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Abstract

We consider the polynomial representation $S(V^*)$ of the rational Cherednik algebra $H_c(W)$ associated to a finite Coxeter group W at constant parameter c. We show that for any degree d of W and $m \in \mathbb{N}$ the space $S(V^*)$ contains a single copy of the reflection representation V of W spanned by the homogeneous singular polynomials of degree $d-1+hm$, where h is the Coxeter number of W; these polynomials generate an $H_c(W)$ submodule with the parameter $c=(d-1)/h+m$. We express these singular polynomials through the Saito polynomials which are flat coordinates of the Saito metric on the orbit space V/W. We also show that this exhausts all the singular polynomials in the isotypic component of the reflection representation V for any constant parameter c.

1. Introduction

In this paper we relate two remarkable constructions associated with a finite Coxeter group W. The first one is the Frobenius manifold structure on the space of orbits of W acting in its reflection representation V; see [Dub96]. The key ingredient here is the Saito flat metric on the orbit space V/W; see [SYS80]. This metric is defined as a Lie derivative of the standard contravariant (Arnold) metric. The flat coordinates form a distinguished basis in the ring of invariant polynomials $S(V^*)^W$. This basis is now known explicitly for all irreducible groups W. All the cases except for W of type E_7 or E_8 were covered in the original paper [SYS80]. The flat coordinates in the latter two cases were found recently in both [Abr09] and [Tal10].

The other famous construction associated with the group W is the rational Cherednik algebra $H_c(W)$ (see [EG02]). It depends on the W-invariant function c on the set of reflections of W, which we assume to be constant. The key ingredient here is the Dunkl operator ∇_ζ, $\zeta \in V$, which acts in the ring of polynomials as a differential-reflection operator [Dun89]. For particular values of c, the polynomial representation $S(V^*)$ has non-trivial submodules M. These values were completely determined by Dunkl *et al.* in [DJO94], where it was shown that non-trivial submodules exist if and only if c is a non-integer number of the form $c=l/d$ where d is one of the degrees of the Coxeter group W and $l \in \mathbb{Z}_{>0}$. The lowest homogeneous component M_0 of M consists of so-called singular polynomials [DJO94], which are annihilated by Dunkl operators ∇_ζ for any $\zeta \in V$. All singular polynomials were found by Dunkl in the case where W is of type A; see [Dun04, Dun05]. Further, it was established in [ES09] that in this case, any submodule M is generated by its lowest homogeneous component M_0. In general, the structure of submodules of $S(V^*)$ and the corresponding singular polynomials are not known. Some singular polynomials for

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the classical groups W and for the icosahedral group were determined in [CE03] and [Dun03] (see also [BP10]), respectively, while the dihedral case was fully studied in [DJO94] (see also [Chm06]).

In this paper we study singular polynomials that belong to the isotypic component of the reflection representation V of the Coxeter group W. The existence of such singular polynomials is known for the Weyl groups when $c = r/h$, with h being the Coxeter number of W and r a positive integer coprime to h (see [GG09]). It appears that, in general, the corresponding parameter values have to be $c = (d-1)/h + m$, where d is one of the degrees of W and $m \in \mathbb{Z}_{\geqslant 0}$. We explain how to construct all the singular polynomials in the isotypic component of V in terms of the Saito polynomials which are flat coordinates of the Saito metric. We use the theory of Frobenius manifolds, especially Dubrovin's almost duality [Dub04]. We show that the singular polynomials under consideration correspond to the W-invariant polynomial twisted periods of the Frobenius manifold V/W, and we determine all such twisted periods.

Firstly, we prove that the first-order derivatives of the Saito polynomials are singular polynomials at appropriate values of the parameter $c = (d-1)/h$ (Corollary 2.14). Then we explain how to construct further singular polynomials with the parameter c shifted by an integer (Theorem 3.16). We next show in Corollary 4.10 that this construction gives all of the singular polynomials in the isotypic component of the reflection representation.

In the final section we present residue formulae for all the polynomial invariant twisted periods in the case of classical Coxeter groups W. Then we generalize them to get some singular polynomials for the complex reflection group $W = S_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})^n$.

2. Frobenius structures on the orbit spaces

Let $V = \mathbb{C}^n$ with the standard constant metric g given by $g(e_i, e_j) = (e_i, e_j) = \delta_{ij}$, where e_i, $i = 1, \ldots, n$, is the standard basis in V. Let $(x^1, \ldots, x^n)$ be the corresponding orthogonal coordinates. Let W be an irreducible finite Coxeter group of rank n which acts in V by orthogonal transformations such that V is the complexified reflection representation of W. Let $\mathcal{R} \subset V$ be the Coxeter root system with the group W (see [Hum90]). Let $y^1(x), \ldots, y^n(x)$ be a homogeneous basis in the ring of invariant polynomials $S(V^*)^W = \mathbb{C}[x^1, \ldots, x^n]^W = \mathbb{C}[x]^W$. Let d_α be the corresponding degrees $d_\alpha = \deg y^\alpha(x)$, for $\alpha = 1, \ldots, n$. We assume that the polynomials are ordered so that $d_1 \geqslant \cdots \geqslant d_n$, with $d_1 = h$ being the Coxeter number of the group W. The polynomials $y^1, \ldots, y^n$ are coordinates on the orbit space $\mathcal{M} = V/W$. The Euclidean coordinates $x^1, \ldots, x^n$ can also be viewed as local coordinates on $\mathcal{M}\backslash\Sigma$, where Σ is the discriminant set. Denote by $\mathfrak{S} = \{x \in V \mid (\gamma, x) = 0 \text{ for some } \gamma \in \mathcal{R}\}$ the preimage of Σ in the space V.

The metric g is defined on $\mathcal{M}\backslash\Sigma$ owing to its W-invariance. Let $g^{\alpha\beta}$ be the corresponding contravariant metric. Consider its Lie derivative $\eta^{\alpha\beta}(y) = \partial_{y^1} g^{\alpha\beta}(y)$. The metric η is called the *Saito metric*. It is correctly defined (up to proportionality), and it is flat. There exist homogeneous coordinates $t^\alpha \in \mathbb{C}[x]^W$, $1 \leqslant \alpha \leqslant n$, with $\deg t^\alpha = d_\alpha$ such that η is constant and anti-diagonal, more exactly,

$$\eta^{\alpha\beta} = \partial_{t^1} g^{\alpha\beta}(t) = \delta_{n+1}^{\alpha+\beta} \quad \text{for } 1 \leqslant \alpha, \beta \leqslant n,$$

where $\delta^i_j = \delta_{ij}$ is the Kronecker symbol.[1] Such coordinates are called *Saito polynomials*.

