BULL. AUSTRAL. MATH. SOC. 17B63, 16W20
VOL. 71 (2005) [505–516]

THE GROUP OF POISSON AUTOMORPHISMS OF POISSON SYMPLECTIC SPACE

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The group of Poisson automorphisms of the coordinate ring of Poisson symplectic $2n$-space is isomorphic to the algebraic torus $(\mathbf{k}^*)^{n+1}$ and it confirms that the algebra constructed by K.L. Horton (2003) is a quantisation of the coordinate ring of Poisson symplectic $2n$-space.

INTRODUCTION

In [**6**], Horton constructed a class of algebras $K_{n,\Gamma}^{P,Q}$ which includes the multiparameter quantised coordinate rings of symplectic and Euclidean $2n$-spaces, the graded quantised Weyl algebra, the quantised Heisenberg space, and is similar to a class of iterated skew polynomial rings constructed by Gómez-Torrecillas and Kaoutit in [**4**]. As a Poisson case, in [**9**], the first author constructed a class of Poisson algebras $A_{n,\Gamma}^{P,Q}$ which includes the coordinate rings of Poisson symplectic and Euclidean $2n$-spaces and whose quantisation is $K_{n,\Gamma'}^{P',Q'}$ for suitable Γ', P' and Q'. Moreover Gómez-Torrecillas and Kaoutit proved in [**5**] that the group of automorphisms of the coordinate ring of quantum symplectic $2n$-space is isomorphic to the algebraic torus $(\mathbf{k}^*)^{n+1}$. The main purpose of this paper is to find the group H of Poisson automorphisms of the coordinate ring of Poisson symplectic $2n$-space which is isomorphic to the algebraic torus $(\mathbf{k}^*)^{n+1}$. This confirms that $K_{n,\Gamma}^{P,Q}$ is a quantisation of $A_{n,\Gamma}^{P,Q}$.

This paper consists of three sections. In the first section, we review several elementary but basic and important properties which are used in the next sections. In the second section, we define H-actions on $A_{n,\Gamma}^{P,Q}$ which act as Poisson automorphisms and prove that every H-prime Poisson ideal is generated by an admissible set. In the final section, we prove that the group of Poisson automorphisms of the coordinate ring of Poisson symplectic $2n$-space is just H defined in the second section.

Received 10th March, 2005
The first author is supported by the Korea Research Foundation Grant, KRF-2002-015-CP0010.

1. The Poisson algebra $A_{n,\Gamma}^{P,Q}$

1.1 Let $\Gamma = (\gamma_{ij})$ be a skew-symmetric $n \times n$-matrix with entries in $\mathbf{k}$, that is, $\gamma_{ij} = -\gamma_{ji}$ for all $i, j = 1, \dots, n$. Let $P = (p_1, p_2, \dots, p_n)$ and $Q = (q_1, q_2, \dots, q_n)$ be elements of $\mathbf{k}^n$ such that $p_i \neq q_i$ for each $i = 1, \dots, n$. Then, by [**9**, 1.2], the polynomial ring $\mathbf{k}[y_1, x_1, \dots, y_n, x_n]$ has the following Poisson bracket:

$$
(1) \qquad
\begin{aligned}
\{y_i, y_j\} &= \gamma_{ij} y_i y_j && (\text{all } i, j)\\
\{x_i, y_j\} &= (p_j - \gamma_{ij}) y_j x_i && (i < j)\\
\{y_i, x_j\} &= -(q_i + \gamma_{ij}) y_i x_j && (i < j)\\
\{x_i, x_j\} &= (q_i - p_j + \gamma_{ij}) x_i x_j && (i < j)\\
\{x_i, y_i\} &= q_i y_i x_i + \sum_{k=1}^{i-1} (q_k - p_k) y_k x_k && (\text{all } i)
\end{aligned}
$$

The Poisson algebra $\mathbf{k}[y_1, x_1, \dots, y_n, x_n]$ is denoted by $A_{n,\Gamma}^{P,Q}$ or by A_n unless any confusion arises.

One should observe that the class of multi-parameter algebra $K_{n,\Gamma'}^{P',Q'}$ constructed by Horton in [**6**] is a quantisation of the Poisson algebra $A_{n,\Gamma}^{P,Q}$, where P', Q' and Γ' are multiplicative forms for P, Q and Γ, respevtively.

1.2. In A_n, set

$$\Omega_i = \sum_{k=1}^{i} (q_k - p_k) y_k x_k$$

for each $i = 1, \dots, n$ and let $\mathcal{P}_n = \{\Omega_1, y_1, x_1, \dots, \Omega_n, y_n, x_n\} \subseteq A_n$. A subset T of $\mathcal{P}_n$ is said to be admissible if it satisfies the following conditions:

1. y_i or $x_i \in T \Leftrightarrow \Omega_i$ and $\Omega_{i-1} \in T \quad (2 \leqslant i \leqslant n)$
2. y_1 or $x_1 \in T \Leftrightarrow \Omega_1 \in T$.

