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CERTAIN SUMMATION FORMULAE FOR BASIC HYPERGEOMETRIC SERIES

BY
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§1. In 1927, Jackson [5] obtained a transformation connecting a

$$ {}_2\Phi_1\left[\begin{matrix} q^\alpha, q^\beta; q^{\gamma-\alpha-\beta+N} \\ q^\gamma \end{matrix}\right] $$

where N is any integer, with a

$$ {}_3\Phi_1\left[\begin{matrix} q^\alpha, q^\beta, q^N \\ q^{\alpha+\beta-\gamma+1} \end{matrix}; q\right], $$

viz.,

$$ (1) \qquad {}_2\Phi_1\left[\begin{matrix} q^\alpha, q^\beta; q^{\gamma-\alpha-\beta+N} \\ q^\gamma \end{matrix}\right] = \frac{\Gamma_q[\gamma]\Gamma_q[\gamma-\alpha-\beta]}{\Gamma_q[\gamma-\alpha]\Gamma_q[\gamma-\beta]}\, {}_3\Phi_1\left[\begin{matrix} q^\alpha, q^\beta, q^N; q \\ q^{\alpha+\beta-\gamma+1} \end{matrix}\right], $$

where $|q|>1$ and $|q^{\gamma-\alpha-\beta+N}|>1$. $\Gamma_q[X]$ being the q-analogue of the gamma function[2]. Jackson also conjuctured that it might be possible to remove the restriction that N is an integer, altogether.

The result stated by Jackson is not correct as it is unless further conditions on α and β are imposed. In fact (1) is false if neither α, β, nor N is a negative integer because under these conditions the right hand side of (1) is a divergent infinite series for $|q|>1$. Furthermore, the result (1) reduces for $N=0$ to

$$ (2) \qquad {}_2\Phi_1\left[\begin{matrix} q^\alpha, q^\beta; q^{\gamma-\alpha-\beta} \\ q^\gamma \end{matrix}\right] = \frac{\Gamma_q[\gamma]\Gamma_q[\gamma-\alpha-\beta]}{\Gamma_q[\gamma-\alpha]\Gamma_q[\gamma-\beta]}, $$

where $|q|>1$ and $|q^{\gamma-\alpha-\beta}|>1$, which is known to be false if α and β are different from negative integer [See Jackson [4] for details].

Lastly, if neither α nor β is a negative integer and N is a negative integer, the result still remains false in general. As a verification let $\alpha=\gamma$ and $N=-1$, $q=1/p$ the left hand side of (1) becomes $\prod_{s=0}^{\infty}[1-p^{\beta+2+s}/1-p^{2+s}]\neq 0$ (since β is different from a negative integer) whereas the right hand side of (1) becomes zero and therefore the result is false.

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[2] For definition and properties of this function please see Jackson [4, 5] and references therein.

Hence under the conditions $|q|>1$ and $|q^{\gamma-\alpha-\beta+N}|>1$, (1) is false if neither α nor β is a negative integer, whatsoever be N. Jackson got the incorrect result because in his proof for (1) he made use of the incorrect relation (2). It may be remarked that (2) is true only if α or β is a negative integer or $|q|<1$ and $|q^{\gamma-\alpha-\beta}|<1$.

In this paper we prove that if a or b or c is of the form of q^{-n}, n a positive integer, then for $|q|<1$ and $|ec/ab|<1$,

$$(3) \qquad {}_2\Phi_1\begin{bmatrix} a, b; ec/ab \\ e \end{bmatrix} = \prod\begin{bmatrix} e/a, e/b; \\ e, e/ab \end{bmatrix} {}_3\Phi_1\begin{bmatrix} a, b; c; q \\ abq/e \end{bmatrix},$$

where

$$\prod\begin{bmatrix} a_1, a_2, \ldots, a_r; \\ b_1, b_2, \ldots, b_s \end{bmatrix}$$

is defined to be the infinite product

$$\prod_{j=0}^{\infty}\left[\frac{(1-a_1q^j)(1-a_2q^j)\cdots(1-a_rq^j)}{(1-b_1q^j)(1-b_2q^j)\cdots(1-b_sq^j)}\right].$$

In the event of a or b being of the form q^{-n}, n a positive integer, the conditions $|q|<1$ and $|ec/ab|<1$ can be waived off, since under these conditions both series of (3) reduce to polynomials. Hence the result (3) is equivalent to Jackson's result (1) if either α or β is a negative integer.