The pair of metrics g, η forms a flat pencil which defines a Frobenius manifold [Dub96]. We will mainly be concerned with the almost dual Frobenius structure [Dub04]; it is defined by the

[1] We distinguish between upper and lower indices, as we will use the standard differential-geometrical convention of assuming summation over the repeated upper and lower indices.

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prepotential

$$F(x) = \frac{1}{2} \sum_{\gamma \in \mathcal{R}_+} (\gamma, x)^2 \log(\gamma, x),$$

where summation is over the set of positive roots and the roots are normalized so that $(\gamma, \gamma) = 2$. The prepotential is quasi-homogeneous, that is, its Lie derivative is of the form

$$\mathcal{L}_E F = \frac{2}{h} F + \text{quadratic terms in } x,$$

where E is the Euler vector field

$$E = \frac{1}{h} x^i \frac{\partial}{\partial x^i} = E^\alpha \frac{\partial}{\partial t^\alpha}$$

with $E^\alpha = (d_\alpha / h) t^\alpha$.

Define the tensor

$$\overset{*}{C}_{ijk} = \frac{\partial^3 F}{\partial x^i \partial x^j \partial x^k} = \sum_{\gamma \in \mathcal{R}_+} \frac{\gamma_i \gamma_j \gamma_k}{(\gamma, x)}, \tag{2.1}$$

where $x \in V$ and $\gamma_i = (\gamma, e_i)$. Let $\overset{*}{C}{}^i_{jk} = g^{il} \overset{*}{C}_{jkl}$. Then, for any $x \in V \backslash \mathfrak{S}$, the tensor $\overset{*}{C}{}^i_{jk} = \overset{*}{C}{}^i_{jk}(x)$ gives the structure constants of an associative n-dimensional algebra [Dub04].

Define another tensor $C^{ij}_k = \overset{*}{C}{}^{ij}_k = g^{il} \overset{*}{C}{}^j_{kl}$, and consider the corresponding tensor with two low indices $C^\alpha_{\beta\lambda} = \eta_{\lambda\varepsilon} C^{\alpha\varepsilon}_\beta$ in the flat coordinates t^α. The tensor $C^\alpha_{\beta\lambda}$ defines associative multiplication of tangent vectors at any point of the orbit space; the vector field ∂_{t^1} is the unity of this multiplication. Let C_α be the corresponding $n \times n$ matrix with entries $(C_\alpha)^\beta_\lambda = C^\beta_{\alpha\lambda}$. Let U be the matrix $U^\alpha_\beta = g^{\alpha\lambda} \eta_{\lambda\beta}$. The following result plays a key role.

Theorem 2.2 [Dub04, Proposition 3.3]. *A function $p(t^1, \ldots, t^n)$ satisfies the system of equations*

$$\frac{\partial^2 p}{\partial x^i \partial x^j} = \nu \, \overset{*}{C}{}^k_{ij} \frac{\partial p}{\partial x^k}, \quad 1 \leqslant i, j \leqslant n \tag{2.3}$$

if and only if the following equations hold:

$$\xi_\alpha(t) = \partial_{t^\alpha} p(t), \qquad 1 \leqslant \alpha \leqslant n, \tag{2.4}$$

$$\partial_{t^\alpha} \xi(t) \, U = \xi(t)(\nu + \Lambda) C_\alpha, \quad 1 \leqslant \alpha \leqslant n, \tag{2.5}$$

where $\xi(t) = (\xi_1(t), \ldots, \xi_n(t))$ and Λ is the diagonal matrix

$$\Lambda = -\frac{1}{h} \operatorname{Diag}(d_1 - 1, \ldots, d_n - 1).$$

Functions p satisfying the system (2.3) are called *twisted periods* of the Frobenius manifold [Dub04]. We will be dealing with the system (2.4)–(2.5), so we note the following fact.

Lemma 2.6. *For any polynomial solution $\xi(t)$ of the system (2.5) there exists a polynomial $p(t)$ satisfying (2.4).*

Proof. It is sufficient to check the compatibility $\partial_{t^\alpha} \xi_\beta = \partial_{t^\beta} \xi_\alpha$. Taking into account invertibility of the matrix U on $\mathcal{M} \backslash \Sigma$ and using equation (2.5), we rewrite this equality as

$$\xi_\lambda (\nu + \Lambda)^\lambda_\varepsilon C^\varepsilon_{\beta\rho} (U^{-1})^\rho_\alpha = \xi_\lambda (\nu + \Lambda)^\lambda_\varepsilon C^\varepsilon_{\alpha\rho} (U^{-1})^\rho_\beta.$$

Note that it is necessary to check that

$$C^\varepsilon_{\beta\rho} (U^{-1})^\rho_\alpha = C^\varepsilon_{\alpha\rho} (U^{-1})^\rho_\beta. \tag{2.7}$$

The equality (2.7) is equivalent to

$$C^{\varepsilon}_{\lambda\rho}U^{\lambda}_{\delta} = C^{\varepsilon}_{\lambda\delta}U^{\lambda}_{\rho}. \tag{2.8}$$

Indeed, upon multiplying (2.8) by $(U^{-1})^{\rho}_{\alpha}(U^{-1})^{\delta}_{\beta}$ and summing by repeating indices we obtain (2.7). The relation (2.8), in turn, follows from the property $U^{\alpha}_{\beta} = E^{\varepsilon}C^{\alpha}_{\varepsilon\beta}$ (see [Dub96]) and the associativity conditions $C^{\varepsilon}_{\lambda\rho}C^{\lambda}_{\alpha\delta} = C^{\varepsilon}_{\lambda\delta}C^{\lambda}_{\alpha\rho}$. □

We observe that the Saito polynomials themselves satisfy the equations from Theorem 2.2. More exactly, we have the following statement.

Lemma 2.9. *For each $\beta = 1, \ldots, n$, the function $p(t) = t^{\beta}$ is a solution to the system (2.4)–(2.5) with parameter*

$$\nu = \frac{d_{\beta} - 1}{h}.$$

Proof. By substituting $\xi_{\alpha} = \delta^{\beta}_{\alpha}$ into (2.5), we obtain the equations $0 = e^{\beta}(\nu + \Lambda)C_{\alpha}$, where $e^{\beta} = (0, 0, \ldots, 1, \ldots, 0)$ with 1 in the position β. These equations are satisfied if $\nu - (d_{\beta} - 1)/h = 0$. □

Now recall the Dunkl operators associated with the Coxeter group W. We fix a parameter $c \in \mathbb{C}$. The Dunkl operator in the direction e_i (with $i = 1, \ldots, n$) is given by

$$\nabla_i = \partial_{x^i} - c \sum_{\gamma \in \mathcal{R}_+} \frac{\gamma_i}{(\gamma, x)}(1 - s_{\gamma}), \tag{2.10}$$

where s_{γ} denotes the orthogonal reflection with respect to the hyperplane $(\gamma, x) = 0$. The key property of Dunkl operators is their commutativity [Dun89]:

$$[\nabla_i, \nabla_j] = 0 \quad \text{for } 1 \leqslant i, j \leqslant n.$$

Proposition 2.11. *Suppose that a W-invariant polynomial $p(x^1, \ldots, x^n)$ satisfies the system (2.3). Then for any $j = 1, \ldots, n$ the polynomial $v_j(x) = \partial_{x^j}p(x)$ satisfies the equations*

$$\nabla_i v_j = 0, \quad i = 1, \ldots, n, \tag{2.12}$$

where ∇_i is the Dunkl operator (2.10) with parameter $c = \nu$.

