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The Operator Amenability of Uniform Algebras

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Abstract. We prove a quantized version of a theorem by M. V. Sheĭnberg: A uniform algebra equipped with its canonical, *i.e.*, minimal, operator space structure is operator amenable if and only if it is a commutative C^*-algebra.

In [Joh 1], B. E. Johnson introduced the notion of an amenable Banach algebra. It is an active area of research to determine, for a particular class of Banach algebras, which algebras in that class are the amenable ones. For example, Johnson himself proved that a locally compact group G is amenable if and only if its group algebra $L^1(G)$ is amenable (this characterization motivates the choice of terminology). The characterization of the amenable C^*-algebras is a deep result due to several authors (see [Run, Chapter 6] for a self-contained exposition): A C^*-algebra is amenable precisely when it is nuclear. The amenability of algebras of compact operators on a Banach space E is related to certain approximation properties of E ([G-J-W]). In [Sheĭ], M. V. Sheĭnberg showed that a uniform Banach algebra is amenable if and only if it is already a commutative C^*-algebra. In this note, we prove a quantized version of Sheĭnberg's theorem (and thus answer [Run, Problem 31]).

Our reference for the theory of operator spaces is [E-R], whose notation we adopt.

A Banach algebra which is also an operator space is called *a completely contractive Banach algebra* if multiplication is a completely contractive bilinear map. For any Banach algebra $\mathfrak{A}$, the maximal operator space $\max \mathfrak{A}$ is a completely contractive Banach algebra. In [Rua 1], Z.-J. Ruan introduced a variant of Johnson's definition of amenability for completely contractive Banach algebras called *operator amenability* (see [E-R, Section 16.1] and [Run, Chapter 7]). A Banach algebra $\mathfrak{A}$ is amenable in the sense of [Joh 1] if and only if $\max \mathfrak{A}$ is operator amenable ([E-R, Proposition 16.1.5]). Nevertheless, operator amenability is generally a much weaker condition than amenability. For any locally compact group G, the Fourier algebra $A(G)$ carries a natural operator space structure as the predual of $\mathrm{VN}(G)$. In [Rua 1], Ruan showed that $A(G)$—equipped with this natural operator space structure—is operator amenable if and only if G is amenable; on the other hand, there are even compact groups G for which $A(G)$ fails to be amenable ([Joh 2]).

Let $\mathfrak{A}$ be a uniform algebra, *i.e.*, a closed subalgebra of a commutative C^*-algebra. The canonical operator space structure $\mathfrak{A}$ inherits from this C^*-algebra turns it into

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a completely contractive Banach algebra. By [E-R, Proposition 3.3.1], this canonical operator space structure is just $\min \mathfrak{A}$.

We have the following operator analogue of Sheĭnberg's theorem:

Theorem *Let* $\mathfrak{A}$ *be a uniform algebra such that* $\min \mathfrak{A}$ *is operator amenable. Then* $\mathfrak{A}$ *is a commutative* C^**-algebra.*

Proof Without loss of generality suppose that $\mathfrak{A}$ is unital with compact character space Ω. We assume towards a contradiction that $\mathfrak{A} \subsetneq \mathcal{C}(\Omega)$. Combining the Hahn-Banach theorem with the Riesz representation theorem, we obtain a complex Borel measure $\mu \neq 0$ on Ω such that

$$\int_\Omega f \, d\mu = 0 \quad (f \in \mathfrak{A}).$$

Let $H := L^2(|\mu|)$. The canonical representation of $\mathcal{C}(\Omega)$ on H as multiplication operators turns H into a left Banach $\mathcal{C}(\Omega)$-module. Let H_c denote H with its column space structure (see [E-R, p. 54]). Then H_c is a left operator $\mathcal{C}(\Omega)$-module.

Let K denote the closure of $\mathfrak{A}$ in H; clearly, K, is an $\mathfrak{A}$-submodule of H. Trivially, K is complemented in H, and by [E-R, Theorem 3.4.1], K_c is completely complemented in H_c, *i.e.* the short exact sequence

$$(*) \qquad \{0\} \to K_c \to H_c \to H_c/K_c \to \{0\}$$

of left operator $\mathfrak{A}$-modules is admissible. Since $\min \mathfrak{A}$ is operator amenable, $(*)$ even splits: We obtain a (completely bounded) projection $P \colon H_c \to K_c$ which is also a left $\mathfrak{A}$-module homomorphism. (The required splitting result can easily be proven by a more or less verbatim copy of the proof of its classical counterpart [Run, Theorem 2.3.13].)

The remainder of the proof is like in the classical case.

For $f \in \mathcal{C}(\Omega)$, let M_f denote the corresponding multiplication operator on H. The fact that P is an $\mathfrak{A}$-module homomorphism means that

$$M_f P = P M_f \quad (f \in \mathfrak{A}).$$

Since each M_f is a normal operator with adjoint $M_{\bar{f}}$, the Fuglede-Putnam theorem implies that

$$M_{\bar{f}} P = P M_{\bar{f}} \quad (f \in \mathfrak{A})$$

as well, and from the Stone-Weierstraß theorem, we conclude that

$$M_f P = P M_f \quad \big(f \in \mathcal{C}(\Omega)\big).$$

Since $\mathfrak{A}$ is unital, this implies that $K = H$.

Let $f \in \mathfrak{C}(\Omega)$ be arbitrary. Then there is a sequence $(f_n)_{n=1}^\infty$ in $\mathfrak{A}$ such that $\|f - f_n\|_2 \to 0$. Hence, we have

$$\left| \int_\Omega f \, d\mu \right| = \lim_{n\to\infty} \left| \int_\Omega (f - f_n) \, d\mu \right| \leq \lim_{n\to\infty} \int_\Omega |f - f_n| \, d|\mu|$$
$$\leq \lim_{n\to\infty} |\mu|(\Omega)^{\frac{1}{2}} \|f - f_n\|_2 \to 0,$$

which is impossible because $\mu \neq 0$. ■

Corollary *The following are equivalent for a uniform algebra $\mathfrak{A}$:*

(i) $\min \mathfrak{A}$ *is operator amenable.*
(ii) $\mathfrak{A}$ *is operator amenable for any operator space structure on* $\mathfrak{A}$ *turning* $\mathfrak{A}$ *into a completely contractive Banach algebra.*
(iii) $\mathfrak{A}$ *is amenable.*
(iv) $\mathfrak{A}$ *is a commutative C^*-algebra.*

Proof (i) $\Rightarrow$ (iv) is the assertion of the theorem, and (iv) $\Rightarrow$ (iii) is well known.

(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i): The amenability of $\mathfrak{A}$ is equivalent to the operator amenability of $\max \mathfrak{A}$. Let $\mathfrak{A}$ be equipped with any operator space structure turning it into a completely contractive Banach algebra. Since

$$\max \mathfrak{A} \stackrel{\mathrm{id}}{\to} \mathfrak{A} \stackrel{\mathrm{id}}{\to} \min \mathfrak{A},$$

are surjective completely contractive algebra homomorphisms, it follows from basic hereditary properties of operator amenability ([Rua 2, Proposition 2.2]) that the operator amenability of $\max \mathfrak{A}$ entails that of $\mathfrak{A}$ and, in turn, that of $\min \mathfrak{A}$. ■

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