In consequence, however, of two of these solutions being given in 2 ways each, the actual solutions got are only 22, or 8 in addition to the corner solutions. There are thus 24 solutions for the 9-board which cannot be got from solutions for the 8-board. This method of treating the question therefore does not lead us to a law.

A study of the diagrams suggests several interesting propositions for investigation; for instance, in no one of the solutions we have got for the 6-, 7-, 8-, and 9-boards, is there an arrangement of 4 men such as we have them in the 4-board solution; and I am inclined to think it is impossible there should be, but I have not succeeded in proving it. Similarly, I am inclined to think it is impossible there should be an arrangement of 4 men on the outside columns and rows of a board, as shown in Fig. 52; but this also I have not succeeded in proving.

On the equations of Vortex motion, with special reference to the use of polar co-ordinates.

By C. Chree, M.A., King's College, Cambridge.

In several previous communications* to the Society, I have considered the equations of vortex motion in two dimensions in a compressible fluid. In the present communication I propose to consider certain forms of the hydro-dynamical equations of a more general kind. In certain cases the fluid will be supposed to be rotating, prior to the introduction of the vortex motion, with uniform angular velocity about a fixed axis.

Using the same notation as in my previous papers, and supposing the axes of x and y rotating with uniform angular velocity ω about the axis of z, which is supposed fixed, we can easily prove by the method of my paper "*On vortex motion in a rotating Fluid*" † that the equations of three dimensional motion are the following—

$$\frac{\delta u}{\delta t} - 2\omega v - \omega^2 x = -\frac{1}{\rho}\frac{dp}{dx} + X \quad \ldots \quad \ldots \quad (1),$$

$$\frac{\delta v}{\delta t} + 2\omega u - \omega^2 y = -\frac{1}{\rho}\frac{dp}{dy} + Y \quad \ldots \quad \ldots \quad (2),$$

$$\frac{\delta w}{\delta t} = -\frac{1}{\rho}\frac{dp}{dz} + Z \quad \ldots \quad \ldots \quad (3).$$

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* *Proceedings*, vol. V., p. 52; vol. VI., p. 59; vol. VII., p. 29.

† *Proceedings*, vol. VII., p. 29.

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The component velocities u and v are velocities relative to the moving axes. It will also be remembered that

$$\frac{\delta}{\delta t} \equiv \frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz} \quad \ldots \quad \ldots \quad (4)$$

denotes differentiation following the fluid ; while $\frac{d}{dt}$ denotes variation at a point fixed relative to the moving axes.

The equation of continuity is simply

$$\frac{1}{\rho}\frac{\delta\rho}{\delta t} + \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (5).$$

The *form* of this equation is, it will be noticed, the same whether ω exist or not.

Employing the usual notation for the components of the *apparent** vorticity—

$$\xi = \tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right), \text{ etc.}, \quad \ldots \quad \ldots \quad \ldots \quad (6),$$

we easily deduce from the preceding equations in the case when the components X, Y, Z of the external forces vanish or are derivable from a potential, the following equations † for the variation of the vorticity—

$$\rho\frac{\delta(\xi/\rho)}{\delta t} = \left[\xi\frac{d}{dx} + \eta\frac{d}{dy} + (\zeta+\omega)\frac{d}{dz}\right]u \quad \ldots \quad (7),$$

$$\rho\frac{\delta(\eta/\rho)}{\delta t} = \left[\xi\frac{d}{dx} + \eta\frac{d}{dy} + (\zeta+\omega)\frac{d}{dz}\right]v \quad \ldots \quad (8),$$

$$\rho\frac{\delta(\zeta+\omega)/\rho}{\delta t} = \left[\xi\frac{d}{dx} + \eta\frac{d}{dy} + (\zeta+\omega)\frac{d}{dz}\right]w \quad \ldots \quad (9).$$

In the left-hand side of (9) we have introduced $\frac{\delta\omega}{\delta t}$ which is of course zero. The components ξ, η are of course absolute as well as apparent vorticity components along the instantaneous directions of the corresponding axes. These equations will remain unaltered when we transfer the origin to any point fixed relative to the original rotating axes, the new axes of x and y being supposed to rotate with the same angular velocity ω as the original. In proof of this it is sufficient to remark that the co-ordinates of any point, whether fixed or moving, referred to the new axes will differ from those referred to

* See *Proceedings*, vol. VII., p. 32.

† Lamb's *Motion of Fluids*, Note D.

the old only by constant quantities independent of the time, and so all velocity and vorticity components are unaffected by the change.

The components of velocity and vorticity must satisfy (7), (8), (9) and in addition the ordinary boundary conditions and the equation), (5), of continuity. This last equation requires *inter alia* that two contiguous elements of fluid the one inside and the other outside a vortex filament have the same component of velocity along the normal to the surface of the filament.

As the fluid is supposed frictionless it is not necessary, unless it be involved in the equations (7) – (9), for the component velocities in elements just inside and just outside the surface of a vortex filament to be identical in any direction which lies in the tangent plane to the surface of the filament.

Suppose now that when ω is zero the components of velocity and vorticity in the fluid constituting a vortex ring, whose centre is in the axis of z and whose plane is parallel to xy, are given by

$$\left.\begin{array}{lll} u=\nu\cos\theta & v=\nu\sin\theta & w=w_0 \\ \xi=-\Omega\sin\theta & \eta=\Omega\cos\theta & \zeta=0 \end{array}\right\} \quad \ldots \quad (10),$$

where $\theta=\tan^{-1}(y/x)$, and ν, w_0 and Ω are independent of θ.

These components will thus satisfy (7), (8), (9) and other necessary conditions when ω is zero. When, however, ω ceases to be zero the equations (7), (8), and (9) are no longer satisfied by (10). These equations may, however, be all satisfied by adding to the components (10) the additional terms

$$\left.\begin{array}{lll} u=\omega r\sin\theta & v=-\omega r\cos\theta & w=0 \\ \xi=0 & \eta=0 & \zeta=-\omega \end{array}\right\} \quad \ldots \quad (11),$$

where $r^2=x^2+y^2$.

The additional terms in the velocity components add nothing to the components ξ and η of vorticity, and they are consistent with $\zeta=-\omega$.

To prove our statement we must substitute in (7), (8), and (9) the system of velocities and vorticities obtained by combining (10) and (11). Doing so and remembering that (9) is satisfied by the values (10) when $\omega=0$, we find that it is still identically satisfied, and that (7) and (8) respectively lead to

$$-\sin\theta\frac{\delta\Omega}{\delta t}-\Omega\cos\theta\frac{\delta\theta}{\delta t}+\frac{\Omega\sin\theta}{\rho}\frac{\delta\rho}{\delta t}=\frac{\Omega}{r}\frac{d}{d\theta}(\nu\cos\theta+\omega r\sin\theta) \quad \ldots \quad (12),$$

$$\cos\theta\frac{\delta\Omega}{\delta t}-\Omega\sin\theta\frac{\delta\theta}{\delta t}-\frac{\Omega\cos\theta}{\rho}\frac{\delta\rho}{\delta t}=\frac{\Omega}{r}\frac{d}{d\theta}(\nu\sin\theta-\omega r\cos\theta) \quad \ldots \quad (13).$$

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Multiplying (12) by $\cos\theta$, and (13) by $\sin\theta$, then adding and dividing out by Ω we simply get

$$\frac{\delta\theta}{\delta t} = -\omega.$$

This signifies that the fluid in the vortex ring moves relative to the rotating axes with angular velocity $-\omega$ about the axis of the ring, which is exactly the motion indicated by (11).