Proof. By using the W-invariance of $p(x)$, we rearrange the left-hand side of equation (2.12) as

$$\begin{aligned}\nabla_i v_j &= \partial_{x^i}\partial_{x^j}p(x) - \nu \sum_{\gamma \in \mathcal{R}_+} \frac{\gamma_i}{(\gamma, x)}\left(\partial_{x^j}p(x) - \frac{\partial}{\partial(s_{\gamma}e_j)}p(x)\right) \\ &= \partial_{x^i}\partial_{x^j}p(x) - \nu \sum_{\gamma \in \mathcal{R}_+} \frac{2\gamma_j\gamma_i}{(\gamma, \gamma)(\gamma, x)}\left(\gamma, \frac{\partial}{\partial x}\right)p(x).\end{aligned}$$

By using $(\gamma, \partial/\partial x) = \sum_i \gamma_i\partial_{x^i}$ and the formula (2.1), we obtain

$$\nabla_i v_j = \partial_{x^i}\partial_{x^j}p(x) - \nu \overset{*}{C}{}^{k}_{ij}\partial_{x^k}p(x). \tag{2.13}$$

Thus the property (2.12) follows from (2.3). □

Corollary 2.14. *Consider the Saito polynomial $t^{\beta} = t^{\beta}(x)$ for some $\beta = 1, \ldots, n$. Then the derivatives $v_j = \partial_{x^j}t^{\beta}(x)$ satisfy the relations (2.12), that is,*

$$\nabla_i\nabla_j t^{\beta}(x) = \nabla_i\partial_{x^j}t^{\beta}(x) = 0 \quad \text{for } i, j = 1, \ldots, n, \tag{2.15}$$

if the Dunkl operators have parameter $c = (d_{\beta} - 1)/h$.

Definition 2.16 [DJO94]. A polynomial $q(x)$ is said to be *singular* if $\nabla_i q(x) = 0$ for all $i = 1, \ldots, n$.

Thus Corollary 2.14 deals with the singular polynomials v_j. The W-module $\langle v_1, \ldots, v_n \rangle$ is isomorphic to the reflection representation of the Coxeter group W.

3. Shifting

In the previous section we established that derivatives of the Saito polynomials t^β are singular polynomials for the appropriate values of the parameter $c = c_\beta = (d_\beta - 1)/h$. In this section we explain how to generate further singular polynomials starting with Saito polynomials. The corresponding parameters c differ from c_β by integers.

We start with a known property of the solutions of system (2.5).

Lemma 3.1 [Dub04, Lemma 3.6]. *If $\xi(t)$ is a solution of the system (2.5), then $\widetilde{\xi}(t) = \partial_{t^1}\xi(t)$ is a solution of the same system with ν replaced by $\nu - 1$:*

$$\partial_{t^\alpha}\widetilde{\xi}(t)\, U = \widetilde{\xi}(t)\, (\nu - 1 + \Lambda)C_\alpha, \quad 1 \leqslant \alpha \leqslant n.$$

Note also that if a function $p(t)$ is a solution of the system (2.4)–(2.5), then $\partial_{t^1}p(t)$ is a solution of the same system with ν replaced by $\nu - 1$. Indeed, the partial derivatives of the function $\partial_{t^1}p(t)$ are $\partial_{t^\alpha}\partial_{t^1}p(t) = \partial_{t^1}\xi_\alpha(t)$, so they satisfy the system (2.5) with $\nu - 1$.

The new solution $\widetilde{\xi}(t)$ given in Lemma 3.1 can be expressed as $\widetilde{\xi} = \xi(\nu + \Lambda)U^{-1}$. If $\nu \neq (d_\alpha - 1)/h$ for all $\alpha = 1, \ldots, n$, then the matrix $\nu + \Lambda$ is invertible and we can rewrite this relation as $\xi = \widetilde{\xi}\, U(\nu + \Lambda)^{-1}$. This suggests a way of inverting Lemma 3.1 in order to generate solutions with an increased value of ν.

Lemma 3.2. *Let $\xi(t)$ be a solution of the system (2.5). Assume that $\nu \neq (d_\alpha - 1)/h - 1$ for all $\alpha = 1, \ldots, n$. Then*

$$\widehat{\xi}(t) = \xi(t)U(\nu + 1 + \Lambda)^{-1} \tag{3.3}$$

is a solution of the system (2.5) with ν replaced by $\nu + 1$:

$$\partial_{t^\alpha}\widehat{\xi}(t)\, U = \widehat{\xi}(t)\, (\nu + 1 + \Lambda)C_\alpha, \quad 1 \leqslant \alpha \leqslant n. \tag{3.4}$$

Proof. Let t_0 be a generic point in $\mathcal{M}$ and let $\xi(t_0)$ be the value of $\xi(t)$ at this point. Then the value of $\widehat{\xi}(t)$ at this point is $\widehat{\xi}(t_0) = \xi(t_0)U(t_0)(\nu + 1 + \Lambda)^{-1}$. There exists a solution $\widehat{\zeta}(t)$ of the system (3.4) such that $\widehat{\zeta}(t_0) = \widehat{\xi}(t_0)$. By Lemma 3.1, the covector $\zeta(t) = \widehat{\zeta}(t)(\nu + 1 + \Lambda)U^{-1}$ is a solution of (2.5). Note that there exists a unique solution of the system (2.5) with a given value at the point t_0. So, upon taking into account $\zeta(t_0) = \widehat{\xi}(t_0)(\nu + 1 + \Lambda)U^{-1}(t_0) = \xi(t_0)$, one gets $\zeta(t) = \xi(t)$. Therefore $\widehat{\zeta}(t) = \zeta(t)U(\nu + 1 + \Lambda)^{-1} = \xi(t)U(\nu + 1 + \Lambda)^{-1} = \widehat{\xi}(t)$ and $\widehat{\xi}(t)$ satisfies equation (3.4). ∎

Remark 3.5. Suppose that all the components of the $\xi(t)$ in Lemma 3.2 are polynomials. Then, by Lemma 2.6, there exists a polynomial $\widehat{p}(t)$ such that $\widehat{\xi}_\alpha(t) = \partial_{t^\alpha}\widehat{p}(t)$. Thus the covector $\widehat{\xi}(t)$ satisfies the whole system (2.4)–(2.5) (with ν replaced by $\nu + 1$) for some polynomial $\widehat{p}(t)$.

