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Abstract

We construct a functor associating a cubical set to a (simple) graph. We show that cubical sets arising in this way are Kan complexes, and that the A-groups of a graph coincide with the homotopy groups of the associated Kan complex. We use this to prove a conjecture of Babson, Barcelo, de Longueville, and Laubenbacher from 2006, and a strong version of the Hurewicz theorem in discrete homotopy theory.

Contents

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Introduction

Discrete homotopy theory, introduced in [BKLW01, BBdLL06], is a homotopy theory in the category of simple graphs. It builds on the earlier work of Atkin [Atk74, Atk76], made precise in [KL98], on the homotopy theory of simplicial complexes, and can also be generalized to the homotopy theory of finite metric spaces [BCW14]. It has found numerous applications [BL05, § 5–6], including in matroid theory, hyperplane arrangements, combinatorial time series analysis, and, more recently, topological data analysis [MZ19].

The key invariants associated to graphs in discrete homotopy theory are the A-groups $A_n(G, v)$, named after Atkin, which are the discrete analogue of homotopy groups $\pi_n(X, x)$, studied in the homotopy theory of topological spaces. In [BBdLL06], Babson, Barcelo, de Longueville, and Laubenbacher construct an assignment $G \mapsto X_G$, taking a graph G to a topological space, constructed as a certain cubical complex, and conjecture that the A-groups of G coincide with the homotopy groups of X_G. They further prove [BBdLL06, Theorem 5.2] their conjecture under an assumption of a cubical approximation property [BBdLL06, Proposition 5.1], a cubical analogue of the simplicial approximation theorem, which remains open.

The assignment $G \mapsto X_G$ arises as a composite

$$\mathsf{Graph} \xrightarrow{\mathrm{N}_1} \mathsf{cSet} \xrightarrow{|-|} \mathsf{Top}.$$

Here cSet denotes a particular category of cubical sets, which are well-studied combinatorial models for the homotopy theory of spaces [Cis06]. (Specifically, cubical sets used in this paper are cubical sets with positive and negative connections; cf. [BH81, BM17, DKLS24].) Informally, a cubical set consists of a family of sets $\{X_n\}_{n \in \mathbb{N}}$ of n-cubes together with a family of structure maps indicating how different cubes 'fit together', e.g. that a certain $(n-1)$-cube is a face of another n-cube. More formally, it is a presheaf on the category $\square$ of combinatorial cubes. The functor $\mathrm{N}_1 \colon \mathsf{Graph} \to \mathsf{cSet}$ is obtained by taking the n-cubes of $\mathrm{N}_1 G$ to be maps $I_1^{\otimes n} \to G$, where $I_1^{\otimes n}$ denotes the n-dimensional hypercube graph. The functor $|-| \colon \mathsf{cSet} \to \mathsf{Top}$, called the geometric realization, assigns the topological n-cube $[0,1]^n$ to each (formal) n-cube of X and then glues these cubes together according to structure maps. (We will of course give precise definitions of all these notions later in the paper.)

A reader familiar with [BBdLL06] will recognize a change in notation: the functor N_1 above was in [BBdLL06, § 4] denoted by M_*. This change is intentional, as we wish to emphasize that this is only the first in a sequence of functors $\mathsf{Graph} \to \mathsf{cSet}$ and natural transformations:

$$\mathrm{N}_1 \longrightarrow \mathrm{N}_2 \longrightarrow \mathrm{N}_3 \longrightarrow \cdots. \tag{1}$$

It is this sequence and its colimit

$$\mathrm{N} := \operatorname{colim}(\mathrm{N}_1 \to \mathrm{N}_2 \to \mathrm{N}_3 \to \cdots)$$

that we investigate. Our main technical results can be summarized by the following theorem.

THEOREM (cf. Theorems 4.1 and 4.6).

(i) *For a graph G, the cubical set $\mathrm{N}G$ is a Kan complex.*
(ii) *The natural transformations in (1) are natural weak equivalences.*

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(iii) *For a based graph* (G, v), *there is a natural group isomorphism* $A_n(G, v) \cong \pi_n(\mathrm{N}G, v)$.

A few words of explanation are in order. A Kan complex is a cubical set satisfying a certain lifting property making it particularly convenient for the development of homotopy theory. In particular, in a companion paper [CK23], we develop the theory of homotopy groups of Kan complexes and show that these agree with their topological analogues under the geometric realization functor.

This establishes the key ingredient required for the proof of the conjecture of Babson, Barcelo, de Longueville, and Laubenbacher, which asks for the commutativity of the outer square in the following diagram.

$$
\begin{tikzcd}
 & \mathsf{cSet}_* \arrow[dr, bend left, "|-|"] & \\
\mathsf{Graph}_* \arrow[ur, bend left, "\mathrm{N}_1"] \arrow[r, "\mathrm{N}"] \arrow[dr, bend right, "A_n"'] & \mathsf{Kan}_* \arrow[u, hook] \arrow[r, "|-|"] \arrow[d, "\pi_n"] & \mathsf{Top}_* \arrow[dl, bend left, "\pi_n"] \\
 & \mathsf{Grp} &
\end{tikzcd}
$$

By the theorem above, the two triangles on the left commute, with the upper one commuting up to a natural weak equivalence (which is sent by the composite functor $\pi_n \circ |-| \colon \mathsf{cSet}_* \to \mathsf{Grp}$ to a natural isomorphism). The upper right triangle commutes on the nose and the lower right triangle commutes since it expresses the compatibility between homotopy groups of cubical sets and those of topological spaces, established in [CK23]. Thus, the A-groups of (G, v) agree with those of $(\mathrm{N}G, v)$, which in turn agree with those of $(|\mathrm{N}_1 G|, v)$, since the map $\mathrm{N}_1 G \to \mathrm{N}G$ is a weak equivalence.

Theorem (Conjectured in [BBdLL06]; cf. Theorem 5.1). *There is a natural group isomorphism* $A_n(G, v) \cong \pi_n(|\mathrm{N}_1 G|, v)$.

The insight of [BBdLL06] cannot be overstated. It is *a priori* not clear, or at the very least it was not clear to us, that the A-groups of a graph should correspond to the homotopy groups of any space. The fact that the space can be obtained from a graph in such a canonical and simple way is what drove us to the problem and to the field of discrete homotopy theory. The title of our paper, ‘Cubical setting for discrete homotopy theory, revisited’, pays tribute to this insight by alluding to the title ‘A cubical set setting for the A-theory of graphs’ of [BBdLL06, § 3].

It is also worth noting that the case of $n = 1$ of Theorem 5.1 was previously proven in [BKLW01, Proposition 5.12] and perhaps helped inspire the statement of the conjecture in the general case.

Our main theorem allows us to derive a few more consequences of interest in discrete homotopy theory. The first of those is a strong form of the Hurewicz theorem for graphs. The Hurewicz theorem relates the first non-trivial homotopy group of a sufficiently connected space to its homology. In discrete homotopy theory, it relates the first non-trivial A-group of a graph to its reduced discrete homology, introduced in [BCW14].