Again subtracting (12) multiplied by $\sin\theta$ from (13) multiplied by $\cos\theta$, we get

$$\frac{\delta\Omega}{\delta t} - \frac{\Omega}{\rho}\frac{\delta\rho}{\delta t} = \frac{\Omega\nu}{r},$$

which is precisely the same relation as when ω is zero.

The additional terms (11) in the velocity of the fluid possessed of vorticity have their resultant in the tangent plane at every point on the surface of the vortex ring, and so their existence in no way affects the equality of the velocity components inside and outside the ring in the direction of the normal to the surface.

This indicates that a circular vortex ring may exist in a rotating fluid with its plane perpendicular to the axis of rotation, provided it have, in addition to the motion existing in a similar ring in a non-rotating fluid, a uniform angular velocity about the axis of the ring which is equal in magnitude but opposite in direction to that of the undisturbed rotating fluid. As in the case of a non-rotating fluid, there may be an infinite plane boundary at right angles to the axis of rotation, the necessary conditions being satisfied by the existence of an image on the remote side of the plane.

If, for instance, we imagine the state of matters in the earth's atmosphere in latitude λ the same as in fluid limited by an infinite plane coincident with the tangent plane to the earth's surface, the fluid rotating with angular velocity $\omega\sin\lambda$ about an axis answering to the vertical at the point considered; then a vortex ring whose plane was horizontal would have an angular motion $\omega\sin\lambda$ in azimuth from east to west through south. In other words, a diameter of the ring connecting two definite material cross sections would appear to an observer at the point to rotate about the vertical in precisely the same way and at the same rate as the plane of vibration of a Foucault's pendulum.

Hitherto I have said nothing as to the motion in the fluid surrounding the vortex ring. When we consider this point a difficulty appears which I do not see my way to answer satifactorily. It seems well worthy of notice.

The equations (7), (8), and (9) apply at every point in the fluid, and so must be satisfied by the fluid surrounding the vortex ring we have just considered. Now the velocity components due to the action of a vortex ring in the surrounding fluid satisfy the conditions of irrotational motion, but they depend both in magnitude and direction on the distance from the instantaneous position of the plane of the ring, Thus $\frac{du}{dz}$, $\frac{dv}{dz}$ and $\frac{dw}{dz}$ do not vanish in the fluid surrounding the ring, and so in accordance with (7), (8) and (9) unless $\zeta + \omega$ vanish everywhere, the whole of the fluid surrounding a vortex ring will take up some form of vorticity of a complicated character.

Theoretically as $\frac{du}{dz}$, etc., do not absolutely vanish, however distant the point considered from the vortex ring may be, in order to avoid setting up vorticity possessed of components ξ, η we require some species of motion set up whereby $\zeta = -\omega$ all through the fluid. This would require either the motion of the rotating fluid entirely to stop, or in addition to the rotation ω about the original axis a rotation $-\omega$ about some parallel axis. The two rotations would give a uniform translational velocity the same at every point occupied by the fluid.

In practice when we consider that every fluid is possessed of viscosity, and that the terms $\frac{du}{dz}$ etc., due to the action of an ordinary vortex ring become insensible at a small distance from the vortex, the conclusion we seem led to is that the vortex ring itself and the immediately adjacent fluid will possess, relative to the moving axes, a rotation about the axis of the ring in the direction opposite to that in which the undisturbed fluid rotates. Throughout the ring itself, if its vorticity be great, this angular velocity may approach indefinitely near $-\omega$, but it will gradually diminish in the external fluid as the distance from the ring increases and altogether vanish at distances where the direct action of the ring becomes insensible.

The previous remarks also indicate that any variation in any velocity component with the distance from a fixed plane perpendicular to the axis of rotation, whatever may be the exciting cause, necessitates the existence of some form or other of apparent vortex motion.

For certain purposes it is desirable to employ polar co-ordinates in the hydrodynamical equations. Such equations are obtained by Basset in his *Treatise on Hydrodynamics.** His proof, however, is not of a very elementary character, and I think the following method shows more clearly the meaning to be attached to the symbols employed. For some of the results I shall refer the reader to a previous paper in the *Proceedings.*†

As in the paper referred to, the element of volume has three of its edges δr, $r\delta\theta$ and $r\sin\theta\delta\phi$ intersecting in P, the point whose co-ordinates referred to a fixed point 0, and the ordinary polar system are r, θ, ϕ. The first of those three line elements and the tangents to the other two at P form a system of orthogonal axes called the *fundamental* axes at P. If, as in the paper referred to, the corner of the element of volume opposite to P be termed S′, the co-ordinates of S′ are $r+\delta r$, $\theta+\delta\theta$, $\theta+\delta\phi$.

Also neglecting squares of the small quantities $\delta\theta$, $\delta\phi$, the cosines of the angles between the fundamental axes at S′ and those at P are given by the following scheme, reproduced from my previous paper,—

		at S′ r	at S′ θ	at S′ ϕ
at P	r	1	$-\delta\theta$	$-\sin\theta\delta\phi$
at P	θ	$\delta\theta$	1	$-\cos\theta\delta\phi$
at P	ϕ	$\sin\theta\delta\phi$	$\cos\theta\delta\phi$	1

Let us first suppose that the axes are fixed, so as to avoid dealing with too many difficulties at once.

Let the velocity of a given element of fluid at the time t at the point P have the components u, v, w along the fundamental axes at P. After a short interval τ let this element be at S′, so that

$$\delta r = u\tau,\ \delta\theta = \frac{v}{r}\tau,\ \delta\phi = \frac{w}{r\sin\theta}\tau.$$

Then the velocity components at time $t+\tau$ along the fundamental axes at S′ are

$$u_1 = u + \frac{du}{dt}\tau + \frac{du}{dr}\delta r + \frac{du}{d\theta}\delta\theta + \frac{du}{d\phi}\delta\phi., \text{ etc.}$$

* Vol. II., Art. 470. † Vol. III., p. 109.