By applying Lemma 3.2 to the first-order derivatives of the Saito polynomials, we get the following result.

Proposition 3.6. *The covector $\widehat{\xi}$ with components*

$$\widehat{\xi}_\alpha = \frac{U_\alpha^\beta}{d_\beta - d_\alpha + h}, \quad \alpha = 1, \ldots, n, \tag{3.7}$$

satisfies the equations

$$\partial_{t^\alpha} \widehat{\xi}(t)\, U = \widehat{\xi}(t)(\widehat{\nu} + \Lambda) C_\alpha, \quad 1 \leqslant \alpha \leqslant n, \tag{3.8}$$

with $\widehat{\nu} = (d_\beta - 1)/h + 1$.

Proof. Let $\nu = (d_\beta - 1)/h$, and consider the solution $p(t) = t^\beta$ of the system (2.4)–(2.5) and the corresponding covector ξ with components $\xi_\alpha = \delta_\alpha^\beta$ (see Lemma 2.9). By Lemma 3.2, the covector $\widehat{\xi}$ given by formula (3.3) is a solution of (3.8) for $\widehat{\nu} = \nu + 1$. By substituting $\xi_\alpha = \delta_\alpha^\beta$ into (3.3) we obtain the components (3.7). □

This leads us to the following result.

Theorem 3.9. *For any $\zeta \in V$ and $\beta = 1, \ldots, n$, the polynomial*

$$q(x) = \sum_{i,\alpha=1}^{n} \frac{1}{d_\beta - d_\alpha + h} \frac{\partial t^\beta}{\partial x^i} \frac{\partial t^{n+1-\alpha}}{\partial x^i} \, \partial_\zeta t^\alpha \tag{3.10}$$

is a singular polynomial for the Dunkl operators with parameter $c = (d_\beta - 1)/h + 1$.

Proof. First, we rearrange to get

$$U_\alpha^\beta = g^{\beta\lambda} \eta_{\lambda\alpha} = g^{\beta, n+1-\alpha} = \sum_{a=1}^{n} \frac{\partial t^\beta}{\partial x^a} \frac{\partial t^{n+1-\alpha}}{\partial x^a}. \tag{3.11}$$

It follows from Proposition 3.6 and Lemma 2.6 that there exists a W-invariant polynomial $p(x)$ such that

$$\partial_{t^\alpha} p = \frac{1}{d_\beta - d_\alpha + h} \sum_{a=1}^{n} \frac{\partial t^\beta}{\partial x^a} \frac{\partial t^{n+1-\alpha}}{\partial x^a}.$$

By Proposition 2.11, the derivative $q(x) = \partial_\zeta p(x)$ is a singular polynomial. It has the required form as $\partial_\zeta p = \partial_{t^\alpha} p \partial_\zeta t^\alpha$. □

As an example, consider the case where $\beta = n$. The corresponding Saito polynomial t^n is proportional to $(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2$. The right-hand side of the equality (3.10) is then proportional to

$$\sum_{\alpha=1}^{n} t^{n+1-\alpha} \partial_\zeta t^\alpha,$$

as the polynomial $t^{n+1-\alpha}$ is homogeneous of degree $d_{n+1-\alpha}$ and $d_\alpha + d_{n+1-\alpha} = h + 2$. We arrive at the following consequence.

Proposition 3.12. *For any $\zeta \in V$, the polynomial*

$$q(x) = \partial_\zeta \sum_{\alpha=1}^{n} t^\alpha t^{n+1-\alpha} \tag{3.13}$$

is a singular polynomial for the Dunkl operators with parameter $c = (h+1)/h$.

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Remark 3.14. For $c = 1/h + m$, where $m \in \mathbb{Z}_{\geqslant 0}$, the existence of the singular polynomials in the isotypic component of the reflection representation is known from [BEG03] (see also [Gor03]). Further, in the case of Weyl groups W and $c = r/h$ where r is a positive integer coprime to h, the existence of singular polynomials in the isotypic component of the reflection representation is known from [GG09].

Example 3.15. Let $\mathcal{R} = \mathcal{A}_n \subset V \subset \mathbb{C}^{n+1}$ be given in its standard embedding, so that $V \subset \mathbb{C}^{n+1}$ is defined by $\sum_{i=1}^{n+1} z_i = 0$ where z_i are the standard coordinates in $\mathbb{C}^{n+1}$. Define the polynomials $s^\alpha = \mathrm{Res}_{z=\infty} \prod_{i=1}^{n+1} (z - z_i)^{(n+1-\alpha)/(n+1)}\, dz$ for $\alpha = 1, \ldots, n$. Then the Saito coordinates satisfy $t^\alpha = s^\alpha|_V/(n - \alpha + 1)$ (see [Dub96, SYS80]). The polynomials s^α satisfy $\sum_{i=1}^{n+1} \partial s^\alpha / \partial z_i = 0$, so for any $\zeta \in \mathbb{C}^{n+1}$ and $i = 1, \ldots, n+1$ Corollary 2.14 gives $\nabla_i^{(n+1-\alpha)/(n+1)} \partial_\zeta s^\alpha = 0$ where

$$\nabla_i^c = \frac{\partial}{\partial z_i} - c \sum_{\substack{j=1 \\ j \neq i}}^{n+1} \frac{1 - s_{ij}}{z_i - z_j},$$

with s_{ij} exchanging z_i and z_j (see also [Dun98] and [Eti07, Proposition 11.14], where this fact was established). Further, Proposition 3.12 gives that the polynomial $q(z_1, \ldots, z_{n+1}) = \partial_\zeta \sum_{\alpha=1}^n t^\alpha t^{n+1-\alpha}$ satisfies $\nabla_i^{(h+1)/h} q = 0$.

By iterating the previous arguments we get the following statement.