Theorem (Discrete Hurewicz theorem; cf. Theorem 5.8). *Let* $n \geq 2$ *and* (G, v) *be a connected pointed graph. Suppose* $A_i(G, v) = 0$ *for all* $i \in \{1, \ldots, n-1\}$. *Then the Hurewicz map* $A_n(G, v) \to \widetilde{DH}_n(G, v)$ *from the* n*th A-group to the* n*th reduced discrete homology group is an isomorphism.*

This generalizes the results of Lutz [Lut21, Theorem 5.10], who proves surjectivity of the Hurewicz map for a more restrictive class of graphs, and complements the result of Barcelo, Capraro, and White [BCW14, Theorem 4.1], who prove the one-dimensional analogue of the Hurewicz theorem, namely that the Hurewicz map $A_1(G,v) \to \widetilde{DH}_1(G,v)$ is surjective with kernel given by the commutator $[A_1(G,v), A_1(G,v)]$ subgroup.

Finally, our main theorem allows us equip the category of graphs with additional structure making it amenable to techniques of abstract homotopy theory and higher category theory.

By our theorem, the functor $\mathrm{N}\colon \mathsf{Graph} \to \mathsf{cSet}$ takes values in the full subcategory Kan of cSet spanned by Kan complexes. This subcategory is known to carry the structure of a fibration category in the sense of Brown [Bro73]. In brief, a fibration category is a category equipped with two classes of maps: fibrations and weak equivalences, subject to some axioms. Fibration categories are one of the main frameworks used in abstract homotopy theory (see, for instance, [Szu16]). We declare a map f of graphs to be a fibration/weak equivalence if $\mathrm{N}f$ is one in the fibration category of Kan complexes. Our theorem guarantees that this gives a well-defined fibration category structure on Graph (Theorem 5.9), hence allowing for the use of techniques from abstract homotopy theory in discrete homotopy theory. Furthermore, it follows that the weak equivalences of this fibration category are precisely graph maps inducing isomorphisms on all A-groups for all choices of the basepoint.

This also allows us to put the results of [BBdLL06, § 6] in the context of abstract homotopy theory by proving that the loop graph functor constructed there is an exact functor in the sense of fibration category theory (Theorem 5.27).

In a different direction, we observe that the functor $\mathrm{N}\colon \mathsf{Graph} \to \mathsf{cSet}$ is lax monoidal (Lemma 5.28). The category of graphs is enriched over itself, meaning that the collection of graph maps between two graphs forms not merely a set, but a graph, and that the composition of graph maps defines a graph homomorphism. Enrichment can be transferred along lax monoidal functors, which means that the category of graphs is, via N, canonically enriched over cubical sets, and, more precisely, over Kan complexes (Theorem 5.36). This establishes a presentation of the $(\infty, 1)$-category of graphs, thus allowing for the use of techniques of higher category theory, developed extensively by Joyal and Lurie [Lur09], in discrete homotopy theory, via the cubical homotopy coherent nerve construction of [KV20, § 2].

Organization of the paper

This paper is organized as follows. In §§ 1, 2, we review the background on discrete homotopy theory and cubical sets, respectively. In § 3, we explain the link between graphs and cubical sets by defining the functors N_m and N, and proving their basic properties. The technical heart of the paper is contained in § 4, where we prove our main results. In § 5, we proceed to deduce the consequences of our main theorem, as described above.

1. Discrete homotopy theory

The category of graphs

We define the category of simple undirected graphs without loops as a reflective subcategory of a presheaf category.

Let G be the category generated by the diagram

$$V \underset{t}{\overset{s}{\underset{\longleftarrow r \longrightarrow}{\longrightarrow}}} E \circlearrowleft \sigma$$

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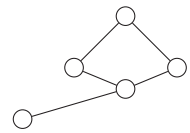

Figure 1. An example depiction of a graph.

subject to the identities

$$\begin{aligned} &rs = rt = \mathrm{id}_V, && \sigma^2 = \mathrm{id}_E,\\ &\sigma s = t, && \sigma t = s,\\ &r\sigma = r. \end{aligned}$$

We write $\widehat{\mathsf{G}}$ for the functor category $\mathsf{Set}^{\mathsf{G}^{\mathrm{op}}}$.

For $G \in \widehat{\mathsf{G}}$, we write G_V and G_E for the sets $G(V)$ and $G(E)$, respectively. Explicitly, such a functor consists of sets G_V and G_E together with the following functions between them:

$$\begin{tikzcd} G_V \arrow[r, "Gr"] & G_E \arrow[l, shift right=2, "Gs"'] \arrow[l, shift left=2, "Gt"] \arrow[loop right, "G\sigma"] \end{tikzcd}$$

subject to the dual versions of identities in G.

Definition 1.1. A *graph* is a functor $G \in \widehat{\mathsf{G}}$ such that the map $(Gs, Gt) \colon G_E \to G_V \times G_V$ is a monomorphism (Figure 1).

Let Graph denote the full subcategory of $\widehat{\mathsf{G}}$ spanned by graphs.

In more concrete terms, a graph G consists of a set G_V of vertices and a set G_E of 'half-edges'. A half-edge $e \in G_E$ has source and target, and these are given by the maps Gs and Gt, respectively. Each half-edge is paired with its other half via the map $G\sigma \colon G_E \to G_E$. Note that the edges paired by $G\sigma$ have swapped source and target, making the pair (i.e. whole edge) undirected. The map $Gr \colon G_V \to G_E$ takes a vertex to an edge whose source and target is that vertex (i.e. a loop). Finally, the condition that (Gs, Gt) is a monomorphism ensures that there is at most one (whole) edge between any two vertices. This is equivalent to specifying a binary 'incidence' relation on G_V which is reflexive and symmetric.

A map of graphs $f \colon G \to H$ is a natural transformation between such functors. However, since (Hs, Ht) is a monomorphism, such a map is completely determined by a function $f_V \colon G_V \to H_V$ that preserves the incidence relation.

We may therefore assume that our graphs have no loops, but the maps between them, rather than merely preserving edges, are allowed to contract them to a single vertex. That is, a graph map $f \colon G \to H$ is a function $f \colon G_V \to H_V$ such that if $v, w \in G_V$ are connected by an edge, then either $f(v)$ and $f(w)$ are connected by an edge or $f(v) = f(w)$.

Proposition 1.2.

(i) *The inclusion* $\mathsf{Graph} \hookrightarrow \widehat{\mathsf{G}}$ *admits a left adjoint.*
(ii) *The category* Graph *is (co)complete.*
(iii) *The functor* $\mathsf{Graph} \to \mathsf{Set}$ *mapping a graph* G *to its set of vertices* G_V *admits both adjoints.*

Proof. (i) The left adjoint is $\mathsf{Im} \colon \widehat{\mathsf{G}} \to \mathsf{Graph}$ given by

$$(\mathsf{Im}\, G)_V = G_V, \quad (\mathsf{Im}\, G)_E = (Gs, Gt)(G_E),$$

where $(Gs, Gt)(G_E)$ is the image of G_E under the map $(Gs, Gt) \colon G_E \to G_V \times G_V$.

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(ii) The category $\widehat{\mathsf{G}}$ is (co)complete as a presheaf category. The conclusion follows from (i) as Graph is a reflective subcategory of a (co)complete category.