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Thus, referring to the above scheme, the components of the velocity of the element at time $t+\tau$, *along the fundamental axes at P* are

$$u_1 - v_1\delta\theta - w_1\sin\theta\delta\phi, \text{ etc.}$$

But the change per unit time in the component of the velocity in any given direction equals the accelerating force per unit mass in that direction. If then U, V, W be the components of the external forces per unit mass at the point P, proceeding to the limit when τ vanishes, we easily deduce for the equations of motion—

$$\frac{\delta u}{\delta t} - \frac{v^2+w^2}{r} = -\frac{1}{\rho}\frac{dp}{dr} + U \quad \ldots \quad \ldots \quad (14),$$

$$\frac{\delta v}{\delta t} + \frac{uv}{r} - \frac{w^2}{r}\cot\theta = -\frac{1}{\rho r}\frac{dp}{d\theta} + V \ldots \quad \ldots \quad (15),$$

$$\frac{\delta w}{\delta t} + \frac{uw}{r} + \frac{vw}{r}\cot\theta = -\frac{1}{\rho r\sin\theta}\frac{dp}{d\phi} + W \quad \ldots \quad (16);$$

where

$$\frac{\delta}{\delta t} \equiv \frac{d}{dt} + u\frac{d}{dr} + \frac{v}{r}\frac{d}{d\theta} + \frac{w}{r\sin\theta}\frac{d}{d\phi} \quad \ldots \quad (17)$$

signifies as usual differentiation following the fluid.

These equations agree with Basset's, except that he has u instead of w in the last term on the left-hand side of (16). I believe, however, that the above is the correct equation.

By employing the method of flux it is very easily proved that the equation of continuity is

$$\frac{1}{\rho}\frac{\delta\rho}{\delta t} + \frac{1}{r^2}\frac{d(ur^2)}{dr} + \frac{1}{r\sin\theta}\frac{d}{d\theta}(v\sin\theta) + \frac{1}{r\sin\theta}\frac{dw}{d\phi} = 0 \quad \ldots \quad (18).$$

Let us next suppose that the entire system of fluid and axes is rotating with uniform angular velocity ω about the fixed axis $\theta = 0$.

Let u', v', w' denote the component velocities at time t at the point P (r, θ, ϕ) relative to the fundamental axes there, which are supposed fixed relative to the rotating axes; and let u, v, w be the absolute velocities in the direction of lines fixed in space with which the fundamental axes at P coincide at the time t. Then

$$u = u', \; v = v', \; w = w' + \omega r\sin\theta \quad \ldots \quad \ldots \quad (19).$$

Let R represent the position at time $t+\tau$, where τ is very small, of a point rigidly connected with the moving axes which at the time t coincided with the point P; then the components of the velocity at R at time $t+\tau$ relative to the fundamental axes there, which move with the fluid, are respectively

$$u' + \frac{du'}{dt}\tau, \quad v' + \frac{dv'}{dt}\tau, \quad w' + \frac{dw'}{dt}\tau;$$

and so the component velocities at R at time $t+\tau$ relative to fixed axes, coinciding with the instantaneous position of the fundamental axes there, are

$$u' + \frac{du'}{dt}\tau, \quad v' + \frac{dv'}{dt}\tau, \quad w' + \frac{dw'}{dt}\tau + \omega r \sin\theta \ldots \qquad (20).$$

For the last result it must be noticed that referred to axes fixed in space, coinciding with the position of the axes at 0 at time t, the r and θ co-ordinates of R and P are the same. From this and the consideration that the ϕ co-ordinate of R relative to the fixed axes exceeds that of P by $\omega\tau$, it also follows that the components (20) are identical with

$$u + \frac{du}{dt}\tau + \omega\tau\frac{du}{d\phi}, \quad v + \frac{dv}{dt}\tau + \omega\tau\frac{dv}{d\phi}, \quad w + \frac{dw}{dt}\tau + \omega\tau\frac{dw}{d\phi} \quad \ldots \qquad (21).$$

Thus, comparing (20) and (21) and remembering (19), we find

$$\frac{du'}{dt} = \frac{du}{dt} + \omega\frac{du}{d\phi}, \quad \frac{dv'}{dt} = \frac{dv}{dt} + \omega\frac{dv}{d\phi}, \quad \frac{dw'}{dt} = \frac{dw}{dt} + \omega\frac{dw}{d\phi}.$$

But from (19)

$$\frac{du}{d\phi} = \frac{du'}{d\phi}, \quad \frac{dv}{d\phi} = \frac{dv'}{d\phi}, \quad \frac{dw}{d\phi} = \frac{dw'}{d\phi};$$

and therefore

$$\frac{du}{dt} = \frac{du'}{dt} - \omega\frac{du'}{d\phi}, \quad \frac{dv}{dt} = \frac{dv'}{dt} - \omega\frac{dv'}{d\phi}, \quad \frac{dw}{dt} = \frac{dw'}{dt} - \omega\frac{dw'}{d\phi}.$$

Substituting these values and the values

$$\frac{du}{dr} = \frac{du'}{dr}, \qquad \frac{dv}{dr} = \frac{dv'}{dr}, \qquad \frac{dw}{dr} = \frac{dw'}{dr} + \omega\sin\theta,$$

$$\frac{du}{d\theta} = \frac{du'}{d\theta}, \qquad \frac{dv}{d\theta} = \frac{dv'}{d\theta}, \qquad \frac{dw}{d\theta} = \frac{dw'}{d\theta} + \omega r\cos\theta,$$

in the equations (14), (15) and (16) we get after reduction, dropping the dashes so that in the following equations u, v, w are the velocities *relative to the moving axes*,

$$\frac{\delta u}{\delta t} - \frac{v^2}{r} - \frac{(w + \omega r\sin\theta)^2}{r} = -\frac{1}{\rho}\frac{dp}{dr} + U \ldots \quad \ldots \quad \ldots \qquad (22),$$

$$\frac{\delta v}{\delta t} + \frac{uv}{r} - (w + \omega r\sin\theta)^2\frac{\cot\theta}{r} = -\frac{1}{\rho r}\frac{dp}{d\theta} + V \quad \ldots \quad \ldots \qquad (23),$$

$$\frac{\delta w}{\delta t} + \frac{uw}{r} + \frac{vw\cot\theta}{r} + 2\omega(u\sin\theta + v\cos\theta) = -\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{dp}{d\phi} + W \ldots (24).$$

In these equations $\frac{\delta}{\delta t}$ has the same *form* as in (17), and it also as there signifies differentiation following the fluid. The reader will, I think, find no great difficulty in this proof if he clearly realise that $\frac{du'}{dt}$ in (20) signifies the rate of change with the time of the component of the velocity relative to a certain moving axis, at a point whose co-ordinates are fixed relative to the moving axes; while $\frac{du}{dt}$ in (21) signifies the rate of change of the component of the velocity, relative to a certain axis fixed in space, at a point which is absolutely fixed in space.

As a simple illustration of these formulae, let us apply them to the case of the earth's atmosphere regarded as surrounding a rotating sphere of radius R, the acceleration due to whose attraction at distance r from its centre is gR^2/r^2.