Theorem 3.16. *Let $m \in \mathbb{Z}_{\geqslant 0}$, and fix β with $1 \leqslant \beta \leqslant n$. Define the covector $\xi^{(m)} = (\xi_1^{(m)}, \ldots, \xi_n^{(m)})$ by*

$$\xi^{(m)} = \xi^{(0)} \overrightarrow{\prod_{1 \leqslant j \leqslant m}} U\left(\Lambda + \frac{d_\beta - 1}{h} + j\right)^{-1}, \tag{3.17}$$

where $\xi^{(0)}$ has components $\xi_\alpha^{(0)} = \delta_\alpha^\beta$, $\alpha = 1, \ldots, n$, and the factors are ordered as $\overrightarrow{\prod}_{1 \leqslant j \leqslant m} A_j = A_1 A_2 \cdots A_m$. Then, for any $i = 1, \ldots, n$, the polynomials

$$q_i(x) = q_{\beta,i}(x) = \sum_{\alpha=1}^n \xi_\alpha^{(m)} \frac{\partial t^\alpha}{\partial x^i} \tag{3.18}$$

are singular polynomials for the Dunkl operators with parameter $c = (d_\beta - 1)/h + m$. These polynomials are homogeneous of degree $d_\beta - 1 + hm$.

4. Singular polynomials in the reflection representation

We are going to show that the polynomials (3.18) generate all singular polynomials in the isotypic component of the reflection representation of W. First, we note that each copy of the reflection representation spanned by the singular polynomials is governed by a single W-invariant polynomial.

Proposition 4.1. *Let a subspace $M_0 \subset \mathbb{C}[x]$ be spanned by the singular polynomials, and suppose that $M_0 \cong V$ as W-modules. Choose a basis $\{P_1, \ldots, P_n\}$ in M_0 such that each polynomial P_i is mapped to the basis vector $e_i \in V$ under the isomorphism. Then there exists $Q \in \mathbb{C}[x^1, \ldots, x^n]^W$ such that $\partial Q/\partial x^i = P_i$ for all $i = 1, \ldots, n$.*

Proof. We have $\nabla_i P_j = \nabla_j P_i = 0$ for all $i, j = 1, \ldots, n$. Hence

$$\partial_{x^i} P_j - \partial_{x^j} P_i - c \sum_{\gamma \in \mathcal{R}_+} \frac{(\gamma, e_i)}{(\gamma, x)} (1 - s_\gamma) P_j + c \sum_{\gamma \in \mathcal{R}_+} \frac{(\gamma, e_j)}{(\gamma, x)} (1 - s_\gamma) P_i = 0. \tag{4.2}$$

Since $(1 - s_\gamma) e_i = 2(\gamma, e_i)\gamma/(\gamma, \gamma)$, we get

$$(\gamma, e_i)(1 - s_\gamma) P_j = (\gamma, e_j)(1 - s_\gamma) P_i$$

for any $\gamma \in \mathcal{R}_+$. Thus it follows from the relation (4.2) that $\partial_{x^i} P_j = \partial_{x^j} P_i$, and so there exists $Q \in \mathbb{C}[x^1, \ldots, x^n]$ such that $\partial Q/\partial x^i = P_i$ for all $i = 1, \ldots, n$. Let us also check that Q is W-invariant. Fix $\gamma \in \mathcal{R}_+$. Then for any $i = 1, \ldots, n$ we have $s_\gamma P_i = \partial_{s_\gamma e_i}(s_\gamma Q)$; on the other hand, $s_\gamma P_i = \partial_{s_\gamma e_i} Q$. Thus $\partial_{s_\gamma e_i}(Q - s_\gamma Q) = 0$, and so $Q = s_\gamma Q$ as required. □

Corollary 4.3. *The singular polynomials (3.18) can be represented as*

$$q_i = \frac{\partial Q}{\partial x^i} \tag{4.4}$$

where

$$Q = Q_\beta = \frac{1}{d_\beta + hm} \sum_{\alpha=1}^{n} d_\alpha \xi_\alpha^{(m)} t^\alpha, \tag{4.5}$$

for $i = 1, \ldots, n$, keeping the notation of Theorem 3.16.

Proof. By Proposition 4.1 we have the relation (4.4) for some invariant polynomial Q of degree $d_\beta + hm$. Hence $(d_\beta + hm) Q = \sum_{i=1}^n x^i q_i$ □

Remark 4.6. It has recently been explained in [KL11] how an $\mathcal{N} = 4$ multi-particle mechanical system with $D(2, 1; \alpha)$ superconformal symmetry can be constructed based on a solution of the WDVV equations and a particular twisted period. It follows from Theorem 3.16 and Corollary 4.3 that the polynomials q_i and Q given by (3.18) and (4.5) define a superconformal mechanical system with the bosonic potential proportional to $Q^{-2} \sum_{i=1}^n q_i^2$ at the parameter value $\alpha = -(d_\beta + hm)/2$.

Remark 4.7. Let $g(x^1, \ldots, x^n)$ be a homogeneous W-invariant polynomial of positive degree. Let L_g be the differential operator which acts on the W-invariant functions by $g(\nabla_1, \ldots, \nabla_n)$. The operators L_g commute with each other and include the corresponding *Calogero–Moser operator* [Eti07]. It follows from Corollary 4.3 that if $c = (d_\beta - 1)/h + m$, then $L_g Q_\beta = 0$ and so the polynomial Q_β belongs to the joint kernel of the Calogero–Moser operator and its quantum integrals. In particular, the Saito polynomial t^β satisfies $L_g t^\beta = 0$ if $c = (d_\beta - 1)/h$.

Now we move to the main statement of this section, on possible polynomial twisted periods of the Frobenius manifold $\mathcal{M}$.

Theorem 4.8. *Let L be the linear space of solutions $p(x)$ to the system (2.3) such that $p \in \mathbb{C}[x]^W$. Then* $\dim L = 1$ *unless $\nu = (d_\beta - 1)/h + m$ for some degree d_β and $m \in \mathbb{Z}_{\geqslant 0}$. In the latter case,* $\dim L = 2$ *unless $W = D_n$, where n is even and $d_\beta = n$ in which case* $\dim L = 3$.

Proof. Suppose that $p \in \mathbb{C}[x]^W$ is a homogeneous solution of the system (2.3) such that $D = \deg p > 0$. By Proposition 2.11, the polynomials $v_i(x) = \partial p(x)/\partial x^i$ are singular at the

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parameter $c=\nu$. It follows from the relation

$$\sum_{i=1}^{n} x^i \nabla_i = \sum_{i=1}^{n} x^i \partial_{x^i} - \nu \sum_{\gamma\in\mathcal{R}_+} (1-s_\gamma)$$

that $\nu=(D-1)/h \geqslant 0$ (cf. [DJO94]). Consider first the case where $0 \leqslant \nu \leqslant 1$. Equation (2.5) at $\alpha=1$ takes the form $0=\xi(t)(\nu+\Lambda)$, since $C_1=\mathrm{Id}$ and $\deg \xi_\lambda = D-d_\lambda < h$ for any $1\leqslant\lambda\leqslant n$. Hence the matrix $\nu+\Lambda$ is degenerate, so $\nu=(d_\beta-1)/h$ for some β, and $D=d_\beta$. Moreover, $p(x)$ as a polynomial of the Saito coordinates has to be a linear combination of t^λ with $d_\lambda=d_\beta$.