(iii) The left adjoint $\mathsf{Set} \to \mathsf{Graph}$ takes a set A to the discrete graph with vertex set A. The right adjoint takes a set A to the complete graph with vertex set A. □

Remark 1.3. Proposition 1.2 gives a procedure for constructing limits and colimits in Graph. Given a diagram $F\colon J \to \mathsf{Graph}$, the set of vertices of $\lim F$ is the limit $\lim UF$ of the diagram $UF\colon J \to \mathsf{Set}$. The set of edges is the largest set such that the limit projections $\lim F \to F_j$ are graph maps. The set of vertices of $\operatorname{colim} F$ is the colimit $\operatorname{colim} UF$ of the diagram $UF\colon J \to \mathsf{Set}$ and the set of edges is the smallest set such that the colimit inclusions $F_j \to \operatorname{colim} F$ are graph maps.

Examples of graphs

DEFINITION 1.4. For $m \geq 0$,

(i) the *m-interval* I_m is the graph which has
  - as vertices, integers $0 \leq i \leq m$;
  - an edge between i and $i+1$ (see Figure 2);

(ii) the *m-cycle* C_m is the graph which has
  - as vertices, integers $0 \leq i \leq m-1$;
  - an edge between i and $i+1$ and an edge between $m-1$ and 0 (see Figure 2);

(iii) the *infinite interval* I_∞ is the graph which has
  - as vertices, integers $i \in \mathbb{Z}$;
  - an edge between i and $i+1$ for all $i \in \mathbb{Z}$.

Remark 1.5. The graphs I_0 and I_1 are representable when regarded as functors $\mathsf{G}^{\mathrm{op}} \to \mathsf{Set}$, represented by V and E, respectively.

For $m \geq 0$, we have a map $l\colon I_{m+1} \to I_m$ defined by

$$l(v) = \begin{cases} 0 & \text{if } v = 0 \\ v-1 & \text{otherwise.} \end{cases}$$

As well, we have a map $r\colon I_{m+1} \to I_m$ defined by

$$r(v) = \begin{cases} m & \text{if } v = m+1 \\ v & \text{otherwise.} \end{cases}$$

We write $c\colon I_{m+2} \to I_m$ for the composite $lr = rl$. Explicitly, this map is defined by

$$c(v) = \begin{cases} 0 & \text{if } v = 0 \\ m & \text{if } v = m+2 \\ v-1 & \text{otherwise.} \end{cases}$$

We show the inclusion $\mathsf{Graph} \hookrightarrow \widehat{\mathsf{G}}$ preserves filtered colimits and use this to show all finite graphs are compact, i.e. if G is a finite graph then the functor $\mathsf{Graph}(G, -)\colon \mathsf{Graph} \to \mathsf{Set}$ preserves filtered colimits.

PROPOSITION 1.6. *The inclusion* $\mathsf{Graph} \hookrightarrow \widehat{\mathsf{G}}$ *preserves filtered colimits.*

Proof. Fix a filtered category J and a diagram $D\colon J \to \mathsf{Graph}$. Let i denote the inclusion $\mathsf{Graph} \hookrightarrow \widehat{\mathsf{G}}$. Recall that $\operatorname{colim} D$ is computed by $\mathsf{Im}(\operatorname{colim}(iD))$. It suffices to show $\operatorname{colim}(iD) \in \widehat{\mathsf{G}}$ is a graph,

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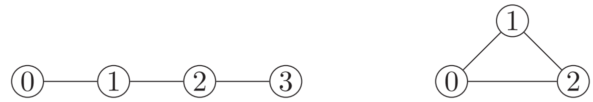

Figure 2. The graphs I_3 and C_3, respectively.

since then the unit map $\operatorname{colim}(iD) \to i\mathsf{lm}(\operatorname{colim}(iD))$ is an isomorphism $\operatorname{colim}(iD) \cong i(\operatorname{colim} D)$ natural in D.

Let $\lambda\colon iD \to \operatorname{colim}(iD)$ denote the colimit cone. Suppose two edges $e, e' \in \operatorname{colim}(iD)_E$ have the same source and target. Regarding e, e' as maps $I_1 \to \operatorname{colim}(iD)$, these maps factor as

$$\begin{tikzcd} I_1 \arrow[rr, "{e,e'}"] \arrow[dr, "{\overline{e},\overline{e}'}"'] & & \operatorname{colim}(iD) \\ & iDx \arrow[ur, "\lambda_x"'] & \end{tikzcd}$$

for some $x \in J$ since $I_1 \in \widehat{\mathsf{G}}$ is representable. Let $s, s' \in (iDx)_V$ denote the sources of $\overline{e}, \overline{e}'$ and let $t, t' \in (iDx)_V$ denote the targets, respectively. Using an explicit description of the colimit (Remark 1.3), since $\lambda_x(s) = \lambda_x(s')$ and $\lambda_x(t) = \lambda_x(t')$, there exist arrows $f, g\colon y \to x$ and $h, k\colon z \to x$ with vertices $v \in iDy$ and $w \in iDz$ such that

$$iDf(v) = s, \quad iDg(v) = s', \quad iDh(w) = t, \quad iDk(w) = t'.$$

As J is filtered, there exists an arrow $l\colon x \to w$ in J such that $lf = lg$ and $lh = lk$. This implies the edges $iDl(\overline{e}), iDl(\overline{e}') \in (iDw)_E$ have the same source and target. As iDw is a graph, it follows that $\overline{e} = \overline{e}'$, thus $e = e'$. □

Corollary 1.7. *For a finite graph G, the functor $\mathsf{Graph}(G, -)\colon \mathsf{Graph} \to \mathsf{Set}$ preserves filtered colimits.*

Proof. Given a filtered category J and a diagram $D\colon J \to \mathsf{Graph}$, we have a natural isomorphism

$$\mathsf{Graph}(G, \operatorname{colim} D) \cong \widehat{\mathsf{G}}(G, \operatorname{colim} D)$$

by Proposition 1.6 since Graph is a full subcategory. This then follows since $G \in \widehat{\mathsf{G}}$ is a finite colimit of representable presheaves. □

Monoidal structure on the category of graphs

Define a functor $\otimes\colon \mathsf{G} \times \mathsf{G} \to \widehat{\mathsf{G}}$ by

$$\begin{array}{ll} (V, V) \mapsto I_0, & (V, E) \mapsto I_1, \\ (E, V) \mapsto I_1, & (E, E) \mapsto C_4. \end{array}$$

Left Kan extension along the Yoneda embedding yields a monoidal product $\otimes\colon \widehat{\mathsf{G}} \times \widehat{\mathsf{G}} \to \widehat{\mathsf{G}}$:

$$\begin{tikzcd} \mathsf{G} \times \mathsf{G} \arrow[r] \arrow[d] & \widehat{\mathsf{G}} \\ \widehat{\mathsf{G}} \times \widehat{\mathsf{G}} \arrow[ur, dotted, "\otimes"'] & \end{tikzcd}$$

Note that Graph is closed with respect to this product, i.e. if $G, H \in \mathsf{Graph}$ then $G \otimes H \in \widehat{\mathsf{G}}$ is a graph. Thus, this product descends to a monoidal product $\otimes\colon \mathsf{Graph} \times \mathsf{Graph} \to \mathsf{Graph}$, called the *cartesian product.*

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Explicitly, the graph $G \otimes H$ has:

– as vertices, pairs (v, w) where $v \in G$ is a vertex of G and $w \in H$ is a vertex of H;
– an edge from (v, w) to (v', w') if either $v = v'$ and w is connected to w' in H or $w = w'$ and v is connected to v' in G.

