SOME RESULTS ON CONFIGURATIONS

JENNIFER WALLIS

(Received 14 May 1969; revised 23 September 1969)

Communicated by B. Mond

A (v, k, λ) configuration is conjectured to exist for every v, k and λ satisfying
$\lambda(v-1) = k(k-1)$
and
$k-\lambda$ is a square if v is even,
$x^2 = (k-\lambda)y^2+(-1)^{(v-1)/2}\lambda z^2$ has a solution in integers x, y and z not all zero for v odd.
See Ryser [5, p. 111] for further discussion.

Necessary conditions for the existence of (b, v, r, k, λ) configurations are that

$$bk = vr$$

$$r(k-1) = \lambda(v-1).$$

We write I for the identity matrix and J for the matrix with every element $+1$. In the case of block matrices, $(X)_{ij}$ means the matrix whose (i, j)th block is X; for example, $(T^{i-j})_{ij}$ is the matrix whose (i, j)th block is T^{i-j}. We define the *Kronecker product* of two matrices $A = (a_{ij})$ of order $m \times n$ and B of any order as the $m \times n$ block matrix

$$A \times B = (a_{ij}B)_{ij}.$$

THEOREM 1. *There exists a $(q(q^2+2), q(q+1), q)$ configuration whenever q is a prime.*

Takeuchi [7] and Ahrens and Szekeres [1] have proven that Theorem 1 holds for all prime powers q. Our method can be extended to $q = 2^2, 2^3, 2^4, 3^2, 3^3$ or 7^3. We include Theorem 1 as our method is entirely different to the others' and closely connected to the proof of Theorem 2.

THEOREM 2. *A $(q(k^2+\lambda), qk, k^2+\lambda, k, \lambda)$ configuration exists whenever a (q, k, λ) configuration exists and q is a prime power.*

THEOREM 3. *If there exists a matrix N of odd order $v-1$ with zero diagonal and every other element $+1$ or -1, such that $NJ = JN = 0$ and*

$$NN^T = (v-1)I_{v-1} - J_{v-1},$$

then there is a $(2(v-1), v, v-1, \frac{1}{2}v, \frac{1}{2}(v-2))$ configuration.

COROLLARY 4: *If v is the order of a skew-Hadamard or n-type matrix (see [8] for definitions) then there is a $(2(v-1), v, v-1, \frac{1}{2}v, \frac{1}{2}(v-2))$ configuration.*

1. Preliminary remark

We require that there exist (0, 1) matrices R_i, $0 \leqq i \leqq q-1$, Q of order q^2 and $\bar{Q}$ which is $kq \times q^2$, k an integer less than q, which together with P (defined in (iv) below) satisfy the following conditions

$$(1)\quad \begin{cases} \text{(i)} & PR_j^T = J \times J \\ \text{(ii)} & R_i R_j^T = J \times J \quad i \neq j \\ \text{(iii)} & \sum_{i=0}^{q-1} R_i R_i^T = q^2 I \times I + q(J-I) \times J \\ \text{(iv)} & P = I \times J, \quad PP^T = qI \times J \\ \text{(v)} & QQ^T = qI \times I + (J-I) \times J \\ \text{(vi)} & \bar{Q}\bar{Q}^T = qI_{kq} + (J_k - I_k) \times J \\ \text{(vii)} & J_{kq}\bar{Q} = k\bar{J} \\ \text{(viii)} & \bar{Q}J_{q^2} = q\bar{J}. \end{cases}$$

In formula (1), unless subscripted otherwise, I and J are of order q and $\bar{J}$ is the $kq \times q^2$ matrix with every element $+1$.

We will show in § 3 some cases where these conditions are satisfied.

2. Constructions

LEMMA 5. *If P, a (0, 1) matrix, is defined as in (1, iv), and if (0, 1) matrices R_i, $0 \leqq i \leqq q-1$ satisfying conditions (1, i, ii, iii) exist then there exists a $(q^2(q+2), q(q+1), q)$ configuration.*

PROOF. It is easily seen that this triplet satisfies the necessary conditions for (v, k, λ) configurations.