1.3. An element a of a Poisson algebra A is said to be normal if $\{a, A\} \subseteq aA$. Note that aA is a Poisson ideal if a is normal.

Lemma. *In A_n, we have the following:*

$$
(2) \qquad
\begin{aligned}
&\{y_i, \Omega_j\} = -q_i y_i \Omega_j, && \{x_i, \Omega_j\} = q_i x_i \Omega_j, && (i \leqslant j)\\
&\{y_i, \Omega_j\} = -p_i y_i \Omega_j, && \{x_i, \Omega_j\} = p_i x_i \Omega_j, && (i > j)\\
&\{\Omega_i, \Omega_j\} = 0, && && (\text{all } i, j)
\end{aligned}
$$

$$
(3) \qquad \Omega_{i-1} = \{x_i, y_i\} - q_i y_i x_i, \qquad \Omega_i = \{x_i, y_i\} - p_i y_i x_i
$$

Hence, all Ω_i, $i \geqslant 1$, are normal elements of A_n and y_i and x_i are normal modulo the ideals $\langle \Omega_i \rangle$ and $\langle \Omega_{i-1} \rangle$.

Proof: See [**9**, 1.3]. ∎

Lemma 1.4.

(a) *For every admissible set T, the ideal $\langle T\rangle$ is a prime Poisson ideal of A_n.*

(b) *For every prime Poisson ideal P of A_n, $P\cap\mathcal{P}_n$ is an admissible set.*

Proof: See [**9**, 1.5 and 1.6]. ∎

1.5. We define an order on the generators of A_n by

$$y_1 < x_1 < y_2 < x_2 < \cdots < y_n < x_n$$

and give the grade lexicographic order on the set of all standard monomials

$$y_1^{r_1}x_1^{r_2}y_2^{r_3}x_2^{r_4}\ldots y_n^{r_{2n-1}}x_n^{r_{2n}}$$

of A_n, where r_i are nonnegative integers. That is,

$$\begin{gathered} y_1^{r_1}x_1^{r_2}y_2^{r_3}x_2^{r_4}\cdots y_n^{r_{2n-1}}x_n^{r_{2n}} \prec y_1^{s_1}x_1^{s_2}y_2^{s_3}x_2^{s_4}\cdots y_n^{s_{2n-1}}x_n^{s_{2n}} \\ \Longleftrightarrow \\ \sum_{1\leqslant i\leqslant 2n} r_i < \sum_{1\leqslant i\leqslant 2n} s_i \text{ or} \\ \sum_{1\leqslant i\leqslant 2n} r_i = \sum_{1\leqslant i\leqslant 2n} s_i,\ r_{2n} = s_{2n},\ldots,r_{i+1}=s_{i+1} \text{ and } r_i < s_i. \end{gathered}$$

Note that, for standard monomials X^α and X^β of A_n with $X^\alpha \prec X^\beta$, we have $X^\alpha X^\gamma \prec X^\beta X^\gamma$ for all standard monomials X^γ, that is, the grade lexicographic order $\prec$ is a monomial order by [**2**, Section 2.2].

Fix an admissible set T of A_n. For each $i = 1,\ldots,n$, set

$$\begin{aligned} S_i &= \{j \mid 1\leqslant j < i, \Omega_j \in T\} \\ i_0 &= \max S_i \quad \text{if } S_i \neq \phi. \end{aligned}$$

For each $i = 1,2,\ldots,n$, define an element $\Omega_i' \in A_n$ by

$$\Omega_i' = \begin{cases} \Omega_i - \Omega_{i_0} & \text{if } S_i \neq \phi \\ \Omega_i & \text{if } S_i = \phi \end{cases}$$

and denote

$$G_T = \{x_i \mid x_i \in T\} \cup \{y_i \mid y_i \in T\} \cup \{\Omega_i' \mid \Omega_i \in T, x_i \notin T, y_i \notin T\}.$$

Note that the leading terms of Ω_i' and Ω_i are equal and if $\Omega_i' \in G_T$, $i > 1$, then $\Omega_{i-1} \notin T$.

In order to find a **k**-basis for $A_n/\langle T\rangle$, we use an argument for the Gröbner basis. Refer to [**2**, Chapter 2] for further background and terminologies on the Gröbner basis.

Lemma.

(a) *For every admissible set T of A_n, G_T is a Gröbner basis for $\langle T\rangle$.*

(b) *The algebra $A_n/\langle T\rangle$ has a* $\mathbf{k}$*-basis $\mathcal{B}_T$ consisting of the natural images of all the standard monomials which are not divided by any leading terms of elements in G_T.*

(c) *The elements $\overline{\Omega}_j \in A_n/\langle T\rangle$, $\Omega_j \notin T$, are algebraically independent over* $\mathbf{k}$.