The result (3) gives the summations of terminating ${}_2\Phi_1$ with arguments q^2, q^3, etc. These results are then used to give alternative proof of some of the summation theorems proved earlier by Lakin [7] by using q-difference equations. The paper is concluded by proving summation formula for terminating ${}_3\Phi_2$ and a curious summation formula for a non-terminating

$${}_2\Phi_1\begin{bmatrix} q^\alpha, q^m; \\ q^\beta \end{bmatrix}; q\Bigg]$$

where m is a positive integer. Both these summations are believed to be new.

§2. Sears [8; equation (10.2)] has shown that if $|ef/abc|<1$ and $|q|<1$

$$(4) \quad {}_3\Phi_2\begin{bmatrix} a, b, c; ef/abc \\ e, f \end{bmatrix} = \prod\begin{bmatrix} e/a, e/b; \\ e, e/ab \end{bmatrix} {}_3\Phi_2\begin{bmatrix} a, b, f/c; q \\ abq/e, f \end{bmatrix}$$

$$+\prod\begin{bmatrix} a, b, f/c, ef/ab; \\ ab/e, f, ef/abc \end{bmatrix} {}_3\Phi_2\begin{bmatrix} e/a, e/b, ec/ab; q \\ qe/ab, ef/ab \end{bmatrix}.$$

In this transformation replacing c by f/c and then letting $f\to 0$ we get that if

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$|ec/ab|<1$ and $|q|<1$

$$(5)\qquad {}_2\Phi_1\begin{bmatrix} a, b; ec/ab \\ e \end{bmatrix} = \prod\begin{bmatrix} e/a, e/b; \\ e, e/ab \end{bmatrix} {}_3\Phi_1\begin{bmatrix} a, b, c; q \\ abg/e \end{bmatrix}$$
$$+\prod\begin{bmatrix} a, b, c; \\ ab/e, ec/ab \end{bmatrix} {}_2\Phi_1\begin{bmatrix} e/a, e/b; q \\ qe/ab \end{bmatrix}.$$

From the above it is clear that the second term vanishes if a or b or c is of the form q^{-n} and in that case we get that

$$(6)\qquad {}_2\Phi_1\begin{bmatrix} a, b; ec/ab \\ e \end{bmatrix} = \prod\begin{bmatrix} e/a, e/b; \\ e, e/ab \end{bmatrix} {}_3\Phi_1\begin{bmatrix} a, b, c; q \\ abq/e \end{bmatrix},$$

where either a or b or c is of the form q^{-n}.

In this result setting $a=q^{-n}$, $c=1/q$, $b=q^{1-\beta-n}$, $e=q^{1-\alpha-n}$ and rewriting the ${}_2\Phi_1$ in the reverse order by using the transformation

$${}_2\Phi_1\begin{bmatrix} q^{-n}, q^b; \\ q^e \end{bmatrix}; z = \frac{(-)^n[q^b]_n}{[q^e]_n} z^n q^{-n(n+1)/2}$$
$$\times {}_2\Phi_1\begin{bmatrix} q^{-n}, q^{1-e-n}; (q/z)^{1+e-b+n} \\ q^{1-b-n} \end{bmatrix},$$

we get

$$(7)\qquad {}_2\Phi_1\begin{bmatrix} q^{-n}, q^{\alpha}; q^2 \\ q^{\beta} \end{bmatrix} = \frac{[q^{\beta-\alpha}]_{n-1}q^{\alpha(n-1)}}{[q^{\beta}]_n}\{1+q^{n+\alpha}-q^{\beta-1+n}-q^n\}.$$