DEFINITION 1.8. Let G and H be graphs. The graph $\hom^{\otimes}(G, H)$ has:

– as vertices, morphisms $G \to H$ in Graph;
– an edge from f to g if there exists $H\colon G \otimes I_1 \to H$ such that $H|_{G\otimes\{0\}} = f$ and $H|_{G\otimes\{1\}} = g$.

This structure makes the category of graphs into a closed symmetric monoidal category.

PROPOSITION 1.9. $(\mathsf{Graph}, \otimes, I_0, \hom^{\otimes}(-,-))$ *is a closed symmetric monoidal category.*

Homotopy theory of graphs

We now review the basics of discrete homotopy theory. Our treatment is brief and more categorically oriented, but the reader can find full details in any of the references [Mal83, BKLW01, BL05, BBdLL06].

DEFINITION 1.10. Let $f, g\colon G \to H$ be graph maps. An *A-homotopy* (or just a *homotopy*) from f to g is a map $\eta\colon G \otimes I_m \to H$ for some $m \geq 0$ such that $\eta|_{G\otimes\{0\}} = f$ and $\eta|_{G\otimes\{m\}} = g$.

Note that one needs to allow the parameter m appearing above to vary, as otherwise the notion of homotopy is not transitive. This is reminiscent of the notion of *Moore path* in topological spaces, which is a path parameterized by an interval of arbitrary length [May75, vdBG12, BR13].

PROPOSITION 1.11 [Mal83, Proposition 2.1]. *For graphs G and H, homotopy is an equivalence relation on the set of graphs maps $G \to H$.*

We also define the A-homotopy groups of a graph. As in topological spaces, this requires the definition of based graph maps and based homotopies between them.

DEFINITION 1.12. Let $A \hookrightarrow G$ and $B \hookrightarrow H$ be graph monomorphisms. A *relative graph map*, denoted $(G, A) \to (H, B)$, is a morphism from $A \hookrightarrow G$ to $B \hookrightarrow H$ in the arrow $\mathsf{Graph}^{[1]}$, where $[1]$ denotes the poset $\{0 \leq 1\}$ viewed as a category.

Explicitly, this data consists of maps $G \to H$ and $A \to B$ such that the following square commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\ G \arrow[r] & H \end{tikzcd}$$

That is, A is a subgraph of G and the map $A \to B$ is the restriction of the map $G \to H$ to A, whose image is contained in the subgraph B.

In the context of relative graph maps, we denote a monomorphism $A \hookrightarrow G$ by (G, A), suppressing the data of the map itself. An exception to this is for monomorphisms of the form $I_0 \to G$. This is exactly the data of a pointed graph, which we denote by (G, v), where v is the unique vertex in the image of the map $I_0 \to G$.

For a relative graph map $(f, g)\colon (G, A) \to (H, B)$, the map g is uniquely determined by f. If a map $h\colon A \to B$ also forms a commutative square with f then $g = h$ as the map $B \hookrightarrow Y$ is monic. Thus, we denote a relative graph map by $f\colon (G, A) \to (H, B)$. We additionally write $f\colon G \to H$ for the bottom map in the square and $f|_A\colon A \to B$ for the top map.

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We define relative homotopies between relative graph maps as follows.

Definition 1.13. Let $f, g\colon (G, A) \to (H, B)$ be relative graph maps. A *relative homotopy* from f to g is a relative map $(G \otimes I_m, A \otimes I_m) \to (H, B)$ for some $m \geq 0$ such that

- the map $A \otimes I_m \to B$ is a homotopy from $f|_A$ to $g|_A$;
- the map $G \otimes I_m \to H$ is a homotopy from f to g.

Explicitly, a relative homotopy from f to g consists of

- a homotopy $\eta\colon G \otimes I_m \to H$ from f to g,
- a homotopy $\eta|_A\colon A \otimes I_m \to B$ from $f|_A$ to $g|_A$,

such that the following square commutes:

$$\begin{tikzcd} A \otimes I_m \arrow[r, "\eta|_A"] \arrow[d, hook] & B \arrow[d, hook] \\ G \otimes I_m \arrow[r, "\eta"] & H \end{tikzcd}$$

Remark 1.14. For graph maps $f, g\colon G \to H$, a path of length m in $\hom^{\otimes}(G, H)$ from the vertex f to the vertex g is exactly a homotopy $G \otimes I_m \to H$ from f to g. For relative graph maps $f, g\colon (G, A) \to (H, B)$, a path of length m from f to g in the pullback graph

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \hom^{\otimes}(A, B) \arrow[d, hook] \\ \hom^{\otimes}(G, H) \arrow[r] & \hom^{\otimes}(A, H) \end{tikzcd}$$

is exactly a relative homotopy from f to g.

It follows that relative homotopy is an equivalence relation on relative graph maps.

Proposition 1.15. *Relative homotopy is an equivalence relation on the set of relative graph maps* $(G, A) \to (H, B)$.

Given a relative graph map $f\colon (G, A) \to (H, B)$, if the subgraph B consists of a single vertex v then we refer to f as a graph map *based at* v or a *based* graph map. We refer to a homotopy between two such maps as a *based* homotopy.

Definition 1.16. Let G be a graph and $n \geq 1$. For $i = 0, \ldots, n$ and $\varepsilon = 0, 1$, a map $f\colon I_\infty^{\otimes n} \to G$ is *stable in direction* (i, ε) if there exists $M \geq 0$ so that for $v_i > M$ we have

$$f(v_1, \ldots, v_{i-1}, (2\varepsilon - 1)v_i, v_{i+1} \ldots, v_n) = f(v_1, \ldots, v_{i-1}, (2\varepsilon - 1)M, v_{i+1}, \ldots, v_n).$$

In words, a map $f\colon I_\infty^{\otimes n} \to G$ is stable in direction $(i, 0)$ if it becomes constant (with respect to change in the ith coordinate) once the ith coordinate is sufficiently large in the negative direction. It is stable in direction $(i, 1)$ if it becomes constant (again, with respect to change in the ith coordinate) once the ith coordinate is sufficiently large in the positive direction.

For $n, M \geq 0$, let $I_{\geq M}^{\otimes n}$ denote the subgraph of $I_\infty^{\otimes n}$ consisting of vertices $(v_1, \ldots, v_n)$ such that $|v_i| \geq M$ for some $i = 1, \ldots, n$. Given a based graph map $f\colon (I_\infty^{\otimes n}, I_{\geq M}^{\otimes n}) \to (G, v)$ we may also regard f as a based graph map $(I_\infty^{\otimes n}, I_{\geq K}^{\otimes n}) \to (G, v)$ for any $K \geq M$. This gives a notion of based homotopy between maps $(I_\infty^{\otimes n}, I_{\geq M}^{\otimes n}) \to (G, v)$ which, for some $M \geq 0$, are based at v.