Proof: (a) Note that $\langle T\rangle = \langle G_T\rangle$ by the definition of admissible set. We use the notation given in [**2**, Chapter 2]. Arrange the elements of G_T by the order induced by the following arrangement

$$x_n, y_n, x_{n-1}, y_{n-1}, \dots, x_1, y_1, \Omega'_n, \Omega'_{n-1}, \dots, \Omega'_1.$$

Now the S-polynomials of the forms $S(x_i, x_j)$, $S(x_i, y_j)$ and $S(y_i, y_j)$ are all zero and the S-polynomials of the forms $S(x_i, \Omega'_j)$ and $S(y_i, \Omega'_j)$ are clearly reduced to zero modulo G_T by the division algorithm. Finally consider the S-polynomial of the form

$$\begin{aligned} S(\Omega'_i, \Omega'_j) &= (q_j - p_j)y_jx_j\Omega'_i - (q_i - p_i)y_ix_i\Omega'_j \\ &= (q_j - p_j)y_jx_j\Omega'_{i-1} - (q_i - p_i)y_ix_i\Omega'_{j-1}. \quad (i < j) \end{aligned}$$

Since Ω'_j is the first element in G_T such that its leading term divides each term appearing in $(q_j - p_j)y_jx_j\Omega'_{i-1}$ and Ω'_i is the first element in G_T such that its leading term divides each term appearing in $(q_i - p_i)y_ix_i\Omega'_{j-1}$, we have that

$$\begin{aligned} S(\Omega'_i, \Omega'_j) &= (q_j - p_j)y_jx_j\Omega'_{i-1} - (q_i - p_i)y_ix_i\Omega'_{j-1} \\ &= (\Omega'_j - \Omega'_{j-1})\Omega'_{i-1} - (\Omega'_i - \Omega'_{i-1})\Omega'_{j-1} \\ &= \Omega'_j\Omega'_{i-1} - \Omega'_i\Omega'_{j-1}. \end{aligned}$$

It follows that $S(\Omega'_i, \Omega'_j)$ is reduced to zero modulo G_T by the division algorithm.

(b) It follows immediately by (a) and [**2**, Section 2.6 Proposition 1].

(c) Let $\{\Omega_{i_1}, \dots, \Omega_{i_k}\} = \{\Omega_i \mid \Omega_i \notin T\}$ and suppose that $0 \neq f \in \mathbf{k}[z_1, \dots, z_k]$ such that $f(\overline{\Omega}_{i_1}, \dots, \overline{\Omega}_{i_k}) = 0$, where $z_1, \dots, z_k$ are indeterminates. Let α be the coefficient of the leading term of f under the grade lexicographic order of monomials for $z_1 < z_2 < \dots < z_k$. Since the coefficient of the leading term of $f(\Omega_{i_1}, \dots, \Omega_{i_k})$ under the grade lexicographic order of standard monomials given in 1.5 is equal to $\alpha\beta$ for some nonzero $\beta \in \mathbf{k}$ and $f(\Omega_{i_1}, \dots, \Omega_{i_k})$ is reduced to zero modulo $\langle T\rangle$ by the division algorithm via G_T, we have $\alpha = 0$, which is a contradiction. Hence $\overline{\Omega}_j \in A_n/\langle T\rangle$, $\Omega_j \notin T$, are algebraically independent over $\mathbf{k}$. ∎

2. H-actions on $A_{n,\Gamma}^{P,Q}$

2.1. Denote

$$H = \{(h_1, \dots, h_{2n}) \in (\mathbf{k}^\times)^{2n} \mid h_{2i-1}h_{2i} = h_{2j-1}h_{2j} \text{ for all } i, j\}.$$

The multiplicative subgroup H of $(\mathbf{k}^\times)^{2n}$ acts on A_n as follows. For $h = (h_1, h_2, \dots, h_{2n}) \in H$ and $f \in A_n$,

$$h \cdot f = f(h_1 y_1, h_2 x_1, \dots, h_{2n-1} y_n, h_{2n} x_n).$$

Note that each element of H acts on A_n by a Poisson automorphism and H is isomorphic to the algebraic torus $(\mathbf{k}^*)^{n+1}$.

2.2. Let A be a Poisson algebra and let a group G act on A by Poisson automorphisms. A proper Poisson ideal Q of A is said to be G-prime Poisson ideal if Q is G-stable such that whenever I, J are G-stable ideals of A with $IJ \subseteq Q$, either $I \subseteq Q$ or $J \subseteq Q$. A Poisson algebra A is said to be G-simple if 0 and A are the only G-stable Poisson ideals of A.

LEMMA. *For each admissible set T, $\langle T \rangle$ is an H-prime Poisson ideal of A_n.*

PROOF: Since every element of T is H-eigenvector, $\langle T \rangle$ is H-stable and thus the result follows from 1.4. ∎

2.3. Let an affine algebraic group G act on a $\mathbf{k}$-algebra A by algebra automorphisms. Remind that the action G on A is said to be rational if A is a direct union of finite dimensional G-invariant subspaces V_i such that the restrictions $G \longrightarrow \operatorname{Aut} A \longrightarrow \mathrm{GL}(V_i)$ are morphisms of algebraic varieties.

