since $\{g_n\}$ is a normal family, we may assume that $g_n \to g$, where g is a holomorphic function on B_j. Clearly, $|g| \le 1$ and $\max |g| \ge 1/k$ on $S(j)$. Thus, $g \not\equiv 0$. Let $q \in U_j$ be a point where $g(q) \ne 0$. Let Q be a compact ball centred at q in U_i on which g is zero free. Then, g_n is zero free on Q for large n, and consequently, f_n is also zero free on Q. By Lemma 2.2, either $f \equiv 0$ on Q or f is zero-free on Q. Since $f \not\equiv 0$ on Ω, it follows that f is zero-free on Q, and in particular $f(q) \ne 0$. We have shown that

$$Z(f|U_i) \subset Z(g).$$

Thus, $g \in \mathcal{O}_{i,j}(u)$ and $X_{i,j,k}$ is closed.

Finally, we claim that each closed set $X_{i,j,k}$ is nowhere dense. This is equivalent to showing that its complement is dense. But its complement contains all functions satisfying the property of the theorem and we have shown that the latter is dense. ■

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