For the undisturbed condition, when the atmosphere is at rest relative to the sphere, we have $u = v = w = 0$. Substituting these values along with $U = -gR^2/r^2$, $V = W = 0$, we reduce the equations (22) – (24) to

$$\left.\begin{aligned} \frac{1}{\rho}\frac{dp}{dr} &= \omega^2 r\sin^2\theta - gR^2/r^2 \\ \frac{1}{\rho}\frac{dp}{d\theta} &= \omega^2 r^2\sin\theta\cos\theta \\ \frac{1}{\rho}\frac{dp}{d\phi} &= 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (25);$$

whose solution is, according as $p = k\rho$ or $p = k\rho^\gamma$,

$$\left.\begin{aligned} \log(\rho/\rho_0) &= \frac{\omega^2}{2k}r^2\sin^2\theta - \frac{g}{k}R\left(1 - \frac{R}{r}\right) \\ \text{or} \quad \rho^{\gamma-1} - \rho_0^{\gamma-1} &= \frac{\gamma-1}{2k\gamma}\left\{\omega^2 r^2\sin^2\theta - 2gR\left(1 - \frac{R}{r}\right)\right\} \end{aligned}\right\} \quad (26).$$

In both results ρ_0 represents the density of the atmosphere at the surface of the earth at the poles.

These results may of course be obtained in a more elementary manner by the introduction of the idea of "centrifugal force," as in Besant's "*Hydromechanics.*"

In a paper* already referred to, I obtained the following expres-

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* *Proceedings*, Vol. III., p. 113. *Cf.* Basset's *Treatise on Hydrodynamics*, Vol. I., Art. 18.

sions for the components of vorticity at the point (r, θ, ϕ) with respect to the fundamental axes there,

$$\left.\begin{aligned}\xi &= \frac{1}{2r\sin\theta}\left\{\frac{d}{d\theta}(w\sin\theta) - \frac{dv}{d\phi}\right\}\\ \eta &= \frac{1}{2r}\left\{\frac{1}{\sin\theta}\frac{du}{d\phi} - \frac{d}{dr}(wr)\right\}\\ \zeta &= \frac{1}{2r}\left\{\frac{d}{dr}(vr) - \frac{du}{d\theta}\right\}\end{aligned}\right\} \quad \ldots \quad (27).$$

When u, v, w are velocities referred to fixed axes, the above are components of the absolute vorticity, but when the velocities are relative to moving axes they are components of what I have called the *apparent* vorticity.

In order to apply our polar equations to vortex motion we require to obtain from (22), (23), (24) a series of equations corresponding to (7), (8), (9), when as there the external forces vanish or are derivable from a potential. The necessary algebraic operations are somewhat tedious and lengthy, so I shall merely indicate the method of procedure and give the results. The *verification* of the results will present no great difficulty to any one who keeps their form in view in grouping his terms, and who does not allow the length of his intermediate expressions to alarm him.

Replace in every case $\frac{\delta}{\delta t}$ by its equivalent in (17) and notice that, as the external forces have a potential,

$$\mathrm{U} = \frac{d\mathrm{F}}{dr},\ \mathrm{V} = \frac{1}{r}\frac{d\mathrm{F}}{d\theta},\ \mathrm{W} = \frac{1}{r\sin\theta}\frac{d\mathrm{F}}{d\phi},$$

where F is some function of r, θ, ϕ.

To obtain the first equation multiply (24) by $\sin\theta$, then differentiate with respect to θ, and from the result subtract (23) after differentiating it with respect to ϕ. Finally multiply up by $r/2\sin\theta$.

To obtain the second equation multiply (22) by $\operatorname{cosec}\theta$, then differentiate with respect to ϕ, and from the result subtract the result obtained by multiplying (24) by r and then differentiating it with respect to r. Finally divide out by $2r^2$.

To obtain the third equation multiply (23) by r, then differentiate with respect to r, and from the result subtract (22) differentiated with respect to θ. Finally divide out by $2r^2\sin\theta$.

In each case one of the objects of the preliminary operations is to eliminate p and the external forces.

In putting the resulting expressions into the following concise forms the identity (17), the equation of continuity (18), and the identities (27) are alone required. We thus finally obtain

$$\rho\frac{\delta}{\delta t}\left(\frac{r^2\xi}{\rho}\right)=\left[\xi\frac{d}{dr}+\frac{\eta}{r}\frac{d}{d\theta}+\frac{\zeta}{r\sin\theta}\frac{d}{d\phi}\right](r^2u)$$
$$-\frac{\omega}{\sin\theta}\left\{\frac{d.}{d\theta}r\sin\theta(u\sin\theta+v\cos\theta)+r\cos\theta\frac{dw}{d\phi}\right\}\quad\ldots(28),$$

$$\rho\frac{\delta}{\delta t}\left(\frac{\eta}{r\rho}\right)=\left[\xi\frac{d}{dr}+\frac{\eta}{r}\frac{d}{d\theta}+\frac{\zeta}{r\sin\theta}\frac{d}{d\phi}\right]\left(\frac{v}{r}\right)$$
$$+\frac{\omega}{r^2\sin\theta}\left\{\frac{d.}{dr}r\sin\theta(u\sin\theta+v\cos\theta)+\sin\theta\frac{dw}{d\phi}\right\}\ldots(29),$$

$$\rho\frac{\delta}{\delta t}\left(\frac{\zeta}{r\sin\theta\rho}\right)=\left[\xi\frac{d}{dr}+\frac{\eta}{r}\frac{d}{d\theta}+\frac{\zeta}{r\sin\theta}\frac{d}{d\phi}\right]\left(\frac{w}{r\sin\theta}\right)$$
$$+\omega\left[\cos\theta\frac{d}{dr}-\frac{\sin\theta}{r}\frac{d}{d\theta}\right]\left(\frac{w}{r\sin\theta}\right)\ldots\quad\ldots\quad(30).$$

So long as ω is not zero these equations apply to the apparent not the absolute vorticities.

In the ordinary case when ω is zero and the axes are fixed, the following very concise form of the equations seems worth recording. Let Ω denote the resultant vorticity, now absolute, and ds an element in the direction of the axis of the resultant vorticity at any point in the fluid, then it is obvious that

$$\xi\frac{d}{dr}+\frac{\eta}{r}\frac{d}{d\theta}+\frac{\zeta}{r\sin\theta}\frac{d}{d\phi}\equiv\Omega\frac{d}{ds}\ldots\quad\ldots\quad(31);$$

and so the equations of vortex motion may be written

$$\rho\frac{\delta}{\delta t}\left(\frac{r^2\xi}{\rho}\right)=\Omega\frac{d}{ds}(r^2u)\quad\ldots\quad\ldots\quad(28a),$$

$$\rho\frac{\delta}{\delta t}\left(\frac{\eta}{r\rho}\right)=\Omega\frac{d}{ds}\left(\frac{v}{r}\right)\quad\ldots\quad\ldots\quad(29a),$$

$$\rho\frac{\delta}{\delta t}\left(\frac{\zeta}{r\sin\theta\rho}\right)=\Omega\frac{d}{ds}\left(\frac{w}{r\sin\theta}\right)\quad\ldots\quad(30a).$$

In employing polar co-ordinates it will also be convenient to possess equations answering to those in Cartesian co-ordinates of which in Lamb's* notation the type is

$$u=\frac{dP}{dx}+\frac{dN}{dy}-\frac{dM}{dz}.$$

* *Treatise on the Motion of Fluids*, Art. 129.