LEMMA. *Every H-prime Poisson ideal of A_n is a prime Poisson ideal.*

PROOF: Note that H is an irreducible affine algebraic group since its coordinate ring is the prime ring

$$\mathbf{k}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_{2n}^{\pm 1}] / \langle z_1 z_2 - z_{2i-1} z_{2i} \mid i = 2, \dots, n \rangle.$$

Moreover H acts rationally on A_n by algebra automorphisms since H acts semisimply with rational eigenvalues. Hence every H-prime Poisson ideal is a prime Poisson ideal by [**1**, II.2.9 Proposition]. (In the proof of [**1**, II.2.9 Proposition], replace the $\mathbf{k}$-torus by an irreducible affine algebraic group.) ∎

THEOREM 2.4. *Every H-prime Poisson ideal of A_n is generated by an admissible set.*

PROOF: Let P be an H-prime Poisson ideal of A_n such that $P \cap \mathcal{P}_n = T$. Then T is an admissible set by 2.3 and 1.4. By way of contradiction, suppose that $P \neq \langle T \rangle$. Express each element of $\overline{A}_n = A_n / \langle T \rangle$ by a linear combination of elements of $\mathcal{B}_T$ given in 1.5 Lemma (b). Choose a nonzero element $f \in \overline{P}$ such that f has the shortest length among those elements and let

$$H_i = \left\{ (h_1, h_1^{-1}, h_2, h_2^{-1}, \dots, h_n, h_n^{-1}) \in H \mid h_j = 1 \text{ for all } j \neq i \right\}.$$

Suppose that $y_i \in T$ and $x_i \notin T$. Applying H_i on f, the degrees of all nonzero terms of f with respect to $\overline{x}_i$ are equal since each element $z \in \mathcal{B}_T$ is an eigenvector of H_i with

https://doi.org/10.1017/S000497270003851X Published online by Cambridge University Press

eigenvalue depending on the degree of z with respect to $\overline{x}_i$. Hence we may assume that the degree of f with respect to $\overline{x}_i$ is zero since $x_i \notin P$ and P is a prime ideal by 2.3. By the same way the degree of f with respect to $\overline{y}_i$ is zero, where $y_i \notin T$ and $x_i \in T$.

Suppose that $\Omega_i \in T, y_i \notin T, x_i \notin T$. For $z \in \mathcal{B}_T$, let r and s be the respective degrees of z with respect to $\overline{y}_i$ and $\overline{x}_i$. Then $rs = 0$ and, for each $h \in H_i$, z is an eigenvector of h with eigenvalue c^{r-s}, where c is the $(2i-1)$-th component of h. Hence, applying H_i on f, the degrees of all nonzero terms of f with respect to $\overline{y}_i$ (respectively, $\overline{x}_i$) are equal. Therefore we may assume that the degree of f with respect to $\overline{y}_i$ (respectively, $\overline{x}_i$) is zero since $y_i, x_i \notin P$ and P is a prime ideal by 2.3.

Finally suppose that $\Omega_i \notin T$. For $z \in \mathcal{B}_T$, let r and s be the respective degrees of z with respect to $\overline{y}_i$ and $\overline{x}_i$. Then z is an eigenvector of $h \in H_i$ with eigenvalue c^{r-s}, where c is the $(2i-1)$-th component of h. Thus, applying H_i on f, the differences between the degrees of all nonzero terms of f with respect to $\overline{y}_i$ and those with respect to $\overline{x}_i$ are equal. Hence f is the product of $\overline{y_i^r}$ (or $\overline{x_i^r}$), $r \geqslant 0$, and a linear combination of elements $\overline{z}$, $z \in \mathcal{B}_T$, such that the degree of z with respect to $\overline{y}_i$ is equal to that of z with respect to $\overline{x}_i$. Since $y_i, x_i \notin P$ and P is a prime ideal by 2.3, we may assume that f is a linear combination of elements $z \in \mathcal{B}_T$, such that the degree of z with respect to $\overline{y}_i$ is equal to that of z with respect to $\overline{x}_i$. Replace $\overline{y}_j\overline{x}_j$ in f by $(q_j - p_j)^{-1}\overline{\Omega}_j - \overline{\Omega}_{j-1}$ for each $\Omega_j \notin T$ since $(q_j - p_j)y_jx_j = \Omega_j - \Omega_{j-1}$. Then we have that f is a polynomial with variables $\overline{\Omega}_j$ such that $\Omega_j \notin T$.