Now let $\nu>1$. The polynomial $\partial_{t^1} p(t)$ is non-constant as the matrix $\nu+\Lambda$ is non-degenerate. By Lemma 3.1, the polynomial $\partial_{t^1} p(t)$ is a solution of the system (2.3) with ν replaced by $\nu-1$. It follows from Lemma 3.2 and Remark 3.5 that the spaces of positive-degree homogeneous invariant solutions of (2.3) for the parameter ν and for when ν is replaced by $\nu-1$ are isomorphic. □

The arguments from the proof of Proposition 2.11 can be reversed, from which it follows that the polynomial (4.5) is a twisted period with $\nu=(d_\beta-1+hm)/h$ of the Frobenius manifold $\mathcal{M}$. Thus Theorem 4.8 implies the following statement.

COROLLARY 4.9. *Let $p\in\mathbb{C}[x]^W$ be a non-constant twisted period of the Frobenius manifold $\mathcal{M}$. Then $\nu=(d_\beta-1+hm)/h$ for some degree d_β and $m\in\mathbb{Z}_{\geqslant 0}$. Further, $p=\lambda Q_\beta$ unless $W=D_{2r}$ and $d_\beta=2r$ for some $r\in\mathbb{N}$, in which case $p=\lambda Q_r+\mu Q_{r+1}$ where the Q_β are given by (4.5) and $\lambda,\mu\in\mathbb{C}$.*

Consider now a linear space M_0 of singular polynomials such that $M_0\cong V$ as W-modules. Then $Q\in\mathbb{C}[x]^W$ defined by Proposition 4.1 is a twisted period and hence Corollary 4.9 allows us to describe all such W-modules M_0.

COROLLARY 4.10. *Let q be a homogeneous singular polynomial. Suppose that the linear space spanned by the polynomials wq, $w\in W$, is isomorphic to V as a W-module. Then $\deg q=d_\beta-1+hm$ for some degree d_β of W, $m\in\mathbb{Z}_{\geqslant 0}$, and $c=(d_\beta-1+hm)/h$.*

Further, $q=\sum_{i=1}^n \eta_i q_{\beta,i}$ unless $W=D_{2r}$ and $d_\beta=2r$ for some $r\in\mathbb{N}$, in which case $q=\sum_{i=1}^n \eta_i(\lambda q_{r,i}+\mu q_{r+1,i})$ where the $q_{\beta,i}$ are given by (3.17)–(3.18) and $\lambda,\mu,\eta_i\in\mathbb{C}$.

In the former cases, all the homogeneous singular polynomials for $c=(d_\beta-1+hm)/h$ in the isotypic component of the reflection representation are described as linear combinations $\sum_{i=1}^n \eta_i q_{\beta,i}$, while in the latter case the homogeneous singular polynomials in the isotypic component of the reflection representation form the $2n$-dimensional subspace of polynomials $\sum_{i=1}^n \lambda_i q_{r,i}+\mu_i q_{r+1,i}$ where $\lambda_i,\mu_i\in\mathbb{C}$.

5. Further examples for classical series

While we express singular polynomials in the isotypic component of the reflection representation of W through the Saito polynomials, in certain cases direct formulae exist. We refer to [Chm06, DJO94] for the case of dihedral groups, and to [Dun98, CE03] for the case where W is of classical type. For instance, it follows from [Dun98] and Corollary 4.9 that *all* the polynomial invariant twisted periods for $W=A_n$ are given by

$$Q=\mathrm{Res}_{z=\infty}\prod_{j=1}^{n+1}(z-z_j)^\nu\,dz\Big|_{\sum z_j=0},$$

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where $\nu = s/(n+1) + m$ with $s = 1, \ldots, n$ and $m \in \mathbb{Z}_{\geqslant 0}$ (c.f. [Dub04]). Further, let

$$Q = \mathrm{Res}_{z=\infty}\, z^a \prod_{j=1}^{n} (z^2 - x_j^2)^\nu \, dz, \tag{5.1}$$

where $x_1, \ldots, x_n$ are the standard coordinates in $\mathbb{C}^n$. Then *all* the polynomial invariant twisted periods for $W = B_n$ have the form (5.1) where $a = 0$ and $\nu = (2s-1)/2n + m$ with $s = 1, \ldots, n$ and $m \in \mathbb{Z}_{\geqslant 0}$. Similarly, (5.1) is the twisted period for $W = D_n$ if $a = -2\nu$ and $\nu = (2s-1)/2(n-1) + m$ with $s = 1, \ldots, n-1$ and $m \in \mathbb{Z}_{\geqslant 0}$. *All* the remaining polynomial invariant twisted periods for $W = D_n$ have the form

$$Q = \mathrm{Res}_{z=0}\, z^{-2m-1} \prod_{j=1}^{n} (z^2 - x_j^2)^{m+1/2} \, dz, \tag{5.2}$$

where $m \in \mathbb{Z}_{\geqslant 0}$ (cf. [EYY93]). We note that for even n, the polynomial $Q = Q_{m,\infty}$ given by (5.1) with $a = -2m-1$ and $\nu = m + 1/2$ where $m \in \mathbb{Z}_{\geqslant 0}$ has the same degree as the polynomial $Q = Q_{m,0}$ given by (5.2) with the same m. These polynomials are not proportional, as for $m > 0$ the polynomial $\partial_{t^1} Q_{m,\infty}$ is a non-zero multiple of the polynomial $Q_{m-1,\infty}$ and $\partial_{t^1} Q_{m,0}$ is a non-zero multiple of $Q_{m-1,0}$.

This leads to the following proposition, which can also be checked directly.