On the other hand, if we set $a=q^{-n}$, $c=1/g^2$, $b=q^{1-\beta-n}$, $e=q^{1-\alpha-n}$ and rewrite the resulting ${}_2\Phi_1$ in the reverse order, we get

$${}_2\Phi_1\begin{bmatrix} q^{-n}, q^{\alpha}; q^3 \\ q^{\beta} \end{bmatrix} = \frac{[q^{\beta-\alpha}]_{n-2}q^{n(\alpha+2)}}{[q^{\beta}]_n}\{[q^{\beta-\alpha+n-2}]_2$$
$$+q^{1-\alpha-n}(1+q)(1-q^{\beta-\alpha+n-2})(1-q^n)(1-q^{\beta+n-1})$$
$$+q^{-2n-2\alpha}[q^{n-1}]_2[q^{\beta+n-2}]_2\}.$$

From the above it is clear that the sum of a terminating ${}_2\Phi_1$ with argument q^3, q^4, etc. could be written out without any difficulty.

It might be of interest to point out that setting $c=1/q$ in (6), we get a summation for a

$${}_2\Phi_1\begin{bmatrix} a, b; e/abq \\ e \end{bmatrix}$$

mentioned by Bailey [2].

Next, we show that an alternative simple proof can be given for some of the summation theorems derived by Lakin by operator methods. In fact we will give an alternative proof (Lakin [7; (27)])

$$(8)\qquad {}_3\Phi_2\left[\begin{matrix} q^a, q^b, q^{-N}; q \\ q^{1+d}, q^{1+e} \end{matrix}\right] = \frac{[q^{1+d-a}]_N q^{Na} [q^{1+e-a}]_{N-1}}{[q^{1+d}]_N [q^{1+e}]_N (1-q^{e-b})} k_N,$$

where

$$d+e=a+b-N$$

and

$$k_N = q^{e+N(a-e)-N^2}(1+q^{-e}-q^{-a}-q^N)-q^{e-b}(1-q^{e-a+N}).$$

$$\text{LHS} = \sum_{n=0}^{N} \frac{[q^a]_n [q^{-N}]_n}{[q]_n [q^{1+d}]_n} q^n \sum_{r=0}^{n} \frac{[q^{-n}]_r [q^{1+e-b}]_r}{[q]_r [q^{1+e}]_r} q^{r(b+n)}$$

$$= \sum_{r=0}^{N} \frac{[q^{1+e-b}]_r [q^a]_r [q^{-N}]_r}{[q]_r [q^{1+e}]_r [q^{1+d}]_r} (-)^r q^{r(b+r/2+1/2)} {}_2\Phi_1\left[\begin{matrix} q^{-N+r}, q^{a+r} \\ q^{1+d+r} \end{matrix}; q\right]$$

$$= \frac{[q^{1+d-a}]_N}{[q^{1+d}]_N} q^{Na} \sum_{r=0}^{N} \frac{[q^a]_r [q^{-N}]_r [q^{e+1-b}]_r}{[q]_r [q^{1+e}]_r [q^{e-b}]_r} q^{r(b-d)}$$

$$= \frac{[q^{1+d-a}]_N}{[q^{1+d}]_N} \frac{q^{Na}}{(1-q^{e-b})} \left\{ {}_2\Phi_1\left[\begin{matrix} q^{-N}, q^a; q^{b-d} \\ q^{1+e} \end{matrix}\right] - q^{e-b} {}_2\Phi_1\left[\begin{matrix} q^{-N}, q^a; q^{1+b-d} \\ q^{1+e} \end{matrix}\right] \right\}$$