To transfer to polar co-ordinates let as usual

$$x^2+y^2+z^2\equiv r^2,\ z/r\equiv\cos\theta,\ y/x\equiv\tan\phi.$$

Then $$\frac{dr}{dx}=\sin\theta\cos\phi,\quad \frac{d\theta}{dx}=\frac{\cos\theta\cos\phi}{r},\quad \frac{d\phi}{dx}=-\frac{\sin\phi}{r\sin\theta},\quad \text{etc.}$$

Using these and taking the resultant along the three fundamental axes at each point of the components of the velocity in Cartesians, and denoting now the velocities along the fundamental axes by u, v, w, we easily find

$$\begin{aligned} u &= \frac{dP}{dr}+\frac{1}{r}\left[\cos\phi\frac{d}{d\theta}-\sin\phi\cot\theta\frac{d}{d\phi}\right]M \\ &\quad -\frac{1}{r}\left[\sin\phi\frac{d}{d\theta}+\cos\phi\cot\theta\frac{d}{d\phi}\right]L+\frac{1}{r}\frac{dN}{d\phi}, \\ v &= \frac{1}{r}\frac{dP}{d\theta}+\left[\sin\phi\frac{d}{dr}+\frac{\cos\phi}{r}\frac{d}{d\phi}\right]L-\left[\cos\phi\frac{d}{dr}-\frac{\sin\phi}{r}\frac{d}{d\phi}\right]M \\ &\quad +\frac{\cot\theta}{r}\frac{dN}{d\phi}, \\ w &= \frac{1}{r\sin\theta}\frac{dP}{d\phi}+\left[\cos\theta\frac{d}{dr}-\frac{\sin\theta}{r}\frac{d}{d\theta}\right]\left[L\cos\phi+M\sin\phi\right] \\ &\quad -\left[\sin\theta\frac{d}{dr}+\frac{\cos\theta}{r}\frac{d}{d\theta}\right]N. \end{aligned}$$

Putting

$$\left.\begin{aligned} R&=\sin\theta(L\cos\phi+M\sin\phi)+N\cos\theta \\ S&=\cos\theta(L\cos\phi+M\sin\phi)-N\sin\theta \\ T&=M\cos\phi-L\sin\phi \end{aligned}\right\}\quad \ldots \quad (32),$$

the above equations reduce to

$$\left.\begin{aligned} u&=\frac{dP}{dr}+\frac{1}{r^2\sin\theta}\left\{\frac{d}{d\theta}(Tr\sin\theta)-\frac{d}{d\phi}(Sr)\right\} \\ v&=\frac{1}{r}\frac{dP}{d\theta}+\frac{1}{r\sin\theta}\left\{\frac{dR}{d\phi}-\frac{d}{dr}(Tr\sin\theta)\right\} \\ w&=\frac{1}{r\sin\theta}\frac{dP}{d\phi}+\frac{1}{r}\left\{\frac{d(Sr)}{dr}-\frac{dR}{d\theta}\right\} \end{aligned}\right\}\quad \ldots \quad (33).$$

If for shortness we represent the element of volume by dV, and employ the equation of continuity we can replace equation (10) of Lamb's Art. 129 by

$$P=\frac{1}{4\pi}\iiint\frac{1}{\rho'}\frac{\delta\rho'}{\delta t}\frac{1}{r_1}dV' \quad \ldots \quad \ldots \quad (34),$$

a form applying to polar as well as Cartesian co-ordinates.

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The integration is to be extended to every portion of space wherein the fluid is varying in density, and r_1 denotes the distance of the element dV' wherein the density is ρ' from the point where P is being calculated. This point of course is fixed and its co-ordinates constants so far as the above integration is concerned.

When it is desired to introduce polar co-ordinates explicitly into the above and subsequent integrals we know that

$$dV' = r'^2 \sin\theta' dr' d\theta' d\phi'$$
$$r_1^2 = r^2 + r'^2 - 2rr'\{\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')\}.$$

Lamb in the above-mentioned article also shows that if

$$\frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz} = 0 \qquad \ldots \qquad \ldots \qquad (35)$$

throughout the fluid, then L, M, N are like P expressible by integrals representing potentials. This condition holds if over every surface bounding the fluid the vorticity either vanishes or has its axis in the tangent plane to the surface. To represent these integrals in a form suitable for our present purpose, let Ω represent the resultant vorticity at any point and (PQ) the angle between any two directed quantities denoted by P and Q. Then the integrals are

$$\left.\begin{array}{c} L = \dfrac{1}{2\pi}\displaystyle\iiint \frac{\Omega'\cos(\Omega' x)}{r_1} dV' \\ \text{etc.} \end{array}\right\} \qquad \ldots \qquad \ldots \qquad (36).$$

We have now to find the equivalent in polar co-ordinates of equations (35) and (36).

Replacing differentiations with respect to x, y, z by differentiations with respect to r, θ, ϕ, and having regard to (32) after differentiation, we transform (35) into

$$\frac{1}{r^2}\frac{d}{dr}(Rr^2) + \frac{1}{r\sin\theta}\frac{d}{d\theta}(S\sin\theta) + \frac{1}{r\sin\theta}\frac{dT}{d\phi} = 0 \qquad \ldots \qquad (37).$$

This result may also be obtained as follows. Suppose L, M, N to represent the components in the directions of the Cartesian axes of a certain vector quantity. Then under the conditions existing in (35) the vector quantity must satisfy the same condition as the velocity in an incompressible fluid. But by (32), R, S, T are the components of this same vector quantity along the fundamental directions at the point r, θ, ϕ. Thus they must satisfy the same condition as the velocity components in those directions in an in compressible fluid. Thus (18) should be satisfied when

$$\frac{\delta\rho}{\delta t}=0,\ u=\mathrm{R},\ v=\mathrm{S},\ w=\mathrm{T}\ ;$$

and when these substitutions are made we simply get equation (37).