Let S be the $q^2(q+2)$ block matrix given by

$$S = \begin{bmatrix} 0 & P & R_0 & R_1 & \cdots & R_{q-3} & R_{q-2} & R_{q-1} \\ R_{q-1} & 0 & P & R_0 & \cdots & R_{q-4} & R_{q-3} & R_{q-2} \\ & \vdots & & & & & \vdots & \\ R_0 & R_1 & R_2 & R_3 & \cdots & R_{q-1} & 0 & P \\ P & R_0 & R_1 & R_2 & \cdots & R_{q-2} & R_{q-1} & 0 \end{bmatrix}$$

https://doi.org/10.1017/S1446788700009848 Published online by Cambridge University Press

then

$$SS^T = I_{q+2} \times \{PP^T + \sum_{i=0}^{q-1} R_i R_i^T\} + (J_{q+2} - I_{q+2}) \times qJ \times J$$
$$= q^2 I_r + qJ_r,$$

where $r = q^2(q+2)$.

Every element of s is 0 or 1 so s is the incidence matrix of a $(q^2(q+2), q(q+1), q)$ configuration.

LEMMA 6. *If there exists a* (0, 1) *matrix* $\bar{Q}$ *satisfying the conditions* (1, *vi*, *vii*, *viii*) *and a* (q, k, λ) *configuration exists then there exists a* $(q(k^2+\lambda), qk, k^2+\lambda, k, \lambda)$ *configuration.*

PROOF. *A* (q, k, λ) configuration exists, so

$$\lambda(q-1) = k(k-1);$$

hence it is easily verified that the five numbers satisfy the necessary conditions for (b, v, r, k, λ) configurations.

Let V be the incidence matrix of the (q, k, λ) configuration. Then A defined by

$$A^T = [I_k \times V, \bar{Q}, \bar{Q}, \cdots, \bar{Q}]$$

($\bar{Q}$ occuring λ times), has k non-zero elements in every row and $\lambda q + k = k^2 + \lambda$ non-zero elements in each column. Now

$$A^T A = I_k \times VV^T + \lambda \bar{Q}\bar{Q}^T$$
$$= (k - \lambda + \lambda q) I_{kq} + \lambda J_{kq}$$
$$= k^2 I_{qk} + \lambda J_{kq};$$

so A is the incidence matrix of the required configuration.

PROOF OF THEOREM 3. Since N has zero diagonal and every other element $+1$ or -1, C and D defined (with I and J of order $v-1$) by

$$C = \tfrac{1}{2}(N+I+J)$$
$$D = \tfrac{1}{2}(N-I+J)$$

are (0, 1) matrices. Now

$$CC^T + DD^T = \tfrac{1}{2}(NN^T + I + (v-1)J) = \tfrac{1}{2}vI + \tfrac{1}{2}(v-2)J$$

and

$$JC = \tfrac{1}{2}vJ = CJ$$
$$JD = \tfrac{1}{2}(v-2)J = DJ.$$

https://doi.org/10.1017/S1446788700009848 Published online by Cambridge University Press

We define ω_v, ω_b and e to be the vectors of v, b and $(v-1)$ l's respectively and A^T by

$$A^T = \begin{bmatrix} D & C \\ e & 0 \end{bmatrix}.$$

A is $2(v-1)\times v$, and

$$\omega_v A^T = \tfrac{1}{2}v\omega_b, \quad A^T\omega_b^T = (v-1)\omega_v^T,$$

$$A^TA = \begin{bmatrix} D & C \\ e & 0 \end{bmatrix}\begin{bmatrix} D^T & e^T \\ C^T & 0 \end{bmatrix} = \begin{bmatrix} DD^T+CC^T & \tfrac{1}{2}(v-2)e^T \\ \tfrac{1}{2}(v-2)e & v-1 \end{bmatrix} = \frac{v}{2}I_v + \frac{v-2}{2}J_v.$$

So A is the incidence matrix of a $(2(v-1), v, v-1, \tfrac{1}{2}v, \tfrac{1}{2}(v-2))$ configuration.