$$= \frac{[q^{1+d-a}]_N}{[q^{1+d}]_N} \frac{q^{Na}}{(1-q^{e-b})} \left\{ {}_2\Phi_1\left[\begin{matrix} q^{-N}, q^a; q^{e-a+N} \\ q^{1+e} \end{matrix}\right] - q^{e-b} {}_2\Phi_1\left[\begin{matrix} q^{-N}, q^a; q^{1+e-a+N} \\ q^{1+e} \end{matrix}\right] \right\}$$

$$= \frac{[q^{1+d-a}]_N q^{Na}}{[q^{1+d}]_N (1-q^{e-b})} \left\{ \frac{[q^{1+e-a}]_{N-1} q^{e+N(a-e)-N^2}}{[q^{1+e}]_N} (q^{-e}+1-q^N-q^{-a}) - q^{e-b} \frac{[q^{1+e-a}]_N}{[q^{1+e}]_N} \right\}.$$

The first of the ${}_2\Phi_1$ is summed by (7), whereas the second of the ${}_2\Phi_1$ is summed by the q-analogue of Gauss' theorem [9; p. 247]. We then get (8) on some reduction.

Following exactly similar procedure alternative proofs for the summation theorems 28 and 29 of Lakin [7] can be furnished.

Next we show that: If l is any positive integer or zero then

(i) $${}_2\Phi_1\begin{bmatrix} q^{-n}, bq^k; q^l \\ b \end{bmatrix} = 0 \quad \text{for} \quad k \leq n$$

(ii) $${}_2\Phi_1\begin{bmatrix} q^{-n}, bq^{n+1}; 1 \\ b \end{bmatrix} = \frac{[q]_n}{[b]_{n+1}}(-)^{n+1}q^{n(n+1)/2}b^{n+1}$$

(iii) $${}_2\Phi_1\begin{bmatrix} q^{-n}, bq^{n+2}; 1 \\ b \end{bmatrix} = \frac{[q^2]_{n-1}}{[b]_{n+2}}(-)^{n+1}b^{n+1}q^{\frac{1}{2}n^2+n} \times\{1-q^{2+n}-bq^{\frac{1}{2}n+1}(1-q^{n+1})\}.$$

To prove these summations let us rewrite

$${}_2\Phi_1\begin{bmatrix} q^{-n}, bq^k; q^l \\ b \end{bmatrix}$$

as

$$\frac{1}{[b]_k}\sum_{s=0}^{n}\frac{[q^{-n}]_s}{[q]_s}\frac{[bq^s]_\infty}{[bq^{s+k}]_\infty}q^{sl} = \frac{1}{[b]_k}\sum_{s=0}^{n}\frac{[q^{-n}]_s}{[q]_s}\sum_{j=0}^{k}\frac{[q^{-k}]_j}{[q]_j}b^jq^{(k+s)j+sl}$$

$$= \frac{1}{[b]_k}\sum_{j=n+1}^{k}\frac{[q^{-k}]_j}{[q]_j}(b^jq^{kj})\frac{[q^{-n+j+l}]_\infty}{[q^{j+l}]_\infty},$$

since the terms corresponding to $j = 0, 1, \ldots, n$ are all zeros. From the above simple transformations the three summations (i), (ii), and (iii) follow readily.

On this score it might be worth mentioning that if we define

$${}_2\Phi_2\begin{bmatrix} q^a : q_1^b; z \\ q^d : q_1^c \end{bmatrix} = \sum_{m=0}^{\infty}\frac{[q^a]_{m,q}[q_1^b]_{m,q_1}}{[q^d]_{m,q}[q_1^c]_{m,q_1}}z^m; |q|, |q_1|, |z| < 1,$$

then following the above procedure it is possible to show that:

If n, k, l, m are positive integers and $q_1 = q^l$ then

(i) $${}_2\Phi_2\begin{bmatrix} q^{-nl} : q_1^{c+k}; q_1^m \\ q : q_1^c \end{bmatrix} = 0 \quad \text{if} \quad m+k<n$$

(ii) $${}_2\Phi_2\begin{bmatrix} q^{-nl} : q_1^{c+n+1}; 1 \\ q : q_1^c \end{bmatrix} = (-)^{n+1}q^{(\frac{1}{2}n^2+n(c+1)+c)}\frac{[q_1]_{nl,q}}{[q_1^c]_{n+1,q_1}}$$

The result similar to (iii) can also be written out.