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Proposition 1.17 [Mal83, Proposition 3.2]. *Based homotopy is an equivalence relation on the set of based maps*

$$\{(I_\infty^{\otimes n}, I_{\geq M}^{\otimes n}) \to (G, v) \mid M \geq 0\}.$$

Definition 1.18. Let $n \geq 0$ and $v \in G$ be a vertex of a graph G. The nth *A-homotopy group* of G at v is the set of based homotopy classes of maps $(I_\infty^{\otimes n}, I_{\geq M}^{\otimes n}) \to (G, v)$ based at v for some $M \geq 0$.

Let $n \geq 1$ and $i = 1, \ldots, n$. Given $f\colon (I_\infty^{\otimes n}, I_{\geq M}^{\otimes n}) \to (G, v)$ and $g\colon (I_\infty^{\otimes n}, I_{\geq M'}^{\otimes n}) \to (G, v)$, we define a binary operation $f \cdot_i g\colon (I_\infty^{\otimes n}, I_{\geq M+M'}^{\otimes n}) \to (G, v)$ by

$$(f \cdot_i g)(v_1, \ldots, v_n) := \begin{cases} f(v_1, \ldots, v_n) & v_i \leq M \\ g(v_1, \ldots, v_i - M - M', \ldots, v_n) & v_i > M. \end{cases}$$

This induces a group operation on homotopy groups $\cdot_i\colon A_n(G, v) \times A_n(G, v) \to A_n(G, v)$. For $n \geq 2$ and $1 \leq i < j \leq n$, it is straightforward to construct a homotopy witnessing that

$$[[f] \cdot_i [g]] \cdot_j [[f] \cdot_i [g]] = [[f] \cdot_j [g]] \cdot_i [[f] \cdot_j [g]]$$

for any $[f], [g] \in A_n(G, v)$. The Eckmann–Hilton argument gives $[f] \cdot_i [g] = [f] \cdot_j [g]$ and that this operation is abelian.

Path and loop graphs

Definition 1.19. For a graph G, we define the *path graph* PG to be the induced subgraph of $\hom^{\otimes}(I_\infty, G)$ consisting of maps which stabilize in the $(1, 0)$ and $(1, 1)$ directions.

As vertices of PG are paths that stabilize, we have graph maps $\partial_{1,0}, \partial_{1,1}\colon PG \to G$ which send a vertex $v\colon I_\infty \to G$ to its left and right endpoints, respectively.

Proposition 1.20. *For a graph G, we have an isomorphism*

$$PG \cong \operatorname{colim}\left(\hom^{\otimes}(I_1, G) \xrightarrow{l^*} \hom^{\otimes}(I_2, G) \xrightarrow{r^*} \hom^{\otimes}(I_3, G) \xrightarrow{l^*} \cdots\right)$$

natural in G.

Proof. For each $m \geq 1$, we write $m = 2k + b$, where $k \geq 0$ and $b \in \{0, 1\}$, and define a graph map $I_\infty \to I_m$ by

$$v \mapsto \begin{cases} 0 & v \leq -k \\ k + v & -k \leq v \leq k + b \\ m & v \geq k + b. \end{cases}$$

Geometrically, this function maps the subinterval $[-k, k+b]$ surjectively onto I_m, and collapses all other vertices to the endpoints. By pre-composition, this induces a cone

$$\begin{tikzcd} \hom^{\otimes}(I_1, G) \arrow[r, "l^*"] \arrow[dr] & \hom^{\otimes}(I_2, G) \arrow[r, "r^*"] \arrow[d] & \hom^{\otimes}(I_3, G) \arrow[r, "l^*"] \arrow[dl] & \cdots \\ & PG & & \end{tikzcd}$$

which one verifies is a colimit cone. □

Definition 1.21. For a pointed graph (G, v), the *loop graph* $\Omega(G, v)$ is the subgraph of PG of paths $I_\infty \to G$ whose left and right endpoints are v.

From the definition, the following proposition is immediate.

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Proposition 1.22. *For a pointed graph (G, v), the square*

$$\begin{tikzcd} \Omega(G,v) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near start] & PG \arrow[d, "{(\partial_{1,0},\partial_{1,1})}"] \\ I_0 \arrow[r, "{(v,v)}"] & G \times G \end{tikzcd}$$

is a pullback.

The loop graph of a pointed graph (G, v) has a distinguished vertex which is the constant path at v. This gives an endofunctor $\Omega \colon \mathsf{Graph}_* \to \mathsf{Graph}_*$. From this, we define the notion of nth loop graphs.

Definition 1.23. For $n \geq 0$, we define the *nth loop graph* to be

$$\Omega^n(X, x) := \begin{cases} (X, x) & n = 0 \\ \Omega(\Omega^{n-1}(X, x)) & \text{otherwise.} \end{cases}$$

In [BBdLL06], it is shown that the nth homotopy groups of a graph correspond to the connected components of $\Omega^n(G, v)$.

Proposition 1.24 [BBdLL06, Proposition 7.4]. *For $n \geq 0$, we have an isomorphism*

$$A_n(G, v) \cong A_0(\Omega^n(G, v)).$$

2. Cubical sets and their homotopy theory

Cubical sets

We begin by defining the box category $\square$. As used in this paper, the box category will include both positive and negative connections. This variant of the box category was introduced in [BH81] and was later used in [DKLS24] to model $(\infty, 1)$-categories. The objects of $\square$ are posets of the form $[1]^n = \{0 \leq 1\}^n$ and the maps are generated (inside the category of posets) under composition by the following four special classes:

- *faces* $\partial^n_{i,\varepsilon} \colon [1]^{n-1} \to [1]^n$ for $i = 1, \ldots, n$ and $\varepsilon = 0, 1$ given by
$$\partial^n_{i,\varepsilon}(x_1, x_2, \ldots, x_{n-1}) = (x_1, x_2, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_{n-1});$$
- *degeneracies* $\sigma^n_i \colon [1]^n \to [1]^{n-1}$ for $i = 1, 2, \ldots, n$ given by
$$\sigma^n_i(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n);$$
- *negative connections* $\gamma^n_{i,0} \colon [1]^n \to [1]^{n-1}$ for $i = 1, 2, \ldots, n-1$ given by
$$\gamma^n_{i,0}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{i-1}, \max\{x_i, x_{i+1}\}, x_{i+2}, \ldots, x_n).$$
- *positive connections* $\gamma^n_{i,1} \colon [1]^n \to [1]^{n-1}$ for $i = 1, 2, \ldots, n-1$ given by
$$\gamma^n_{i,1}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{i-1}, \min\{x_i, x_{i+1}\}, x_{i+2}, \ldots, x_n).$$