3. Matrices satisfying condition (1)

We shall show that (1) can be satisfied for all primes q and that matrices Q and $\bar{Q}$ can be found for q any prime power. These facts together with lemmas 5 and 6 complete the proofs of Theorems 1 and 2.

In this section T will be used for the circulant matrix of order q given by

$$T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}. \tag{2}$$

3.1 *The case of q prime*

Choose q block matrices R_i of order q^2, $0 \leqq i \leqq q-1$, thus

$$R_i = \begin{bmatrix} I & T^i & T^{2i} & \cdots & T^{(q-1)i} \\ T^{(q-1)i} & I & T^i & \cdots & T^{(q-2)i} \\ \vdots & & & & \vdots \\ T^i & T^{2i} & T^{3i} & \cdots & I \end{bmatrix} = (T^{(m-s)i})_{sm}$$

and let

$$Q = \begin{bmatrix} I & I & I & \cdots & I \\ I & T & T^2 & \cdots & T^{q-1} \\ I & T^2 & T^{2\cdot 2} & \cdots & T^{(q-1)2} \\ \vdots & & & & \vdots \\ I & T^{q-1} & T^{2(q-1)} & \cdots & T^{(q-1)(q-1)} \end{bmatrix} = (T^{(i-1)(j-1)})_{ij}$$

https://doi.org/10.1017/S1446788700009848 Published online by Cambridge University Press

and

$$\bar{Q} = \begin{bmatrix} I & I & I & \cdots & I \\ I & T & T^2 & & T^{q-1} \\ I & T^2 & T^{2\cdot 2} & \cdots & T^{(q-1)2} \\ \vdots & & & & \vdots \\ I & T^{k-1} & T^{2(k-1)} & \cdots & T^{(q-1)(k-1)} \end{bmatrix}.$$

We now verify that these matrices satisfy the conditions (1). Note that $JT^i = J$ for all i, so (i), (vii) and (viii) are immediate.

(ii)
$$\begin{aligned} R_i R_j^T &= \Big(\sum_{m=0}^{q-1} T^{(m-s)i} T^{(n-m)j}\Big)_{s,n} \\ &= \Big(\sum_{m=0}^{q-1} T^{m(i-j)+nj-si}\Big)_{s,n} \\ &= \Big(\sum_{r=0}^{q-1} T^r\Big)_{s,n} = (J)_{s,n} = J \times J \qquad \text{for } i \neq j. \end{aligned}$$

(iii)
$$\begin{aligned} R_i R_i^T &= \Big(\sum_{m=0}^{q-1} T^{(m-s)i} T^{(n-m)i}\Big)_{s,n} \\ &= (qT^{(n-s)i})_{s,n} \\ &= qR_i; \end{aligned}$$

$$\sum_{i=0}^{q-1} R_i = \begin{bmatrix} qI & J & \cdots & J \\ J & qI & \cdots & J \\ \vdots & & & \vdots \\ J & J & \cdots & qI \end{bmatrix} = qI \times I + (J-I) \times J,$$

so the result follows.

(v)
$$\begin{aligned} QQ^T &= \Big(\sum_{m=1}^{q} T^{(i-1)(m-1)} T^{-(m-1)(j-1)}\Big)_{ij} \\ &= \Big(\sum_{m=1}^{q} T^{(m-1)(i-j)}\Big)_{ij} \end{aligned}$$

then if $i = j$ we have $\sum_{m=1}^{q} I = qI$, and if $i \neq j$, we have $\sum_{m=1}^{q} T^{(m-1)(i-j)} = J$, which gives the result.

(vi) This follows since we have chosen $\bar{Q}$ as the first kq rows of Q.

3.2 *The case of q a prime power*

In this case, unless stated otherwise, I, J are of order q.

It is known that a $(q^2+q+1, q+1, 1)$ configuration exists whenever q is a

https://doi.org/10.1017/S1446788700009848 Published online by Cambridge University Press

prime power. If we form the incidence matrix of this configuration then we may rearrange its rows and columns until the following matrix is obtained:

$$A = \begin{bmatrix} 1 & e & 0 \\ e^T & 0 & I \times e \\ 0 & I \times e^T & N \end{bmatrix}$$

where $e = [1, 1, \cdots, 1]$ is of size $1 \times q$ and N is of size p^2.