§3. In this section we have proved the summation formula

(9) $${}_3\Phi_2\begin{bmatrix} q^{p+1+n}, q^{1-n}, q^{p+j}; q \\ q^{p+1}, q^{p+j+1} \end{bmatrix} = \frac{[q]_{n-1}q^{(n-1)(p+j)}}{[q^{1+p+j}]_{n-1}[q^{1+p}]_n} \times\{[q^{1-j}]_n+(-)^{n+1}[q^{p+j}]_nq^{(n+1-2j)/2}\}.$$

We begin by mentioning a q-analogue of an expansion due to Fox [3].

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(It may be remarked that the result of Fox has been proved recently by Karlsson [2].)

$$(10)\qquad {}_{A+1}\Phi_{B+1}\left[\begin{matrix} q^{b+\alpha}, q^{(a)} \\ q^{b}, q^{(b)} \end{matrix}; z\right] = \sum_{j=0}^{\infty} \begin{bmatrix}\alpha\\ j\end{bmatrix} \frac{[q^{(a)}]_j}{[q^b]_j[q^{(b)}]_j} \times z^j q^{j(j+1)+j(b-2)}\,{}_A\Phi_B\left[\begin{matrix} q^{(a)+j} \\ q^{(b)+j} \end{matrix}; z\right],$$

which follows readily on substituting the series definition for ${}_A\Phi_B$, changing the order of summation and summing the inner ${}_2\Phi_1$ by the q-analogue of Gauss' theorem. In (10) taking $A=2$, $B=1$, $b=p+1$, $\alpha=n$ (a positive integer) $a_1=1-n$, $a_2=p+j$, $b_1=p+j+1$, $z=q$ then summing the inner ${}_2\Phi_1$ by the q-analogue of Vandermonde's theorem [9; p. 247], we get that the LHS of (9)

$$=\frac{[q]_{n-1}q^{(n-1)(p+j)}}{[q^{1+p+j}]_{n-1}}\sum_{r=0}^{n-1}\frac{[q^{-n}]_r q^{r(n-j+1)}}{[q]_r[q^{1+p}]_r}[q^{p+j}]_r$$

$$=\frac{[q]_{n-1}q^{(n-1)(p+j)}}{[q^{1+p+j}]_{n-1}}\left\{{}_2\Phi_1\left[\begin{matrix} q^{-n}, q^{p+j}; q^{n-j+1} \\ q^{1+p} \end{matrix}\right]-(-)^n q^{(n+1-2j)/2}\frac{[q^{p+j}]_n}{[q^{1+p}]_n}\right\}.$$

Once again summing the ${}_2\Phi_1$ by the q-analogue of Gauss' theorem, we get (9).

Lastly, we show that

$$(11)\qquad {}_2\Phi_1\left[\begin{matrix} q^{\alpha}, q^{m}; q \\ q^{\beta} \end{matrix}\right]=\frac{[q^{\alpha}]_\infty}{[q^{\beta}]_\infty}\frac{q^{-m\alpha}}{[q^{\beta-\alpha-m}]_m}\left\{\frac{[q^{\beta-m}]_\infty}{[q^{\alpha}]_\infty}-\sum_{r=0}^{m-1}\frac{[q^{\beta-\alpha-m}]_r}{[q]_r}q^{r\alpha}\right\},$$

where $|q|<1$ and m is a positive integer.