That is, $P/\langle T\rangle$ contains a nonzero element which is a polynomial with variables $\overline{\Omega}_j$ such that $\Omega_j \notin T$. Note that $\overline{\Omega}_j$'s, $\Omega_j \notin T$, are algebraically independent over $\mathbf{k}$ by 1.5 Lemma (c). Suppose that $g \in P/\langle T\rangle$ has the smallest length among such elements. Denote

$$g = \sum_{i=1}^{m} a_i \overline{\Omega}_{j_1}^{r_{i1}} \overline{\Omega}_{j_2}^{r_{i2}} \cdots \overline{\Omega}_{j_k}^{r_{ik}},$$

where

$$a_i \neq 0, \quad \{\Omega_{j_1}, \ldots, \Omega_{j_k}\} = \{\Omega_i \mid \Omega_i \notin T\}$$

and

$$j_1 < j_2 < \cdots < j_k.$$

For each x_j such that $\Omega_j \notin T$, there exists a derivation ψ_{x_j} on the localisation $(A_n/\langle T\rangle)[\overline{x}_j{}^{-1}]$ defined by

$$\psi_{x_j}(a) = \{\overline{x}_j, a\}\overline{x}_j{}^{-1}$$

for all

$$a \in (A_n/\langle T\rangle)[\overline{x}_j{}^{-1}].$$

Moreover the extension $\overline{P}^e$ is stable under ψ_{x_j} and g is an eigenvector of ψ_{x_j} by (2). Now acting H on g and applying $\psi_{x_{j_k}}, \psi_{x_{j_{k-1}}}, \ldots, \psi_{x_{j_2}}$ to g, by (2), we have the following linear system

$$(4)\qquad \begin{aligned} r_{i1} + r_{i2} + \cdots + r_{ik} &= c \\ (r_{i1} + r_{i2} + \cdots + r_{i,k-1})p_{j_k} + r_{ik}q_{j_k} &= c_k \\ (r_{i1} + r_{i2} + \cdots + r_{i,k-2})p_{j_k} + (r_{i,k-1} + r_{ik})q_{j_{k-1}} &= c_{k-1} \\ &\cdots \\ r_{i1}p_{j_2} + (r_{i2} + r_{i3} + \cdots + r_{ik})q_{j_2} &= c_2 \end{aligned}$$

for all $i = 1, 2, \ldots, m$, where $c, c_k, c_{k-1}, \ldots, c_2$ are constants which are independent to i. Since $p_i \neq q_i$ for all i, we have from (4) that $r_{1\ell} = r_{2\ell} = \cdots = r_{m\ell}$ for all $\ell = 1, 2, \ldots, k$ and thus g is of the form $\alpha\overline{\Omega}_{j_1}^{r_1}\overline{\Omega}_{j_2}^{r_2}\cdots\overline{\Omega}_{j_k}^{r_k}$ for some nonzero $\alpha \in \mathbf{k}$. It follows that P contains some Ω_j which is not in T, a contradiction. ∎

2.5. For a Poisson algebra A, denote by $\operatorname{pspec}(A)$ the set of all prime Poisson ideals of A. For each ideal I of A_n, denote by $(I : H)$ the largest H-stable ideal contained in I and, for a H-prime Poisson ideal J of A_n and an admissible set T, set

$$\begin{aligned} \operatorname{pspec}_J(A_n) &\doteq \{P \in \operatorname{pspec}(A_n) \mid (P : H) = J\} \\ \operatorname{pspec}_T(A_n) &= \{P \in \operatorname{pspec}(A_n) \mid P \cap \mathcal{P}_n = T\}. \end{aligned}$$

PROPOSITION.

$$\begin{aligned} \operatorname{pspec}(A_n) &= \biguplus_{J\ H\text{-prime Poisson ideal}} \operatorname{pspec}_J(A_n) \\ &= \biguplus_{T\ \text{admissible set}} \operatorname{pspec}_T(A_n). \end{aligned}$$

PROOF: Let P be a prime Poisson ideal of A_n. For H-stable ideals I, J, suppose that $IJ \subseteq (P : H)$. Then P contains I or J because P is a prime ideal, hence $(P : H)$ contains I or J and thus $(P : H)$ is H-prime. Moreover $(P : H)$ is an H-prime Poisson ideal since $(P : H) = \cap_{h\in H} h(P)$ and every element $h \in H$ acts as a Poisson automorphism. Now the statement follows immediately from 2.4 and 1.4. ∎

3. THE GROUP OF POISSON AUTOMORPHISMS OF $A_{n,\Gamma}^{P,Q}$

LEMMA 3.1. *Let A be a finitely generated Poisson algebra, a any normal element which is not a unit, and P a prime Poisson ideal minimal over aA. Then P has height at most 1.*

PROOF: If Q is a prime ideal such that $aA \subseteq Q \subseteq P$ then the maximal Poisson ideal contained in Q is a prime Poisson ideal containing aA by [**3**, 3.3.2], and thus $P = Q$. It follows that P is a prime ideal minimal over aA. Hence P has height at most 1 by [**7**, 4.1.11]. ∎