These maps obey the following *cubical identities*:

$$\partial_{j,\varepsilon'}\partial_{i,\varepsilon} = \partial_{i+1,\varepsilon}\partial_{j,\varepsilon'} \quad \text{for } j \leq i; \quad \sigma_j\partial_{i,\varepsilon} = \begin{cases} \partial_{i-1,\varepsilon}\sigma_j & \text{for } j < i \\ \mathrm{id} & \text{for } j = i \\ \partial_{i,\varepsilon}\sigma_{j-1} & \text{for } j > i; \end{cases}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_{i+1} \quad \text{for } j \leq i; \quad \gamma_{j,\varepsilon'}\gamma_{i,\varepsilon} = \begin{cases} \gamma_{i,\varepsilon}\gamma_{j+1,\varepsilon'} & \text{for } j > i \\ \gamma_{i,\varepsilon}\gamma_{i+1,\varepsilon} & \text{for } j = i, \varepsilon' = \varepsilon; \end{cases}$$

$$\gamma_{j,\varepsilon'}\partial_{i,\varepsilon} = \begin{cases} \partial_{i-1,\varepsilon}\gamma_{j,\varepsilon'} & \text{for } j < i-1 \\ \mathrm{id} & \text{for } j = i-1,\ i,\ \varepsilon = \varepsilon' \\ \partial_{j,\varepsilon}\sigma_j & \text{for } j = i-1,\ i,\ \varepsilon = 1-\varepsilon' \\ \partial_{i,\varepsilon}\gamma_{j-1,\varepsilon'} & \text{for } j > i; \end{cases} \quad \sigma_j\gamma_{i,\varepsilon} = \begin{cases} \gamma_{i-1,\varepsilon}\sigma_j & \text{for } j < i \\ \sigma_i\sigma_i & \text{for } j = i \\ \gamma_{i,\varepsilon}\sigma_{j+1} & \text{for } j > i. \end{cases}$$

This category enjoys many good properties, making it suitable for modeling homotopy theory. Formally speaking, these properties can be encapsulated in saying that $\square$ is an Eilenberg–Zilber category, which in particular implies that the object $[1]^n$ has no non-identity automorphisms. However, we will not explicitly rely on the notion of Eilenberg–Zilber categories.

A *cubical set* is a presheaf $X \colon \square^{\mathrm{op}} \to \mathsf{Set}$. A *cubical map* is a natural transformation of such presheaves. We write cSet for the category of cubical sets and cubical maps.

Given a cubical set X, we write X_n for the value of X at $[1]^n$ and refer to the elements of X_n as *n-cubes* of X. We write cubical operators on the right, e.g. given an n-cube $x \in X_n$ of X, we write $x\partial_{1,0}$ for the $\partial_{1,0}$-face of x.

By a *degenerate cube*, we always mean a cube that is in the image of a degeneracy or a connection map. This nomenclature is borrowed from the theory of Reedy categories, where one can speak abstractly of degenerate elements in a presheaf ([RV14], [Hov99, § 5.2], [Hir03, Ch. 15]).

Definition 2.1. Let $n \geq 0$.

- The *combinatorial n-cube* $\square^n$ is the representable functor $\square(-, [1]^n) \colon \square^{\mathrm{op}} \to \mathsf{Set}$,
- The *boundary of the n-cube* $\partial\square^n$ is the subobject of $\square^n$ defined by

$$\partial\square^n := \bigcup_{\substack{j=0,\ldots,n \\ \eta=0,1}} \operatorname{im} \partial_{j,\eta}.$$

- When $n \geq 1$, given $i = 0, \ldots, n$ and $\varepsilon = 0, 1$, the (i, ε)-open box $\sqcap^n_{i,\varepsilon}$ is the subobject of $\partial\square^n$ defined by

$$\sqcap^n_{i,\varepsilon} := \bigcup_{(j,\eta)\neq(i,\varepsilon)} \operatorname{im} \partial_{j,\eta}.$$

Observe $\square^0 \in \mathsf{cSet}$ is the terminal object in cSet.

Example 2.2. Define a functor $\square \to \mathsf{Top}$ from the box category to the category of topological spaces which sends $[1]^n$ to $[0, 1]^n$, where $[0, 1]$ is the unit interval. The face and degeneracy maps are sent to the face inclusion and product projection maps, respectively. The negative connection $\gamma_{i,0}$ is sent to the map $[0, 1]^n \to [0, 1]^{n-1}$ defined by

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, \max(x_i, x_{i+1}), x_{i+2}, \ldots, x_n),$$

and the image of the positive connection $\gamma_{i,1}$ is defined analogously.

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Left Kan extension along the Yoneda embedding gives the *geometric realization* functor $|-|\colon \mathsf{cSet} \to \mathsf{Top}$:

$$\begin{tikzcd} \square \arrow[r, "{[1]^n \mapsto [0,1]^n}"] \arrow[d] & \mathsf{Top} \\ \mathsf{cSet} \arrow[ur, "|-|"'] & \end{tikzcd}$$

This functor is left adjoint to the *singular cubical complex* functor $\mathrm{Sing}\colon \mathsf{Top} \to \mathsf{cSet}$ defined by

$$(\mathrm{Sing}\, S)_n := \mathsf{Top}([0,1]^n, S).$$

Define a functor $\otimes\colon \square \times \square \to \square$ on the cube category which sends $([1]^m, [1]^n)$ to $[1]^{m+n}$. Post-composing with the Yoneda embedding and left Kan extending gives a monoidal product on cubical sets:

$$\begin{tikzcd} \square \times \square \arrow[r] \arrow[d] & \square \arrow[r] & \mathsf{cSet} \\ \mathsf{cSet} \times \mathsf{cSet} \arrow[urr, "\otimes"'] & & \end{tikzcd}$$

This is the *geometric product* of cubical sets. Although, for $[1]^m, [1]^n \in \square$, there is an isomorphism $[1]^m \otimes [1]^n \cong [1]^n \otimes [1]^m$, this isomorphism is not natural. As a result, the geometric product of cubical sets is not symmetric, i.e. $X \otimes Y$ is not in general isomorphic to $Y \otimes X$.

This product is however biclosed. For a cubical set X, we write $\hom_L(X, -)\colon \mathsf{cSet} \to \mathsf{cSet}$ and $\hom_R(X, -)\colon \mathsf{cSet} \to \mathsf{cSet}$ for the right adjoints to the functors $- \otimes X$ and $X \otimes -$, respectively. As the geometric product is not symmetric, the functors $\hom_L(X, -)$ and $\hom_R(X, -)$ are not naturally isomorphic.

Kan complexes

DEFINITION 2.3.

(i) A cubical map $X \to Y$ is a *Kan fibration* if it has the right lifting property with respect to open box inclusions. That is, if for any commutative square,

$$\begin{tikzcd} \sqcap^n_{i,\varepsilon} \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \square^n \arrow[r] & Y \end{tikzcd}$$

there exists a map $\square^n \to X$ so that the triangles

$$\begin{tikzcd} \sqcap^n_{i,\varepsilon} \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \square^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

commute.

(ii) A cubical set X is a *Kan complex* if the unique map $X \to \square^0$ is a Kan fibration.

We write Kan for the full subcategory of cSet consisting of Kan complexes.