Proof.

$$(12)\qquad \text{LHS}=\frac{[q^{\alpha}]_\infty}{[q^{\beta}]_\infty}\sum_{k=0}^{\infty}\frac{[q^m]_k}{[q]_k}q^k\frac{[q^{\beta+k}]_\infty}{[q^{\alpha+k}]_\infty}$$

$$=\frac{[q^{\alpha}]_\infty}{[q^{\beta}]_\infty}\sum_{k=0}^{\infty}\frac{[q^m]_k}{[q]_k}q^k\sum_{r=0}^{\infty}\frac{[q^{\beta-\alpha}]_r}{[q]_r}q^{r(\alpha+k)}$$

$$=\frac{[q^{\alpha}]_\infty}{[q^{\beta}]_\infty}\sum_{r=0}^{\infty}\frac{[q^{\beta-\alpha}]_r}{[q]_r}q^{r\alpha}\,{}_1\Phi_0\left[\begin{matrix} q^{m}; q^{1+r} \\ \text{---} \end{matrix}\right]$$

$$=\frac{[q^{\alpha}]_\infty}{[q^{\beta}]_\infty[q^{\beta-\alpha-m}]_m}\sum_{r=0}^{\infty}\frac{[q^{\beta-\alpha-m}]_{r+m}}{[q]_{r+m}}q^{r\alpha}$$

$$=\frac{[q^{\alpha}]_\infty q^{-m\alpha}}{[q^{\beta}]_\infty[q^{\beta-\alpha-m}]_m}\left\{{}_1\Phi_0\left[\begin{matrix} q^{\beta-\alpha-m}; q^{\alpha} \\ \text{---} \end{matrix}\right]-\sum_{r=0}^{m-1}\frac{[q^{\beta-\alpha-m}]_r}{[q]_r}q^{r\alpha}\right\}.$$

Now summing the ${}_1\Phi_0$, we get (11).

Some of the interesting special cases of (11) are

(i) Setting $m=1$, in (11) we get

$$\sum_{k=0}^{\infty}\frac{[q^\alpha]_k}{[q^\beta]_k}q^k=\frac{1}{(q^\alpha-q^{\beta-1})}\left\{1-q^{\beta-1}-\frac{[q^\alpha]_\infty}{[q^\beta]_\infty}\right\}$$

(c.f. Andrews et al [1]).

(ii) Letting m tend to infinity in (12), we get

$${}_1\Phi_1\left[\begin{matrix}q^\alpha;q\\ q^\beta\end{matrix}\right]=\frac{[q^\alpha]_\infty}{[q^\beta]_\infty[q]_\infty}\,{}_2\Phi_0[q,q^{\beta-\alpha};-;q^\alpha].$$

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References

1. G. E. Andrews, M. V. Subba Rao, and M. Vidya Sagar, *A family of combinational Identities.* Bull. Canadian Math. Soc. **15** (1972), 11–18.
2. W. N. Bailey, *Identities of the Roger–Ramanujan type.* Proc. London Math. Soc. (2) **50** (1949), 1–10.
3. C. Fox, *The expansion of hypergeometric series in terms of similar series.* Proc. London Math. Soc. (2) **26** (1927), 201–210.
4. F. H. Jackson, *Transformations of q-series.* Mess. of Math. **39** (1910), 145–153.
5. F. H. Jackson, *A new transformation of Heinean series.* Quart. Jour. Math. **50** (1927), 377–384.
6. Per. W. Karlsson, *Hypergeometric functions with integral parameter differences.* Jour. Math. Phys. **12** (1971), 270–271.
7. A. Lakin, *A hypergeometric identity related to Dougall's theorem.* Jour. London Math. Soc. **27** (1952), 229–234.
8. D. B. Sears, *On the transformation theory of basic hypergeometric functions.* Proc. London Math. Soc. (2) **53** (1951), 158–180.
9. L. J. Slater, *Generalised hypergeometric series.* Cambridge University Press (1966).

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