JETSTREAM FORMATION THROUGH INELASTIC COLLISIONS

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An inelastic collision integral is used in a Boltzmann-type equation for a distribution of particles in Kepler orbits. A Fokker-Planck equation is found that leads to radial density clustering.

It has been suggested that in a cloud of grains moving in Kepler orbits in a gravitational field, inelastic collisions will cause the grains to form groups having similar orbits, called jetstreams (Alfvén, 1970). One would expect that a jetstream already formed would contract into a tighter jetstream (Trulsen[1]) because the jetstream would lose total energy because of the inelasticity of collisions, whereas the total angular momentum would be conserved. The grains would move toward a circular orbit because circular orbits have the lowest energy E_0 for a given angular momentum L (fig. 1). We consider the question of whether jetstreams will form from an initially smooth distribution function.

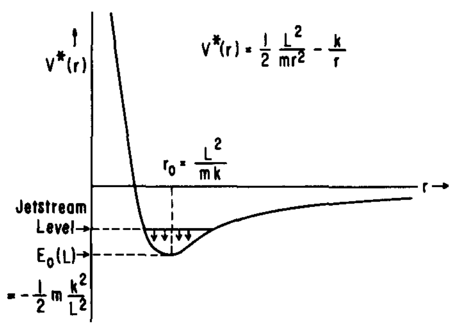

Figure 1.—A jetstream can be thought of as a group of particles filling an effective potential well $V^*(r)$, which arises from gravitational and centrifugal forces. As the particles lose energy E in inelastic collisions, they concentrate at the bottom of the well. m = mass of a single grain.

[1]See p. 327.

THE MODEL

The essential feature of this suggestion is the inelasticity of the collisions. Accordingly, we look at a particularly simple model with particularly simple inelastic collisions; i.e., perfectly inelastic collisions in which colliding particles stick together. We avoid the consideration of accretion of particles by considering the final velocities of colliding particles to be arbitrarily close while the particles maintain their distinct identities. We consider one species of particles moving in coplanar Kepler orbits.

Particles in an arbitrary distribution of exactly circular orbits would never collide, so such a distribution would be stationary (fig. 2). A thermal equilibrium distribution

$$f(r, \theta, p, L) = A \exp\left[-\beta\left(\frac{p^2}{2m} + \frac{L^2}{2mr^2} - \frac{k}{r}\right) - \alpha L\right] \tag{1}$$

yields a radial density distribution

$$n(r) = \iiint f \, d\theta dp dL$$

$$= A(2\pi)^2 \frac{mr}{\beta} \exp\left(\beta \frac{k}{r} + \frac{mr^2\alpha}{2\beta}\right) \tag{2}$$

which diverges exponentially at $r = 0$ and $r = \infty$, where r and θ are polar coordinates, p and L are the corresponding canonical momenta (radial and angular momentum, respectively), m is the mass of a single grain, and $\Phi(r) = -k/r$ is the gravitational potential energy. A distribution with minimum

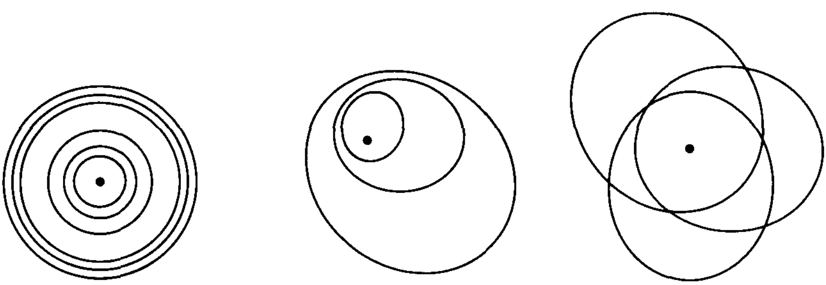

Figure 2.–Particles in circular orbits (or in nonintersecting orbits with $\epsilon \neq 0$) experience no collisions. Collisional evolution only occurs when these elliptical orbits intersect. Kepler orbit: $\Phi(r) = -k/r$.

energy for a given total angular momentum would be one in which a single grain, having all the angular momentum and almost no energy, moves very slowly in an orbit with very large radius, whereas all the other particles collapse into the central body.

We consider an initial distribution that depends arbitrarily on angular momentum, has orbits of generally small eccentricity, and is axisymmetric:

$$f_0(r, \theta, p, L) = A(\epsilon_0)F(L) \exp \frac{-\epsilon^2}{\epsilon_0{}^2} \tag{3}$$

where

$$\epsilon^2(r, p, L) = \frac{p^2/2m + L^2/2mr^2 - k/r + mk^2/2L^2}{mk^2/2L^2}$$

is the spatial eccentricity of an orbit that passes through the point (r, θ, p, L) in phase space, and $-mk^2/2L^2$ is the energy of a circular orbit with angular momentum L. We assume that essentially all of the particles are orbiting in the same direction. We use a smooth function $F(L)$ so that

$$F(L + \delta L) \simeq F(L) + \delta L \frac{dF}{dL} + \frac{\delta L^2}{2} \frac{d^2F}{dL^2} \tag{4}$$