Example 2.4. For any $S \in \mathsf{Top}$, the cubical set $\mathrm{Sing}\, S$ is a Kan complex. A map $\sqcap^n_{i,\varepsilon} \to \mathrm{Sing}\, S$ is, by adjointness, a map $|\sqcap^n_{i,\varepsilon}| \to S$. The inclusion $|\sqcap^n_{i,\varepsilon}| \hookrightarrow |\square^n|$ has a retract in Top. Pre-composing

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with this retract gives a map $|\square^n| \to S$ which restricts to the open box map $|\sqcap^n_{i,\varepsilon}| \to S$:

$$\begin{tikzcd} {|\sqcap^n_{i,\varepsilon}|} \arrow[r] \arrow[d, hook, shift right] & S \\ {|\square^n|} \arrow[u, bend right, shift right] & \end{tikzcd}$$

By adjointness, this gives a suitable map $\square^n \to \operatorname{Sing} S$.

DEFINITION 2.5. A map $f\colon X \to Y$ is a *weak equivalence* if the map $|f|\colon |X| \to |Y|$ is a weak homotopy equivalence, i.e. for any $n \geq 0$ and $x \in |X|$, the map $\pi_n|f|\colon \pi_n(|X|, x) \to \pi_n(|Y|, |f|(x))$ is an isomorphism.

We move towards describing the fibration category of Kan complexes.

DEFINITION 2.6 [Bro73, Definition 1.1]. A *fibration category* is a category $\mathcal{C}$ with two subcategories of *fibrations* and *weak equivalences* such that (in what follows, an *acyclic fibration* is a map that is both a fibration and a weak equivalence) the following statements hold:

(i) weak equivalences satisfy the two-out-of-three property, i.e. given two composable morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

if two of f, g, gf are weak equivalences then all three are;

(ii) all isomorphisms are acyclic fibrations;

(iii) pullbacks along fibrations exist; fibrations and acyclic fibrations are stable under pullback;

(iv) $\mathcal{C}$ has a terminal object 1; the canonical map $X \to 1$ is a fibration for any object $X \in \mathcal{C}$ (i.e. all objects are *fibrant*);

(v) every map can be factored as a weak equivalence followed by a fibration.

Example 2.7 [Hov99, Theorem 2.4.19]. The category Top of topological spaces is a fibration where:

- fibrations are Serre fibrations;
- weak equivalences are weak homotopy equivalences, i.e. maps $f\colon S \to S'$ such that, for all $s \in S$ and $n \geq 0$, the map $\pi_n f\colon \pi_n(S, s) \to \pi_n(S', f(s))$ is an isomorphism.

DEFINITION 2.8. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between fibration categories is *exact* if it preserves fibrations, acyclic fibrations, pullbacks along fibrations, and the terminal object.

Given a fibration category $\mathcal{C}$ with finite coproducts and a terminal object, the category of pointed objects $1 \downarrow \mathcal{C}$ is a fibration category as well.

PROPOSITION 2.9. *Let $\mathcal{C}$ be a fibration category with finite coproducts and a terminal object $1 \in \mathcal{C}$.*

(i) *The slice category $1 \downarrow \mathcal{C}$ under 1 is a fibration category where a map is a fibration/weak equivalence if the underlying map in $\mathcal{C}$ is.*

(ii) *The projection functor $1 \downarrow \mathcal{C} \to \mathcal{C}$ is exact.*

Proof. Statement (i) follows from [Hov99, Proposition 1.1.8].

For statement (ii), the projection functor preserves fibrations/weak equivalences by definition. It is a right adjoint to the functor $- \sqcup 1\colon \mathcal{C} \to 1 \downarrow \mathcal{C}$ which adds a disjoint basepoint, hence preserves finite limits. $\square$

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Theorem 2.10.

(i) *The category* Kan *of Kan complexes is a fibration category where fibrations are Kan fibrations and weak equivalences are as defined above.*
(ii) $\mathrm{Sing} : \mathsf{Top} \to \mathsf{Kan}$ *is an exact functor*
(iii) *The category* Kan_* *of pointed Kan complexes is a fibration category where a map is a fibration/weak equivalence if the underlying map in* Kan *is.*

Proof. (i) This is shown in [CK23, Theorem 2.17].
(ii) This is [CK23, Corollary 2.25].
(iii) This follows from Proposition 2.9. □

Anodyne maps

We move towards defining anodyne maps of cubical sets, i.e. maps which are both monomorphisms and weak equivalences.

Definition 2.11. A class of morphisms S in a cocomplete category $\mathcal{C}$ is *saturated* if it is closed under

– pushouts: if $s\colon A \to B$ is in S and $f\colon A \to C$ is any map then the pushout $C \to B \cup C$ of s along f is in S;

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & C \arrow[d, "\in S"] \\ B \arrow[r] & B \cup C \end{tikzcd}$$

– retracts: if $s\colon A \to B$ is in S and $r\colon C \to D$ is a retract of s in $\mathcal{C}^{[1]}$ then r is in S;

$$\begin{tikzcd} A \arrow[r] \arrow[d, "r"'] \arrow[rr, bend left, "\mathrm{id}_A"] & B \arrow[r] \arrow[d, "s"] & A \arrow[d, "r"] \\ C \arrow[r] \arrow[rr, bend right, "\mathrm{id}_C"'] & D \arrow[r] & C \end{tikzcd}$$

– transfinite composition: given a limit ordinal λ and a diagram $\lambda \to \mathcal{C}$ whose morphisms lie in S,

$$A_1 \xrightarrow{s_1} A_2 \xrightarrow{s_2} A_3 \xrightarrow{s_3} \cdots$$

writing A for the colimit of this diagram, the components of the colimit cone $\lambda_i\colon A_i \to A$ are in S.

$$\begin{tikzcd} A_1 \arrow[r] \arrow[dr, "\lambda_1"'] & A_2 \arrow[r] \arrow[d, "\lambda_2"] & A_3 \arrow[r] \arrow[dl, "\lambda_3"] & \cdots \arrow[dll, bend left, "\cdots"] \\ & A & & \end{tikzcd}$$

Definition 2.12. For a set of morphisms S in a cocomplete category $\mathcal{C}$, the *saturation* $\mathrm{Sat}\, S$ of S is the smallest saturated class containing S.

For a saturated class of maps, closure under pushouts and transfinite composition gives the following proposition.

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Proposition 2.13 [Hov99, Lemma 2.1.13]. *Saturated classes of maps are closed under coproduct. That is, given a collection* $\{s_i\colon A_i \to B_i \mid i \in I\}$ *of morphisms in a saturated class* S*, the coproduct*

$$\coprod_{i\in I} s_i\colon \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$$

is in S*.*

Definition 2.14. A map of cubical sets is *anodyne* if it is in the saturation of open box inclusions

$$\mathrm{Sat}\{\sqcap^n_{i,\varepsilon} \hookrightarrow \square^n \mid n \geq 1, i = 0, \ldots, n, \varepsilon = 0, 1\}.$$

We use the following property of anodyne maps, which follows since saturations are closed under the left lifting property.

Theorem 2.15 [DKLS24, Theorem 1.34]. *Let* $g\colon A \to B$ *be an anodyne map and* $f\colon X \to Y$ *be a Kan fibration. Given a commutative square,*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

there exists a map $B \to X$ *so that the triangles*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

commute.

Theorem 2.16. *A cubical map is anodyne if and only if it is a monomorphism and a weak equivalence.*

Proof. This follows from [DKLS24, Theorem 1.34], as the saturation of open box inclusions is exactly the class of maps which have the left lifting property with respect to fibrations. $\square$

In particular, we use that anodyne maps are sent to weak homotopy equivalences under geometric realization.