Proposition 5.3. *Let the polynomial $Q(x)$ be given by (5.2) with $m \in \mathbb{Z}_{\geqslant 0}$. For any $\zeta \in V$, the polynomial $\partial_\zeta Q$ is singular for $W = D_n$ with parameter $c = m + 1/2$.*

Proposition 5.3 can be generalized to the case of the complex reflection group $W = S_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})^n$ where $\ell \in \mathbb{Z}$ satisfies $\ell \geqslant 2$. This group is generated by the reflections $\sigma_{ij}^{(a)}$ and s_i acting on the standard basis by

$$\sigma_{ij}^{(a)} e_i = \omega^{-a} e_j, \quad \sigma_{ij}^{(a)} e_j = \omega^{a} e_i, \quad \sigma_{ij}^{(a)} e_k = e_k, \tag{5.4}$$

$$s_i e_i = \omega e_i, \qquad s_i e_k = e_k, \tag{5.5}$$

where $\omega = e^{2\pi i/\ell}$ is the ℓth primitive root of unity, $i, j, k = 1, \ldots, n$ with $i \neq j \neq k \neq i$, and $a = 0, \ldots, \ell - 1$. The Dunkl operators in this case have the form

$$\nabla_i = \partial_{x_i} - \nu \sum_{a=0}^{\ell-1} \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{x_i - \omega^a x_j} (1 - \sigma_{ij}^{(a)}) - \sum_{b=1}^{\ell-1} c_b \sum_{a=0}^{\ell-1} \frac{\omega^{-ab}}{x_i} s_i^a \tag{5.6}$$

where $\nu, c_1, \ldots, c_{\ell-1} \in \mathbb{C}$ (see [DO03]; we do not suppose any more that parameters of the Dunkl operators are equal). Define $c_0 = 0$ and $c_{a\ell+b} = c_b$ for $a \in \mathbb{Z}$ and $b = 0, 1, \ldots, \ell-1$.

The next statement generalizes Proposition 5.3. The form of the singular polynomials is suggested by [CE03], where some singular polynomials for the group W were found using the residues at infinity.

Proposition 5.7. *Let $q \in \mathbb{Z}$ satisfy $1 \leqslant q \leqslant \ell - 1$. Suppose that $\nu = m + (\ell - q + s)/\ell$, $c_{q-s} = 0$ and $c_{-s} = s/\ell$ for some $m, s \in \mathbb{Z}_{\geqslant 0}$. Then the formulae*

$$f_j = \prod_{i=1}^{n} x_i^{\ell\nu}\, \mathrm{Res}_{z=0}\, z^{-\ell m - 1} \prod_{i=1}^{n} \left(1 - \frac{z^\ell}{x_i^\ell}\right)^{\nu} \frac{x_j^q \, dz}{x_j^\ell - z^\ell} \tag{5.8}$$

define singular polynomials; that is, $\nabla_i f_j = 0$ for any $i, j = 1, \ldots, n$, where the operator ∇_i is given by (5.6) (and in (5.8) it is assumed that $x_i \neq 0$ for all i). These polynomials are homogeneous *of degree* $(n-1)(m\ell+\ell-q)+ns$*, and they span an irreducible* n*-dimensional representation of* W*.*

Proof. Calculating the residue in (5.8) explicitly yields

$$f_j=(-1)^m\sum_{\substack{k_1,\ldots,k_n=0\\k_1+\cdots+k_n=m}}^{\infty}\binom{\nu-1}{k_j}x_j^{\ell(m-k_j)+s}\prod_{\substack{i=1\\i\neq j}}^{n}\binom{\nu}{k_i}x_i^{\ell(\nu-k_i)},$$

where $\binom{\alpha}{k}=\alpha(\alpha-1)\cdots(\alpha-k+1)/k!$ and $\binom{\alpha}{0}=1$. Since $\nu>m$, the function f_j is a homogeneous polynomial of $x_1,\ldots,x_n$, with degree $(n-1)\ell\nu+s=(n-1)(m\ell+\ell-q)+ns$. Note that all the coefficients in this expression do not vanish, and hence $f_j\neq 0$. The generators of W act on the polynomials (5.8) by the formulae

$$\sigma_{ij}^{(a)}f_i=\omega^{aq}f_j,\quad \sigma_{ij}^{(a)}f_j=\omega^{-aq}f_i,\quad \sigma_{ij}^{(a)}f_k=f_k, \tag{5.9}$$

$$s_if_i=\omega^{-s}f_i,\qquad s_if_k=\omega^{q-s}f_k, \tag{5.10}$$

where $i,j,k=1,\ldots,n$ with $i\neq j\neq k\neq i$ and $a=0,\ldots,\ell-1$. It follows that the space spanned by f_j, $j=1,\ldots,n$, is an n-dimensional irreducible representation of W.

Next, let us show that the polynomial f_j is singular. Let $i\neq j$; then, using the formulae (5.9)–(5.10), we find

$$\nabla_if_j=\partial_{x_i}f_j-\nu\sum_{a=0}^{\ell-1}\frac{1}{x_i-\omega^ax_j}(f_j-\omega^{-aq}f_i)-\sum_{b=1}^{\ell-1}c_b\sum_{a=0}^{\ell-1}\frac{\omega^{-ab}}{x_i}\omega^{a(q-s)}f_j. \tag{5.11}$$

The last term in (5.11) equals $-\ell c_{q-s}x_i^{-1}f_j=0$, as $\sum_{a=0}^{\ell-1}\omega^{ab}=0$ for $b\notin\ell\mathbb{Z}$. Thus $\nabla_if_j=\prod_{i=1}^n x_i^{\ell\nu}\operatorname{Res}_{z=0}z^{-\ell m-1}\prod_{i=1}^n(1-z^\ell/x_i^\ell)^\nu F_{ij}\,dz$, where

$$\begin{aligned}F_{ij}&=\nu\frac{\ell x_i^{\ell-1}x_j^q}{(x_i^\ell-z^\ell)(x_j^\ell-z^\ell)}-\nu\sum_{a=0}^{\ell-1}\frac{1}{x_i-\omega^ax_j}\left(\frac{x_j^q}{x_j^\ell-z^\ell}-\frac{\omega^{-aq}x_i^q}{x_i^\ell-z^\ell}\right)\\&=-\frac{\nu}{(x_i^\ell-z^\ell)(x_j^\ell-z^\ell)}\left(-\ell x_i^{\ell-1}x_j^q+\sum_{a=0}^{\ell-1}\frac{z^\ell\omega^{-aq}(x_i^q-\omega^{aq}x_j^q)+x_i^qx_j^q(x_i^{\ell-q}-\omega^{-aq}x_j^{\ell-q})}{x_i-\omega^ax_j}\right)\\&=-\frac{\nu}{(x_i^\ell-z^\ell)(x_j^\ell-z^\ell)}\Bigg(-\ell x_i^{\ell-1}x_j^q\\&\quad+\sum_{a=0}^{\ell-1}z^\ell\omega^{-aq}\sum_{b=0}^{q-1}x_i^b\omega^{a(q-1-b)}x_j^{q-1-b}+\sum_{a=0}^{\ell-1}x_i^qx_j^q\sum_{b=0}^{\ell-q-1}x_i^{\ell-q-1-b}\omega^{ab}x_j^b\Bigg).\end{aligned}$$

Thus the first double sum in F_{ij} vanishes and the last double sum equals $\ell x_i^{\ell-1}x_j^q$. Therefore $F_{ij}=0$ and $\nabla_if_j=0$ for $i\neq j$.