(where $\delta L = 0(\epsilon L)$) is a valid expression. We can rewrite equation (3) as

$$f_0 = \widetilde{A}F(L) \exp [-\beta(L)E] \tag{5}$$

which has a superficial resemblance to the equation for thermal equilibrium. Note that the initial axisymmetry demands that the final state will be axisymmetric (i.e., only circular jetstreams are possible). We will also have to consider the functions

$$h(L) = \iiint f\, dr d\theta dp \simeq (2\pi)^2 \epsilon_0{}^2 \frac{L}{2} AF(L) \tag{6}$$

$$A(\epsilon_0) = N \left(\iiiint f\, dr d\theta dp dL\right)^{-1}$$

$$= N\left[\int h(L)\, dL\right]^{-1} \tag{7}$$

where $h(L)$ is the density in angular momentum space, $A(\epsilon_0)$ is the normalization constant, and N is the total number of grains.

THE CALCULATION

We wish to find a differential equation that describes the evolution of $h(L)$, the distribution function in angular momentum space. The equation describing the time evolution of the phase space distribution function is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = I(f, f) \tag{8}$$

where $\mathbf{x}$ is the position vector, $\mathbf{v} = d\mathbf{x}/dt$, $\mathbf{a} = d\mathbf{v}/dt$, and $I(f, f)$ is the collision integral. Our initial distribution f_0, being a function of constants of the motion only, is stationary in the absence of collisions. We assume that the mean free path is long compared with the orbital path so that collisions are treated as a perturbation, whence

$$f = f_0 + \delta f \tag{9}$$

where δf is the perturbation distribution caused by collisions. Linearizing equation (8) we get

$$\frac{df}{dt} = \frac{d\delta f}{dt} = I(f_0, f_0) \tag{10}$$

The collision integral for completely inelastic collisions is

$$I(f, f) = -\iint \frac{\sigma}{r}(g)|g| \Bigg[f(r, \theta, p, L) f(r, \theta, p + p', L + L') - f\left(r, \theta, p - \frac{p'}{2}, L - \frac{L'}{2}\right) f\left(r, \theta, p + \frac{p'}{2}, L + \frac{L'}{2}\right)\Bigg] dL'dp' \tag{11}$$

where p' and L' are relative momenta between colliding grains,

$$g = \left[\left(\frac{p'}{m}\right)^2 + \left(\frac{L'}{mr}\right)^2\right]^{1/2} \tag{12}$$

is the relative speed between the two grains, and $\sigma(g)$ is the collisional cross section. (Note that a (p, L); $(p + p', L + L')$ collision scatters a particle out of the phase space volume element at (r, θ, p, L) and a $(p - p'/2, L - L'/2)$; $(p + p'/2, \; L + L'/2)$ collision scatters two particles into a phase space volume element at (r, θ, p, L).)

Inserting equation (3) into equation (11), Taylor-expanding $F(L)$, assuming that $\sigma(g)$ is a constant, and integrating over p' and L', we get

$$\frac{df}{dt}(r, \theta, p, L) = \epsilon_0{}^2 mk \frac{\sigma}{r} \sqrt{\frac{\epsilon_0{}^2 k}{mr}}\, A^2$$

$$\times \left\{ EF^2(L) + \frac{\epsilon_0{}^2 L^2}{8} B \left[F(L) \frac{d^2F}{dL^2} - \left(\frac{dF}{dL}\right)^2 \right] \right\} \exp\left(-\frac{2\epsilon^2}{\epsilon_0{}^2}\right)$$

$$- \left[CF^2(L) + \left(D\delta L + G\frac{\epsilon_0 L}{2}\right) \frac{dF^2}{dL} \right.$$

$$+ \frac{1}{2}\left(H\delta L^2 + J\delta L \;\frac{\epsilon_0 L_0}{2} \;+ K\frac{\epsilon_0{}^2 L_0{}^2}{4}\right) \frac{d^2F^2}{dL^2}$$

$$\left. + \frac{1}{2}\left(M\delta L^2 + N\delta L\frac{\epsilon_0 L_0}{2} + P\frac{\epsilon_0{}^2 L_0{}^2}{4}\right)\left(\frac{dF}{dL}\right)^2 \right] \exp\left(-\frac{\epsilon^2}{\epsilon_0{}^2}\right) \qquad (13)$$

where $\delta L = L - L_0$; $L_0 = \sqrt{mkr}$ is the angular momentum of the circular orbit at radius r; E and B are constants; and C, D, G, N, J, K, M, N, and P are polynomials in ϵ/ϵ_0 with coefficients of order unity. The expressions $\epsilon_0{}^2 mk$ and $\sqrt{\epsilon_0{}^2 k/mr}$ can be thought of as effective available relative momentum space and mean relative velocity, respectively. To find

$$\frac{dh}{dt} = \iiint \frac{df}{dt}\, dr d\theta dp = \iiint I(f_0, f_0)\; dr d\theta dp \qquad (14)$$

we transform (r, p) to (ϵ^2, χ) where χ is the orientation of the major axis of the ellipse through the point (r, θ, p, L) in phase space:

$$r = \frac{L^2}{mk}[1 + \epsilon \cos(\theta - \chi)]^{-1}$$

$$p = \frac{mk}{L}\epsilon \sin(\theta - \chi) \qquad (15)$$

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The Jacobian is

$$\frac{\partial(r, p)}{\partial(\epsilon^2, \chi)} = -\frac{L}{2}[1 + \epsilon \cos(\theta - \chi)]^{-2}$$

so

$$\frac{dh}{dt} = 2\pi \frac{L}{2} \int \frac{d\epsilon^2 d\theta}{(1 + \epsilon \cos\theta)^2} \frac{df}{dt}$$

$$= -2\pi\epsilon_0{}^2 mk\sigma \sqrt{\frac{\epsilon_0{}^2 k}{mr_0}} \frac{L}{2} A^2 \epsilon_0{}^2$$

$$\times \left\{ aF^2(L) + \frac{\epsilon_0 L}{2} b \frac{dF^2}{dL} + \frac{\epsilon_0{}^2 L^2}{8}\left[c \frac{d^2F^2}{dL^2} + d\left(\frac{dF}{dL}\right)^2 \right] \right\} \tag{16}$$

where a, b, c, and d are polynomials in ϵ_0. Because the total number of particles and the total angular momentum are conserved, we have

$$\int \frac{dh}{dt} dL = 0 \qquad \int L \frac{d}{dt} h(L)\, dL = 0 \tag{17}$$

with the result that

$$a = b = 0 \tag{18}$$

therefore

$$\frac{dh}{dt} = -\pi\epsilon_0{}^2 m^2 k^2 \sigma \epsilon_0 k \frac{\epsilon_0{}^4}{8} c \frac{d^2}{dL^2} A^2 F^2 \tag{19}$$

It can be shown that $c > 0$. Using equation (6), we find that

$$\frac{dh}{dt} = -\frac{\epsilon_0{}^3 m^2 k^3 \sigma}{32\pi^3} c \frac{d^2}{dL^2}\left(\frac{h^2}{L^2}\right) \tag{20}$$

CONCLUSION AND INTERPRETATION

Equation (20) looks something like a diffusion equation with a negative diffusion coefficient. Thus $h(L)$ grows at maxima of $L^{-2}h^2(L)$ and decreases where $L^{-2}h^2(L)$ is at a minimum as shown in figure 3(a). Note that this

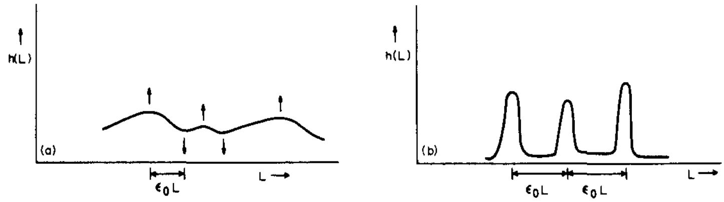

Figure 3.—(*a*) Initial smooth distribution function with local maxima separated by approximately $\epsilon_0 L$. (*b*) A distribution function in which most of the particles are in groups, separated in angular momentum by about $\epsilon_0 L$.

equation will never allow $h(L)$ to become negative. The fastest growth is experienced by the narrowest peaks. These fine scale peaks eventually dominate the distribution function, and the particles concentrate at the angular momenta where the narrowest peaks were originally, as in figure 3(*b*). We initially used a distribution function that was smooth on a scale length $\epsilon_0 L$ (i.e., initial peaks in $F(L)$ and, consequently, in $h(L)$ and $h^2(L)$ were separated in angular momentum by distances $\epsilon_0 L$). Because ϵ_0 is small compared to unity, L is slowly varying by comparison and peaks in $L^{-2}h^2(L)$ are separated by $\epsilon_0 L$. Thus grains concentrate in orbits separated in angular momentum by $\epsilon_0 L$; these jetstreams must be circular because of the initial axisymmetry of our distribution. As the grains lose energy because of inelastic collisions, the orbits themselves become more circular.

A finer grain distribution function would have finer scale peaks, but our result would not necessarily apply in that case because the calculation depended critically on the Taylor expansion of the original distribution. The fine scale peaks might evolve into distinct subjetstreams, or they might merge into a single jetstream.

Now consider what effect this has on the radial density distribution. Because the radius of a circular orbit is related to its angular momentum by

$$r = \frac{L^2}{mk} \tag{21}$$

then the radial separation is given by

$$\frac{\delta r}{r} = 2\frac{\delta L}{L} = 2\epsilon_0 \tag{22}$$

(In our solar system $\delta r/r$ is roughly 0.4 to 0.6 corresponding to $\epsilon_0 \cong 0.25 \pm 0.05$ (Jeans, 1944).)

There are many other properties that may influence the collisional evolution of an orbiting cloud of grains. Although we have neglected size, shape, and mass differences among grains, effects of rotational degrees of freedom, self-gravitation, actual accretion, or even shattering of particles, our calculation indicates that the inelasticity tends to cause jetstreams.

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