3.2. Let $\mathbf{N}$ be the set of all nonnegative integers. Now we give an order $\ll$ on the set $\mathbf{N}^{2n}$ defined by

$$(r_1, r_2, \dots, r_{2n}) \ll (s_1, s_2, \dots, s_{2n})$$
$$\Longleftrightarrow$$
$$r_{2n} = s_{2n}, r_{2n-1} = s_{2n-1}, \dots, r_{i+1} = s_{i+1}, \text{ and } r_i < s_i.$$

Every element of A_n can be uniquely written as a linear combination of standard monomials $X^\alpha = y_1^{r_1} x_1^{r_2} y_2^{r_3} x_2^{r_4} \cdots y_n^{r_{2n-1}} x_n^{r_{2n}}$. For $0 \neq f \in A_n$ expressed by

$$f = \sum_{\alpha \in \mathbf{N}^{2n}} c_\alpha X^\alpha,$$

define $\exp(f)$ to be the maximal element in the set $\{\alpha \mid c_\alpha \neq 0\}$ under the order $\ll$. It is easy to see that $\exp(fg) = \exp(f) + \exp(g)$ for all nonzero elements $f, g \in A_n$.

Let $\mathbf{V}_n$ be the $\mathbf{k}$-vector space spanned by $\Omega_1, \Omega_2, \dots, \Omega_n$. Note that $y_1x_1, y_2x_2, \dots, y_nx_n$ form a k-basis for $\mathbf{V}_n$ since $\Omega_i = (q_i - p_i)y_ix_i + \Omega_{i-1}$.

LEMMA 3.3. *Suppose that the set of all prime Poisson ideals of A_n with height 1 is*

$$\mathbf{P} = \{\langle y_1\rangle, \langle x_1\rangle, \langle \Omega_2\rangle, \langle \Omega_3\rangle, \dots, \langle \Omega_n\rangle\}$$

and let $q_1 \neq 0$. If σ is a Poisson automorphism of A_n then for any $i \in \{2, 3, \dots, n\}$, there exists $j \in \{2, 3, \dots, n\}$ such that

$$\sigma(\Omega_i) = \lambda_{ij}\Omega_j, \ \sigma(y_1) = r_1y_1, \ \sigma(x_1) = s_1x_1$$

for some $\lambda_{ij}, r_1, s_1 \in \mathbf{k}^*$.

PROOF: For each $P \in \mathbf{P}$, P is generated by a normal element x which is not a unit. Since $0 \neq \sigma(P)$ is generated by a normal element $\sigma(x)$, $\sigma(P)$ is a prime Poisson ideal with height 1 by 3.1, and thus $\sigma(P) \in \mathbf{P}$. That is, $\mathbf{P}$ is invariant by σ. Let $x \in \{y_1, x_1, \Omega_2, \dots, \Omega_n\}$. Then there exist $h, h' \in A_n \setminus \{0\}$ and $y \in \{y_1, x_1, \Omega_2, \dots, \Omega_n\}$ such that $\sigma(x) = hy$ and $\sigma^{-1}(y) = h'x$. Hence $h\sigma(h') = \sigma(h')h = 1$, and so h, h' are invertible. It follows that $h, h' \in \mathbf{k}^*$.

Suppose now that there exist $i \in \{2, 3, \dots, n\}$ and $j \in \{2, 3, \dots, n\}$ such that $\sigma(\Omega_i) = \alpha y_1$, $\sigma(\Omega_j) = \beta x_1$ for some $\alpha, \beta \in \mathbf{k}^*$. Since $q_1 \neq 0$ and $\{\Omega_i, \Omega_j\} = 0$ by (2), applying σ to this equality we get $\alpha\beta = 0$. So for each $i \in \{2, 3, \dots, n\}$ there exists $j \in \{2, 3, \dots, n\}$ such that $\sigma(\Omega_i) = \lambda_{ij}\Omega_j$ for some $\lambda_{ij} \in \mathbf{k}^*$.

If $\sigma(y_1) = \alpha x_1$, $\sigma(x_1) = \beta y_1$, $\alpha, \beta \in \mathbf{k}^*$ then $2q_1\alpha\beta = 0$ by applying σ to the equality $\{x_1, y_1\} = q_1x_1y_1$ given in (1). Therefore $\sigma(y_1) = r_1y_1$, $\sigma(x_1) = s_1x_1$ for some $r_1, s_1 \in \mathbf{k}^*$. ∎

LEMMA 3.4. *Let $f, g \in A_n \setminus \mathbf{k}$ such that $fg = \sum_{1 \leqslant i \leqslant n} c_iy_ix_i \in \mathbf{V}_n$ with $c_i \in \mathbf{k}$ and $c_n \neq 0$. Then there exist $\lambda, \lambda' \in \mathbf{k}^*$ such that $f = \lambda y_n$ and $g = \lambda' x_n$ (or $f = \lambda' x_n$ and $g = \lambda y_n$).*

