EQUICARDINALITY OF BASES IN B-MATROIDS

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It is very well known that any two bases of a finitary matroid (see [2] for definitions) have the same cardinality. As Dlab has shown in [1], the same does not hold for arbitrary transitive exchange spaces; indeed, since the examples Dlab constructs in [1] are matroids, it does not even hold for arbitrary matroids. Nevertheless with the aid of the generalized continuum hypothesis (G. C. H.) we are able to prove the result for B-matroids.

THEOREM 1. *Let* $\mathfrak{B}$ *be a set of subsets of a set* E *satisfying*

(i) *no one member of* $\mathfrak{B}$ *is properly contained in another, and*

(ii) *if* B_1 *and* B_2 *are in* $\mathfrak{B}$ *and* A, C *are subsets of* E *such that* $A \subseteq B_1, B_2 \subseteq C$, *and* $A \subseteq C$ *then there exists* B *in* $\mathfrak{B}$ *such that* $A \subseteq B \subseteq C$.

Then if the G. C. H. *is true the members of* $\mathfrak{B}$ *all have the same cardinality.*

Proof. Let B_1 and B_2 be in $\mathfrak{B}$. If B_1 is infinite then, using Sierpinski's construction [3] and the G. C. H. (see also Wolk [4]), we obtain a chain $\mathcal{C}$ of subsets of B_1 such that $|\mathcal{C}| = 2^{|B_1|}$. For each C in $\mathcal{C}$, (ii) shows that there exists a subset D of B_2 such that $C \cup D$ is in $\mathfrak{B}$ and $C \cap D = \phi$. If we select exactly one D for each C then by (i) the resulting D's will be distinct and B_2 must have at least $2^{|B_1|}$ subsets. From $2^{|B_1|} \leq 2^{|B_2|}$ and the G. C. H. we obtain $|B_1| \leq |B_2|$. If B_1 is finite then a similar (and in this case familiar) argument leads to the same conclusion: take $|\mathcal{C}| = |B_1| + 1$ and choose the D's to form a chain themselves, as is clearly possible when $\mathcal{C}$ is finite. Likewise, we may show that $|B_2| \leq |B_1|$. Q.E.D.

Since the set $\mathfrak{B}$ of all bases of a B-matroid is easily seen to satisfy (i) and (ii), we have the following theorem.

THEOREM 2. *The bases of a B-matroid all have the same cardinality.*

Two questions. Is any $\mathfrak{B} \neq \phi$ satisfying (i) and (ii) the set of bases of some B-matroid? Does Theorem 1 (or Theorem 2) imply the G. C. H. ?

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