We likewise easily find from (32) and (36), noticing that θ and ϕ are constants so far as the integrations are concerned,

$$\left.\begin{aligned}\mathrm{R}&=\frac{1}{2\pi}\iiint\frac{\Omega'\cos(\Omega'\xi)}{r_1}\,d\mathrm{V}',\\ \mathrm{S}&=\frac{1}{2\pi}\iiint\frac{\Omega'\cos(\Omega'\eta)}{r_1}\,d\mathrm{V}',\\ \mathrm{T}&=\frac{1}{2\pi}\iiint\frac{\Omega'\cos(\Omega'\zeta)}{r_1}\,d\mathrm{V}'\end{aligned}\right\}\quad\ldots\quad\ldots\quad(38).$$

If we denote by K the vector quantity whose components are L, M, N, or R, S, T, we obtain at once from (38)

$$\begin{aligned}\mathrm{R}\xi+\mathrm{S}\eta+\mathrm{T}\zeta&=\mathrm{K}\Omega\cos(\mathrm{K}\Omega)\\ &=\frac{1}{2\pi}\iiint\frac{\Omega\Omega'\cos(\Omega\Omega')}{r_1}\,d\mathrm{V}'\quad\ldots\quad\ldots\quad(39).\end{aligned}$$

Thus

$$\iiint\mathrm{K}\Omega\cos(\mathrm{K}\Omega)d\mathrm{V}=\frac{1}{2\pi}\iiiint\!\!\iint\frac{\Omega\Omega'\cos(\Omega\Omega')}{r_1}\,d\mathrm{V}d\mathrm{V}'\qquad(40).$$

The meaning of the sextuple integral is that the value of a certain function for the fluid occupying an element $d\mathrm{V}$ at the point $(r,\ \theta,\ \phi)$ is derived from an integration throughout space, and that the value of this function is then integrated throughout space.

When the fluid is incompressible and unlimited, being at rest at infinity, and all the vortices are within a finite distance of the origin, Lamb shows in his Art. 135 that a certain expression which is obviously identical with the right-hand side of (40) is equal to E/ρ, where E is the total kinetic energy of the fluid motion.

It may be seen from the same and previous articles that when the fluid, remaining incompressible, is not unlimited, the right-hand side of (40) still equals E/ρ provided over the boundary or boundaries, S, the axis of the resultant vorticity is in the tangent plane to the surface, and

$$\iint\mathrm{K}q\begin{vmatrix}\lambda&\mu&\nu\\ l&m&n\\ l'&m'&n'\end{vmatrix}d\mathrm{S}=0\quad\ldots\quad\ldots\quad(41);$$

where $(\lambda,\ \mu,\ \nu)$ are the direction cosines of K, $(l,\ m,\ n)$ the direction cosines of the normal, and (l', m', n') the direction cosines of the resultant velocity q.

Supposing the above conditions satisfied when there exist a series of vortex filaments of strengths $m_1, \ldots m_p, m_q \ldots m_n$ forming closed curves, $s_1, \ldots s_p, s_q \ldots s_n$ in an incompressible fluid of density ρ, we obtain for the energy E of the fluid motion the equation

$$\mathrm{E} = \frac{\rho}{2\pi}\Sigma_1^n \left[m_p^2 \int\int \frac{\cos(ds_p ds'_p)}{r_{pp}} ds_p ds'_p \right] + \frac{\rho}{\pi}\Sigma \left[m_p\, m_q \int\int \frac{\cos(ds_p ds_q)}{r_{pq}} ds_p ds_q \right]$$

$$= \rho\, \Sigma_1^n \left[m_p \int \mathrm{K}\cos(\mathrm{K} ds_p) ds_p \right] \quad \ldots \quad \ldots \quad \ldots \quad (42).$$

In the first summation ds'_p is an element of the same vortex filament as ds_p, and r_{pp} is the distance between them. In the second summation every pair of the n closed curves must be combined together, so that $\frac{1}{2}n(n-1)$ integrations are included. In the last summation K must of course be regarded as a function of the arc s_p of the curve along which the integration is being taken. The value of K depends on the combined action of the vorticity in the curve over which the integration is being taken and of the vorticity in each of the other $n-1$ closed curves. Basset gives in his Art. 92 formulae which are equivalent to the first of the two expressions given above for the energy; and points out the analogy of the first integrals to the co-efficients of self-induction and of the second integrals to the co-efficients of mutual induction of a series of closed electric currents.

The similarity of the equations of vorticity to those of electrodynamics * may, I think, be brought out very clearly by the following comparison of (39), (40), and (42) with corresponding electrodynamical expressions. We require first to notice that according to the table in Basset's p. 88, the following are analogous quantities in the two subjects :—

Vorticity Ω and electric current $\mathfrak{C}$,

K and electromagnetic momentum $\mathfrak{A}$.

Suppose with Maxwell in his *Electricity*, Art. 616, that the quantity he terms J is zero. Then treating equations (5) † of that

* *Cf.* Basset's Art. 95.

† Second Edition. In the first edition change μ into $1/\mu$ in these equations and in Art. 617.

article precisely as we treated our equations (38), we obtain as the equation corresponding to (39)

$$\mathfrak{A}\mathfrak{C}\cos(\mathfrak{A}\mathfrak{C}) = \mu \iiint \frac{\mathfrak{C}\mathfrak{C}'\cos(\mathfrak{C}\mathfrak{C}')}{r_1}\, dV' \quad \ldots \quad (43).$$

The meaning of the dashes and of dV' and r_1 is exactly the same as in (39). The co-efficient μ is unity in the electromagnetic system and denotes the permeability in the electrostatic. In obtaining the equations from which (43) is derived Maxwell apparently treats μ as constant throughout every portion of space wherein currents are actually flowing.

From (43) we find the following equation answering to (40)

$$\iiint \mu^{-1}\mathfrak{A}\mathfrak{C}\cos(\mathfrak{A}\mathfrak{C})dV = \iiiiiint \frac{\mathfrak{C}\mathfrak{C}'\cos'(\mathfrak{C}\mathfrak{C}')}{r_1}\, dV dV' \ldots \quad (44),$$

the meaning of the sextuple integral being the same as previously.

Suppose now that the current system consists of currents of strengths $i_1 \ldots i_p, i_q \ldots i_n$ traversing closed linear conductors $s_1 \ldots s_p, s_q \ldots s_n$ of small cross section, then we may replace

$$\mathfrak{C}\ dV \text{ by } i\ ds.$$

We thus transform (44) into

$$\tfrac{1}{2}\Sigma_1^n\left[i_p^2 \iint \frac{\cos(ds_p\, ds'_p)}{r_{pp}}\, ds_p\, ds'_p\right] + \Sigma\left[i_p\, i_q \iint \frac{\cos(ds_p\, ds_q)}{r_{pq}}\, ds_p\, ds_q\right]$$

$$= \tfrac{1}{2}\Sigma_1^n\left[i_p \int \mu^{-1}\mathfrak{A}\cos(\mathfrak{A}ds_p)\, ds_p\right] \quad \ldots \quad \ldots \quad (45).$$

The limits of integration and the meanings to be attached to the letters are the same as in (42). As $\mu = 1$ on the electromagnetic system the second side of (45) is, it will be seen, identical with the expression given in equation (14) of Maxwell's article 634 for the electrokinetic energy in the field. The first side is also a known form for the energy, the integrals of the first series being the co-efficients of self-induction, and those of the second series the co-efficients of mutual induction of the several circuits. The resemblance of (45) and (42) both in their final form and in their mode of derivation is very close.

I am not aware that any name has yet been assigned to the quantity K. It will appear, however, from the previous investigations that it is a quantity of considerable importance. It might with considerable fitness, as showing its analogy to the corresponding electromagnetic quantity, be termed the *vector-potential of vorticity*. This name would also fit in with the mode of derivation from the vorticity, as the following geometrical interpretation of equations (38) will show.