Now, to prove that $\nabla_jf_j=0$, it is sufficient to check that $\sum_{i=1}^n x_i\nabla_if_j=0$. Since f_j is homogeneous of order $(n-1)\ell\nu+s$, we have $\sum_{i=1}^n x_i\partial_{x_i}f_j=((n-1)\ell\nu+s)f_j$. Hence, using $c_{q-s}=0$ as previously, one obtains

$$\sum_{i=1}^n x_i\nabla_if_j=((n-1)\ell\nu+s)f_j+\prod_{k=1}^n x_k^{\ell\nu}\operatorname{Res}_{z=0}z^{-\ell m-1}\prod_{k=1}^n(1-z^\ell/x_k^\ell)^\nu F_j\,dz,$$

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where

$$F_j = -\nu \sum_{a=0}^{\ell-1} \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{x_i}{x_i - \omega^a x_j} \left(\frac{x_j^q}{x_j^\ell - z^\ell} - \frac{\omega^{-aq} x_i^q}{x_i^\ell - z^\ell} \right)$$
$$- \nu \sum_{a=0}^{\ell-1} \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{x_j}{x_j - \omega^a x_k} \left(\frac{x_j^q}{x_j^\ell - z^\ell} - \frac{\omega^{aq} x_k^q}{x_k^\ell - z^\ell} \right) - \sum_{b=1}^{\ell-1} c_b \sum_{a=0}^{\ell-1} \frac{\omega^{-ab-as} x_j^q}{x_j^\ell - z^\ell}.$$

By taking into account that $x_i/(x_i - \omega^a x_j) + x_j/(x_j - \omega^{-a} x_i) = 1$, we obtain

$$F_j = -\nu \sum_{a=0}^{\ell-1} \sum_{\substack{i=1 \\ i \neq j}}^{n} \left(\frac{x_j^q}{x_j^\ell - z^\ell} - \frac{\omega^{-aq} x_i^q}{x_i^\ell - z^\ell} \right) - \ell c_{-s} \frac{x_j^q}{x_j^\ell - z^\ell} = - \frac{x_j^q}{x_j^\ell - z^\ell}((n-1)\ell\nu + s),$$

where we have used the assumption that $c_{-s} = s/\ell$. Hence we deduce that $\sum_{i=1}^n x_i \nabla_i f_j = 0$. $\square$

Remark 5.12. In the case where the parameters satisfy $\ell \,|(s-q)$, the singular polynomials (5.8) appeared earlier in [CE03, Proposition 4.1], where they were presented using the residue at infinity. In other cases, the space spanned by (5.8) does not contain the singular polynomials from [CE03, Proposition 4.1], except for the case where $n = 2$ and $s = 0$ and the case where $n = 1$.

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References

Abr09 D. Abriani, *Frobenius manifolds associated to Coxeter groups of type E_7 and E_8*, Preprint (2009), arXiv:0910.5453.

BP10 M. Balagovic and A. Puranik, *Irreducible representations of the rational Cherednik algebra associated to the Coxeter group H_3*, Preprint (2010), arXiv:1004.2108.

BEG03 Yu. Berest, P. Etingof and V. Ginzburg, *Finite-dimensional representations of rational Cherednik algebras*, Int. Math. Res. Not. **2003** (2003), 1053–1088.

Chm06 T. Chmutova, *Representations of the rational Cherednik algebras of dihedral type*, J. Algebra **297** (2006), 542–565.

CE03 T. Chmutova and P. Etingof, *On some representations of the rational Cherednik algebra*, Represent. Theory **7** (2003), 641–650.

Dub96 B. Dubrovin, *Geometry of 2D topological field theories*, in *Integrable systems and quantum groups* (Montecatini Terme, Italy 1993), Lecture Notes in Mathematics, vol. 1620 (Springer, Berlin, 1996), 120–348.

Dub04 B. Dubrovin, *On almost duality for Frobenius manifolds*, in *Geometry, topology, and mathematical physics*, American Mathematical Society Translations Series 2, vol. 212 (American Mathematical Society, Providence, RI, 2004), 75–132.

Dun89 C. F. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311** (1989), 167–183.

Dun98 C. F. Dunkl, *Intertwining operators and polynomials associated with the symmetric group*, Monatsh. Math. **126** (1998), 181–209.

Dun03 C. Dunkl, *Special functions and generating functions associated with reflection groups*, J. Comput. Appl. Math. **153** (2003), 181–190.

Dun04 C. F. Dunkl, *Singular polynomials for the symmetric groups*, Int. Math. Res. Not. **2004** (2004), 3607–3635.

Dun05 C. F. Dunkl, *Singular polynomials and modules for the symmetric groups*, Int. Math. Res. Not. **2005** (2005), 2409–2436.

DJO94 C. F. Dunkl, M. F. E. de Jeu and E. M. Opdam, *Singular polynomials for finite reflection groups*, Trans. Amer. Math. Soc. **346** (1994), 237–256.

DO03 C. F. Dunkl and E. M. Opdam, *Dunkl operators for complex reflection groups*, Proc. Lond. Math. Soc. **86** (2003), 70–108.

EYY93 T. Eguchi, Y. Yamada and S.-K. Yang, *Topological field theories and the period integrals*, Modern Phys. Lett. A **8** (1993), 1627–1637.

Eti07 P. Etingof, *Calogero-Moser systems and representation theory*, Zurich Lectures in Advanced Mathematics (European Mathematical Society, Zürich, 2007).

EG02 P. Etingof and V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), 243–348.

ES09 P. Etingof and E. Stoica, *Unitary representations of rational Cherednik algebras*, Represent. Theory **13** (2009), 349–370 (with an appendix by S. Griffeth).

Gor03 I. Gordon, *On the quotient ring by diagonal invariants*, Invent. Math. **153** (2003), 503–518.

GG09 I. Gordon and S. Griffeth, *Catalan numbers for complex reflection groups*, Preprint (2009), arXiv:0912.1578.

Hum90 J. E. Humphreys, *Reflection groups and Coxeter groups* (Cambridge University Press, Cambridge, 1990).

KL11 S. Krivonos and O. Lechtenfeld, *Many-particle mechanics with $D(2,1;\alpha)$ superconformal symmetry*, J. High Energy Phys. **2011** (2011), 42.

SYS80 K. Saito, T. Yano and J. Sekiguchi, *On a certain generator system of the ring of invariants of a finite reflection group*, Comm. Algebra **8** (1980), 373–408.

Tal10 V. Talamini, *Flat bases of invariant polynomials and $\hat{P}$-matrices of E_7 and E_8*, J. Math. Phys. **51** (2010), 023520.

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