PROOF: It is clear that $\exp(fg) = (0, 0, \ldots, 1, 1)$, so we have

$$\exp(f) = (0, 0, \ldots, 1, 0), \quad \exp(g) = (0, 0, \ldots, 0, 1)$$

or

$$\exp(f) = (0, 0, \ldots, 0, 1), \quad \exp(g) = (0, 0, \ldots, 1, 0).$$

Then we have, for example,

$$f = \lambda y_n + f_0, \ g = \lambda' x_n + g_0$$

for some $\lambda, \lambda' \in \mathbf{k}^*$, $f_0 \in A_{n-1}$ and $g_0 \in A_{n-1}[y_n]$. So

$$fg = \lambda\lambda' y_n x_n + \lambda' f_0 x_n + \lambda g_0 y_n + f_0 g_0$$

and

$$u = \lambda' f_0 x_n + \lambda g_0 y_n + f_0 g_0 \in \mathbf{V}_n.$$

But

$$\exp(\lambda' f_0 x_n) = \exp(u) = (\nu, 0, 1)$$

for some $\nu \in \mathbf{N}^{2(n-1)}$. It follows that $u \notin \mathbf{V}_n$ if $f_0 \neq 0$. Hence we have that $f_0 = g_0 = 0$. By the same way we get the other case. ∎

LEMMA 3.5. *Suppose that the set of all prime Poisson ideals of A_n with height 1 is*

$$\mathbf{P} = \{\langle y_1 \rangle, \langle x_1 \rangle, \langle \Omega_2 \rangle, \langle \Omega_3 \rangle, \ldots, \langle \Omega_n \rangle\}.$$

Let σ be a Poisson automorphism of A_n and let $q_i \neq 0$ for all $i = 1, 2, \ldots, n$. Then for each $i \in \{1, 2, \ldots n\}$, we have

$$\sigma(\Omega_i) = \lambda_i \Omega_i, \ \sigma(y_i) = r_i y_i, \ \sigma(x_i) = s_i x_i$$

for some $\lambda_i, r_i, s_i \in \mathbf{k}^*$.

PROOF: For $i = 1$, we know that $\sigma(y_1) = r_1 y_1$, $\sigma(x_1) = s_1 x_1$, $r_1, s_1 \in \mathbf{k}^*$ by 3.3, and so $\sigma(\Omega_1) = \lambda_1 \Omega_1$, where $\lambda_1 = r_1 s_1$. Suppose that there exist $i \neq j \in \{2, 3, \ldots, n\}$ such that $\sigma(\Omega_i) = \lambda \Omega_j$, $\lambda \in \mathbf{k}^*$. Let m be the maximal element in the set

$$\{j \mid \sigma(\Omega_l) = \lambda \Omega_j \text{ for some } \lambda \in \mathbf{k}^* \text{ and } l \neq j\}.$$

Let $\sigma(\Omega_i) = \lambda_0 \Omega_m$, $\lambda_0 \in \mathbf{k}^*$. Note that

$$1 < i < m \leqslant n, \ \sigma(\Omega_{i-1}) = \lambda' \Omega_r, \ \sigma(\Omega_{i+1}) = \lambda'' \Omega_s$$

for some λ', $\lambda'' \in \mathbf{k}^*$ and $r, s < m$. Applying σ to $\Omega_i = (q_i - p_i) y_i x_i + \Omega_{i-1}$, we get

$$\sigma(y_i)\sigma(x_i) = \sum_{1 \leqslant l \leqslant m} k_l y_l x_l$$

with

$$k_m = (q_i - p_i)^{-1}(q_m - p_m)\lambda_0.$$

By 3.4 (with $m = n$) applied to $\sigma(y_i)\sigma(x_i)$, we have for example $\sigma(y_i) = \mu' x_m$, $\mu' \in \mathbf{k}^*$. If we apply σ to $\Omega_{i+1} = (q_{i+1} - p_{i+1})y_{i+1}x_{i+1} + \Omega_i$, then we get $\sigma(y_{i+1}) = \mu'' x_m$, $\mu'' \in \mathbf{k}^*$(or $\sigma(x_{i+1}) = \mu'' x_m$), which is a contradiction to the injectivity of σ. In conclusion we have $\sigma(\Omega_i) = \lambda_i \Omega_i$ for some $\lambda_i \in \mathbf{k}^*$. Now applying σ to

$$\Omega_i = (q_i - p_i)y_i x_i + \Omega_{i-1},\ i = 2, 3, \dots, n,$$

we get $\sigma(y_i)\sigma(x_i) \in \mathbf{V}_n$. By 3.4 ($i = n$) applied to $\sigma(y_i)\sigma(x_i)$, we have either $\sigma(y_i) = r_i y_i$, $\sigma(x_i) = s_i x_i$ or $\sigma(x_i) = r_i y_i$, $\sigma(y_i) = s_i x_i$ for some $r_i, s_i \in \mathbf{k}^*$.