We take the mean vorticity of each element of volume and divide it by the distance of the element from the point at which K is to be measured. Then starting from some fixed point as origin we draw, as in forming the polygon of forces, a series of lines representing these quantities in magnitude and in direction parallel to the corresponding vorticities. The line joining the origin to the end of the last line so drawn represents K in magnitude and direction.

This is practically the same construction as Maxwell gives for the corresponding quantity $\mathfrak{A}$ in his Art. 617. I fail, however, to see how he obtains the equation

$$\mathfrak{A} = \mu \int\int\int \frac{\mathfrak{C}}{r}\, dx\, dy\, dz$$

in the same article. It seems to me clearly wrong unless all the currents are parallel. Employing our previous notation, I believe the correct forms of this and of the corresponding hydrodynamical equation are

$$\mathfrak{A} = \mu \int\int\int \frac{\mathfrak{C}' \cos(\mathfrak{A}\mathfrak{C}')}{r_1}\, dV' \quad \dots \quad \dots \quad (46),$$

$$\mathrm{K} = \frac{1}{2\pi} \int\int\int \frac{\Omega' \cos(\mathrm{K}\Omega')}{r_1}\, dV' \quad \dots \quad \dots \quad (47).$$

In each case there exist two other equations determining the direction of the vector potential.

If in the equations (28*a*) – (30*a*) for a non-rotating fluid we make w, ξ and η vanish, and suppose along the vortex element r, u and v constants, we get the case of a circular vortex ring whose centre lies in the line $\theta = 0$ and whose plane is perpendicular to this line. The equations show that $\zeta/\rho r \sin\theta$ must be an absolute constant, and this may easily be seen to follow from the equation of continuity. Such a vortex ring, as I shall immediately show, can exist outside a sphere of which the line $\theta = 0$ is a diameter. Assuming this I shall first consider whether such a ring can exist when the fluid is rotating about the line in which the axis of the ring lies.

As in the similar case in Cartesian co-ordinates, it may easily be found that the equations (28) and (29) cannot be satisfied by supposing w, ξ, η to vanish, and u and v to be independent of ϕ. Suppose, however, that when ω is zero these equations are satisfied by

$$\left.\begin{array}{lll} u=u_0 & v=v_0 & w=0 \\ \xi=0 & \eta=0 & \zeta=\zeta_0 \end{array}\right\} \quad \ldots \quad \ldots \quad (48.$$

Then when ω ceases to vanish they will be found satisfied by the following, which are consistent values,

$$\left.\begin{array}{lll} u=u_0 & v=v_0 & w=-\omega r\sin\theta \\ \xi=-\omega\cos\theta & \eta=\omega\sin\theta & \zeta=\zeta_0 \end{array}\right\} \quad \ldots \quad (49),$$

supposing ρ to have the same value as when ω vanishes.

The method of proof is the same in all three of equations (28) – (30), so it will be sufficient to exemplify it by treating (28) only.

Substitute the values (49) in (28), and then noticing that when ω is zero (49) satisfies the equation, we are left under the necessity of proving

$$\rho\frac{\delta}{\delta t}\left(\frac{\omega r^2\cos\theta}{\rho}\right)=\omega\left[\cos\theta\frac{d}{dr}-\frac{\sin\theta}{r}\frac{d}{d\theta}\right](r^2u_0)$$
$$+\frac{\omega r}{\sin\theta}\left\{\frac{d}{d\theta}\cdot\sin\theta(u_0\sin\theta+v_0\cos\theta)\right\}.$$

Substituting for $\frac{1}{\rho}\frac{\delta\rho}{\delta t}$ from (18), and then for $\frac{\delta}{\delta t}$ from (17), we have to prove

$$\left[u_0\frac{d}{dr}+\frac{v_0}{r}\frac{d}{d\theta}\right](\omega r^2\cos\theta)+\omega r^2\cos\theta\left\{\frac{1}{r^2}\frac{d(u_0r^2)}{dr}+\frac{1}{r\sin\theta}\frac{d}{d\theta}(v_0\sin\theta)\right\}$$
$$=\omega\left[\cos\theta\frac{d}{dr}-\frac{\sin\theta}{r}\frac{d}{d\theta}\right](r^2u_0)+\frac{\omega r}{\sin\theta}\left\{\frac{d}{d\theta}\cdot\sin\theta(u_0\sin\theta+v_0\cos\theta)\right\}.$$

Carrying out the differentiations this will be found to be an identity. Similarly it may be shown that (49) satisfies (29) and (30). In the last case indeed the fact is obvious on inspection. This indicates that so far as the ring itself is concerned the equations may all be satisfied if the fluid in the ring possess, in addition to the velocity it would possess in a non-rotating fluid, a rotation with uniform angular velocity ω about the axis in an opposite direction to that in which the fluid and axes are rotating. Each material cross section of the ring in fact would move in a plane fixed in space.

The difficulties that occur in the motion of the fluid surrounding the ring are precisely the same as in the corresponding case in Cartesian co-ordinates. We may, as there, by bringing in the consideration of the existence of viscosity in all actual fluids, arrive at a similar conclusion as to the probable character of the phenomena in nature.

In order to prove that the motion we have just investigated can apply to the case of a vortex ring when the fluid is bounded internally by a spherical surface, a diameter of which forms the axis of rotation and coincides with the axis of the vortex, we still require to show that all the requisite conditions for a vortex ring in this position can be satisfied when the fluid and axes are not rotating. All the data necessary for this already exist.

It has been shown by Mr T. C. Lewis* that if a vortex ring of strength m exist in an incompressible fluid outside a spherical boundary of radius a, with its core at a distance f from the centre, the boundary conditions are satisfied by the velocities deducible from the true ring and from an image ring of strength $-m\sqrt{f/f'}$ inside the sphere with its core at a distance $f'=a^2/f$ from the centre. Corresponding points of the two rings are, it will be noticed, in the same relative position as an electrified point and its image with respect to a spherical conductor.

Again it has been shown by Mr W. M. Hicks† that a "source" of strength m at a distance $f>a$ from the centre O of a spherical surface of radius a has an image inside the sphere, consisting of a "source" of strength ma/f at the point Q which is the inverse of the true source P, and of a line "sink" of strength m/a per unit length extending from Q to O.

Now by a source of strength m at P we simply mean that there exists a term $-m/r_1$ in the velocity potential, where r_1 is the distance from P. But from (34) if a vortex ring, whose element of cross section is $d\sigma$ and element of core ds, exist in a compressible fluid it contributes to P, which there denotes the velocity potential, the expression

$$\frac{1}{4\pi}\int\left\{\int\int\frac{1}{\rho'}\,\frac{\delta\rho'}{\delta t}\,\frac{1}{r_1}\,d\sigma\right\}ds.$$

The double integral inside the bracket extends over σ, and the integration with respect to ds is taken round the core.