Now $AA^T = pI_r + J_r$, where $r = p^2+p+1$, and

$$AA^T = \begin{bmatrix} q+1 & e & e \times e \\ e^T & qI+J & (I \times e)N^T \\ e^T \times e^T & N(I \times e^T) & I \times J + NN^T \end{bmatrix}$$

so

(a) N is of order q^2;

(b) $NN^T = qI \times I + J \times J - I \times J = qI \times I + (J-I) \times J$;

(c) $N(I \times e^T) = J'$ where J' is of size $q^2 \times q$.

This last condition implies that if N is partitioned into q^2 block matrices N_i then each block matrix N_i has exactly one element in each row and column. Now rearrange the columns of N keeping the first $q+1$ rows of A unaltered until the first row of block matrices in the partitioned N are all I_q and similarly alter the rows of N keeping the first $q+1$ columns of A unaltered until the first column of block matrices in the partitioned N are all I_q. Then this new matrix obtained from N satisfies all the conditions for the matrix Q. We again choose $\bar{Q}$ to consist of the first kq rows of Q.

3.3 *The case of q certain prime powers*

We have not been able to derive enough information from the matrix N to ensure the existence of the matrices R_i when q is a general prime power. However, as noted in the introduction, we can construct these matrices for the following value of q:

$$2^2,\ 2^3,\ 2^4,\ 3^2,\ 3^3,\ 7^2.$$

The methods used do not generalize.

4. Remarks on numerical results

The block designs given by Theorem 2 with $k > 4$ all have $r > 20$, and are outside the range of the tables in [2], [3], [4] and [6]. Consequently it is hard to check whether individual designs are new. We observe, however, that the existence of a (16,6, 2) configuration yields a design with parameters (608, 96, 38, 6, 2); this is the multiple by 2 of the design (304, 96, 19, 6, 1) which is listed as unknown by Sprott [6]. Also the (11, 6, 3) configuration yields a (429, 66, 39, 6, 3) configuration, which is a multiple by 3 of a (143, 66, 13, 6, 1) design. The solution of the latter design in [4] does not appear to have arisen as one of a series of designs. We note in passing that Hall [3] mistakenly lists (143, 66, 13, 6, 1) as 'solution unknown'.

Theorem 3 yields a (34, 18, 17, 9, 8) configuration, which was previously unknown according to [6]. It also gives a (26, 14, 13, 7, 6) configuration, which was already known but was completely omitted from Hall's list, as well as a number of apparently new configurations with $r > 20$.

References

[1] R. Ahrens and G. Szekeres, 'On a combinatorial generalization of 27 lines associated with a cubic surface', *J. Australian Math. Soc.* 10 (1969), 485–492.

[2] R. A. Fisher and F. Yates, *Statistiol Tables for Biological, Agricultural, and Medical Research, 2nd* ed. (Oliver and Boyd Ltd., London, 1943).

[3] Marshall Hall Jr., *Combinatorial Theory* (Blaisdell, Waltham, Mass, 1967).

[4] C. Radhaskrishna Rao, 'A study of BIB designs with replications 11 to 15', *Sankhyā*, 23 (1961) 117–127.

[5] H. J. Ryser, *Combinatorial Mathematics* (Carus Monograph No. 14, Wiley, New York, 1963).

[6] D. A. Sprott, 'Listing of BIB designs from $r = 16$ to 20', *Sankhyā, Series A*, 24 (1962), 203–204.

[7] K. Takeuchi, 'On the construction of a series of BIB designs', *Rep. Stat. Appl. Res., JUSE* 10 (1963). 48.

[8] Jennifer Wallis, 'Some (1, −1) matrices', *J. Combinatorial Theory*, (to appear).

University of Newcastle
New South Wales 2308

https://doi.org/10.1017/S1446788700009848 Published online by Cambridge University Press