Finally, if $\sigma(x_i) = r_i y_i$, $\sigma(y_i) = s_i x_i$ then one have

$$\sigma(\{x_i, y_i\}) = q_i r_i s_i y_i x_i + \lambda_{i-1}\Omega_{i-1}$$
$$\{\sigma(x_i), \sigma(y_i)\} = -q_i r_i s_i y_i x_i - r_i s_i \Omega_{i-1}$$

and thus we have $r_i s_i = 0$, a contradiction. It completes the statement. ∎

THEOREM 3.6. *Suppose that the set of all prime Poisson ideals of A_n with height 1 is*

$$\mathbf{P} = \{\langle y_1\rangle, \langle x_1\rangle, \langle \Omega_2\rangle, \langle \Omega_3\rangle, \dots, \langle \Omega_n\rangle\}$$

and $q_i \neq 0$ for each $i = 1, 2, \dots, n$. Then the group of Poisson automorphisms of A_n is equal to the multiplicative group H.

PROOF: Clearly $h = (h_1, h_2, \dots, h_{2n-1}, h_{2n}) \in H$ is a Poisson automorphism. Conversely, if σ is a Poisson automorphism of A_n then $\sigma(y_i) = r_i y_i, \sigma(x_i) = s_i x_i$ for some $r_i, s_i \in \mathbf{k}^*$ by 3.5. Since

$$\sigma(\Omega_n) = \lambda_n \Omega_n = \textstyle\sum_{1 \leqslant i \leqslant n}(q_i - p_i)\lambda_n y_i x_i$$
$$\sigma(\Omega_n) = \sum_{1 \leqslant i \leqslant n}(q_i - p_i)\sigma(y_i x_i) = \sum_{1 \leqslant i \leqslant n}(q_i - p_i) r_i s_i y_i x_i$$

we have that $r_i s_i = \lambda_n$ for each $i = 1, 2, \dots, n$. It follows that σ can be identified to $(r_1, s_1, \dots, r_n, s_n) \in H$, as required. ∎

3.7. Here we assume that

$$q_i = -2, p_i = 0, \gamma_{ij} = 1$$

for each i and for each $j > i$. Then A_n is called the coordinate ring of Poisson symplectic $2n$-space.

COROLLARY. *The group of Poisson automorphisms of the coordinate ring A_n of Poisson symplectic $2n$-space is the multiplicative group H.*

PROOF: By 3.6, it is enough to prove that the set of all prime Poisson ideals of A_n with height 1 is

$$\mathbf{P} = \{\langle y_1\rangle, \langle x_1\rangle, \langle \Omega_2\rangle, \langle \Omega_3\rangle, \dots, \langle \Omega_n\rangle\}.$$

Clearly all elements of $\mathbf{P}$ are prime Poisson ideals of A_n with height 1 by 2.5. Let P be a prime Poisson ideal with height 1 and suppose that $(P : H) = 0$. Then P does not contain any elements y_i, Ω_i, $i = 1, \dots, n$ since they are H-eignevectors. Set

$$B = A_n[y_1^{-1}, y_2^{-1}, \dots, y_n^{-1}, \Omega_1^{-1}, \Omega_2^{-1}, \dots, \Omega_n^{-1}].$$

By [**8**, Section 2] or [**10**, 2.2 and 2.3], the algebra B is presented by the Poisson algebra $\mathbf{k}_u(\mathbb{Z}^{2n})$, where $u : \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \longrightarrow \mathbf{k}$ is an antisymmetric biadditive map defined by the skew-symmetric $2n \times 2n$-matrix

$$(5) \qquad \begin{array}{c} \\ y_1 \\ y_2 \\ \vdots \\ y_n \\ \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_n \end{array} \begin{array}{c} \begin{array}{cccccccc} y_1 & y_2 & \cdots & y_n & \Omega_1 & \Omega_2 & \cdots & \Omega_n \end{array} \\ \begin{pmatrix} 0 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 \\ -1 & 0 & \cdots & 1 & 0 & 2 & \cdots & 2 \\ . & . & \vdots & . & . & . & \vdots & . \\ -1 & -1 & \cdots & 0 & 0 & 0 & \cdots & 2 \\ -2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ -2 & -2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ . & . & \vdots & . & . & . & \vdots & . \\ -2 & -2 & \cdots & -2 & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array}$$

Since the determinant of the matrix (5) is nonzero, the radical of u

$$\{\alpha \in \mathbb{Z}^{2n} \mid u(\alpha, \beta) = 0 \text{ for all } \beta \in \mathbb{Z}^{2n}\}$$

is trivial, thus $\mathbf{k}_u(\mathbb{Z}^{2n})$ has no nonzero prime Poisson ideal by [**8**, 2.3]. But the extension of P to B is a nonzero prime Poisson ideal, a contradiction. Hence $(P : H) \neq 0$. It follows that $P \in \mathbf{P}$ by 2.4 and 2.5, as required. ∎

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