Assuming, as usual, that the radius of the cross section is very small compared to b, the radius of the ring core, we may regard r_1 as

* *Quarterly Journal*, Vol. xvi., p. 338, or Basset's Art. 311, Vol. ii.

† *Phil. Trans.* 1880, Part ii., p, 455, or Basset's Art. 52, Vol. i.

constant over the cross section and equal to the distance of the external point from the core. We may then replace the above expression by

$$-\frac{1}{4\pi}\int\frac{\dot{\sigma}}{r_1}\,ds,$$

where

$$\dot{\sigma} = -\iint\frac{1}{\rho'}\frac{\delta\rho'}{\delta t}\,d\sigma.$$

I have shown on a previous occasion* that when ρ' is uniform over the cross section, b being very large compared to the radius of the section, we may take

$$\dot{\sigma} = \frac{\delta\sigma}{\delta t}.$$

Whether ρ' vary or not over the cross section, provided $\dot{\sigma}$ be the same for every point on the core, a ring of this kind is equivalent to a ring source of uniform strength $\dot{\sigma}/4\pi$ per unit length of core.

Applying Mr Hicks' result, denoting by a the radius of the sphere and by f the distance of the ring core from the centre, we find for the image a ring source at a distance a^2/f from the centre, whose strength per unit length of core is

$$\frac{a}{f}\frac{f^2}{a^2}\frac{\dot{\sigma}}{4\pi} = \frac{f}{a}\frac{\dot{\sigma}}{4\pi},$$

and a sink spread over the surface of a cone which extends from the image source to the centre. The strength of the annulus of the sink by two planes perpendicular to the axis of the cone intercepting unit length on the generators is everywhere

$$-\frac{2\pi b\dot{\sigma}}{4\pi a} = -\frac{b}{2a}\dot{\sigma}.$$

The strength of the cone sink per unit of surface would thus become infinite at its vertex, but that only means that there is a point sink of finite strength at the centre of the sphere. Combining the systems of images for the vorticity and for the compressibility, we clearly possess a solution of the problem whose solubility we had to establish.

While the preceding investigation as to the image system for the compressibility possesses the advantage of bringing out more clearly the physical side of the question than is possible in the analytical method, it would not in general lead very readily to the actual de-

* *Proceedings*, Vol. vi., p. 65.

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termination of the fluid velocities. When the angle 2α of the cone subtended by the ring at the centre of the sphere is small, we could obtain approximate values for the velocity at points at a considerable distance from the ring by regarding it and the image ring as point sources and the image sink as a line sink.

In general recourse had better be had to the analytical expression for the velocity potential given below. Using a, b, α, and f in the same sense as above, we have $b = f\sin\alpha$.

Denoting by $Q_n(\theta)$ the nth zonal harmonic whose pole, $\theta = 0$, lies on the diameter of the sphere which is the axis of the ring, I find for the complete value of the velocity potential at the point (r,θ) referred to the centre of the sphere as origin

$$P = P_1 + P_2,$$

where

$$P_1 = -\frac{\dot{\sigma} a \sin\alpha}{2f}\left\{\frac{1}{2}\frac{a^2}{r^2}Q_1(\alpha)Q_1(\theta) + \dots + \frac{n}{n+1}\frac{a^{2n}}{f^{n-1}r^{n+1}}Q_n(\alpha)Q_n(\theta) + \dots\right\} \quad \dots \quad (50),$$

$$\left.\begin{aligned} P_2 &= -\frac{\dot{\sigma}\sin\alpha}{2}\left\{1 + \frac{r}{f}Q_1(\alpha)Q_1(\theta) + \dots + \frac{r^n}{f^n}Q_n(\alpha)Q_n(\theta) + \dots\right\} \\ &\text{or} \\ &= -\frac{\dot{\sigma} f\sin\alpha}{2}\left\{\frac{1}{r} + \frac{f}{r^2}Q_1(\alpha)Q_1(\theta) + \dots + \frac{f^n}{r^{n+1}}Q_n(\alpha)Q_n(\theta) + \dots\right\}\end{aligned}\right\} (51).$$

The first part of the velocity potential, P_1, may be regarded as answering to the image system. The second part P_2 is the velocity potential due to the ring itself; and the first or the second value for it is to be taken according as the point considered is nearer the centre of the sphere or is more remote than the ring core.

In calculating P_2 the method suggested in Thomson & Tait's *Natural Philosophy*, Art. 546, may be readily applied. The determination of P_1 is based on the vanishing of the normal component of the fluid velocity over the spherical surface. It presents no difficulty.

Over the spherical surface the velocity is everywhere of course along the tangent which lies in the plane through the axis of the vortex ring. At an angular distance θ from the axis its magnitude is

$$\left[\frac{1}{r}\left(\frac{dP_1}{d\theta} + \frac{dP_2}{d\theta}\right)\right]_{r=a} = \frac{\dot{\sigma}\sin\alpha\sin\theta}{2a}\left\{\frac{3a}{2f}Q_1(\alpha)Q_1'(\mu) + \dots + \frac{2n+1}{n+1}\left(\frac{a}{f}\right)^n Q_n(\alpha)Q_n'(\mu) + \dots\right\} \quad \dots \quad (52),$$

where $Q_n'(\mu) = \frac{d}{d\mu} Q_n(\mu)$, denoting $\cos\theta$ by μ, and replacing the notation $Q_n(\theta)$ by $Q_n(\mu)$. The surface velocity thus vanishes, as is obvious from symmetry, where the axis of the ring cuts the spherical surface.

The expressions for the velocity potential in the form given above do not converge rapidly unless the distance of the ring from the centre of the sphere be considerable compared to the radius.

On a problem in permutations.

By R. E. ALLARDICE, M.A.

The problem to be considered may be stated as follows :—How many necklaces may be formed with p pearls, r rubies, and d diamonds ?*

The peculiarity of this problem is that a general solution cannot be given in terms of p, r, and d alone. The form of the solution depends on the nature of the numbers p, r, and d ; and it is necessary in solving the problem to consider whether or not these numbers have a common measure, and how many of them are odd and how many even. All possible cases of the problem are not discussed in this paper ; but enough of them are considerod to illustrate the variety of forms that the solution may assume.

If we put $p + r + d = n$, the number of possible arrangements of the n stones in a line is $n!/p!r!d!$ Hence the question is, how many of these arrangements will give the same necklace ; or, conversely, if we take any one form of the necklace, how many different arrangements of the stones we can get from it by breaking it at different parts and stretching it out straight. It is obvious that if the n stones had been all different, the answer to the second of these questions would have been $2n$; in other words, with n stones all different, we may form $n!/2n$ necklaces. The further question then naturally arises, In what cases, if any, are the $2n$ arrangements of the stones obtained from each form of the necklace all different when the stones are not all different ? Now these $2n$ arrangements comprise the n that are obtained by a cyclical interchange of the stones, one at a time, together with the n that are obtained by exactly reversing each of these n arrangements.

* This problem was suggested to me by Professor Chrystal.

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