Corollary 2.17. *If* $f\colon X \to Y$ *is anodyne then, for all* $n \geq 0$ *and* $x \in |X|$*, the map* $\pi_n|f|\colon \pi_n(|X|, x) \to \pi_n(|Y|, |f|(x))$ *is an isomorphism.*

Homotopies and homotopy groups

Using the geometric product, we may define a notion of homotopy between cubical maps.

Definition 2.18. Given cubical maps $f, g\colon X \to Y$, a *homotopy* from f to g is a map $H\colon X \otimes \square^1 \to G$ such that the diagram

$$\begin{tikzcd} X \otimes \square^0 \arrow[d, "\partial_{1,0}"'] \arrow[dr, "f"] & \\ X \otimes \square^1 \arrow[r, "H"] & Y \\ X \otimes \square^0 \arrow[u, "\partial_{1,1}"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes.

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Let cSet_2 denote the full subcategory of $\mathsf{cSet}^{[1]}$ spanned by monomorphisms. Explicitly, its objects are monic cubical maps $A \hookrightarrow X$. A morphism from $A \hookrightarrow X$ to $B \hookrightarrow Y$ is a pair of maps (f, g) which form a commutative square of the following form:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, hook] & B \arrow[d, hook] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

We refer to the objects and morphisms of cSet_2 as *relative cubical sets* and *relative cubical maps*, respectively. Following our convention for relative graph maps, we denote a relative cubical set $A \hookrightarrow X$ by (X, A), suppressing the data of the map itself. If the domain of the map $A \hookrightarrow X$ is the 0-cube $A = \square^0$, we denote it by (X, x), where x is the unique 0-cube in the image of the map $\square^0 \hookrightarrow X$.

For a relative cubical map $(f, g)\colon (X, A) \to (Y, B)$, the map g is uniquely determined by f since $B \hookrightarrow Y$ is monic. As a result, we denote a relative cubical map by $f\colon (X, A) \to (Y, B)$. Mirroring our convention for relative graph maps, we write f for the map $X \to Y$ and $f|_A$ for the map $A \to B$.

We have a corresponding notion of homotopy between relative cubical maps.

Definition 2.19. Let $f, g\colon (X, A) \to (Y, B)$ be relative cubical maps. A *relative homotopy* from f to g is a morphism $(X \otimes \square^1, A \otimes \square^1) \to (Y, B)$ in cSet_2 such that:

- the map $A \otimes \square^1 \to B$ is a homotopy from $f|_A$ to $g|_A$;
- the map $X \otimes \square^1 \to Y$ is a homotopy from f to g.

Proposition 2.20 [CK23, Proposition 2.30]. *If B, Y are Kan complexes then relative homotopy is an equivalence relation on relative cubical maps $(X, A) \to (B, Y)$.*

It is essential in Proposition 2.20 that B and Y are Kan complexes. If not, the relation of relative homotopy is neither symmetric nor transitive (and in this case, one considers the symmetric transitive closure of relative homotopy).

We write $[(X, A), (Y, B)]$ for the set of relative homotopy classes of relative maps $(X, A) \to (Y, B)$. With this, we define the homotopy groups of a Kan complex.

Definition 2.21 [CK23, Corollary 3.16]. Let (X, x) be a pointed Kan complex. We define the nth homotopy group $\pi_n(X, x)$ of (X, x) as the relative homotopy classes of relative maps $(\square^n, \partial\square^n) \to (X, x)$.

We give an explicit description of multiplication in the first homotopy group $\pi_1(X, x)$.

Definition 2.22. Given two 1-cubes $u, v\colon \square^1 \to X$ in a cubical set X,

(i) a *concatenation square* for u and v is a map $\eta\colon \square^2 \to X$ such that
   - $\eta\partial_{1,0} = u$,
   - $\eta\partial_{1,1} = v\partial_{1,1}\sigma_1$,
   - $\eta\partial_{2,1} = v$.

(ii) a *concatenation* of u and v is a 1-cube $w\colon \square^1 \to X$ which is the $\partial_{2,0}$-face of some concatenation square for u and v.

Proposition 2.23 [CK23, Theorem 3.11]. *If (X, x) is a pointed Kan complex then composition induces a well-defined binary operation*

$$[(\square^1, \partial\square^1), (X, x)] \times [(\square^1, \partial\square^1), (X, x)] \to [(\square^1, \partial\square^1), (X, x)]$$

on relative homotopy classes of relative maps which gives a group structure on $[(\square^1, \partial\square^1), (X, x)]$.

As with spaces, the homotopy groups of a Kan complex are the connected components of its *loop space*, which we define.

Definition 2.24. For a pointed Kan complex (X, x),

– the *loop space* $\Omega(X, x)$ of X is the pullback

$$\begin{tikzcd} \Omega(X,x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \hom(\square^1, X) \arrow[d, "(\partial^*_{1,0}, \partial^*_{1,1})"] \\ \square^0 \arrow[r, "x"] & X \times X \end{tikzcd}$$

with a distinguished 0-cube $x\sigma_1 \colon \square^0 \to \Omega(X, x)$.

– for $n \geq 0$, the nth *loop space* $\Omega^n(X, x)$ of X is defined to be

$$\Omega^n(X,x) := \begin{cases} (X,x) & n = 0 \\ \Omega(\Omega^{n-1}(X,x)) & n > 0. \end{cases}$$

Proposition 2.25 [CK23, Corollary 3.16]. *For a pointed Kan complex* (X, x) *and* $0 \leq k \leq n$, *we have an isomorphism*

$$\pi_n(X,x) \cong \pi_{n-k}(\Omega^k(X,x))$$

natural in X.

Using Proposition 2.25, the group structure on higher homotopy groups is induced by the bijection $\pi_n(X, x) \cong \pi_1(\Omega^{n-1}(X, x))$.

This definition of homotopy groups agrees with the homotopy groups of its geometric realization.

Theorem 2.26 [CK23, Theorem 3.25]. *There is an isomorphism*

$$\pi_n(X,x) \cong \pi_n(|X|, x)$$

natural in X.

We know that the loop space functor is exact.

Theorem 2.27 [CK23, Theorem 3.6]. *The loop space functor* $\Omega \colon \mathsf{Kan}_* \to \mathsf{Kan}_*$ *is exact.*

3. Cubical nerve of a graph

Let $m \geq 1$. Geometrically, we view the graph $I_m^{\otimes n}$ as an n-dimensional cube. Making this intuition formal, we have face, degeneracy, and connection maps defined as follows.

– the face map $\partial^n_{i,\varepsilon} \colon I_m^{\otimes n-1} \to I_m^{\otimes n}$ for $1 \leq i \leq n$ and $\varepsilon = 0$ or 1 is given by

$$\partial^n_{i\varepsilon}(v_1, \ldots, v_n) = (v_1, \ldots, v_{i-1}, \varepsilon m, v_i, \ldots, v_n);$$

– the degeneracy map $\sigma^n_i \colon I_m^{\otimes n+1} \to I_m^{\otimes n}$ for $1 \leq i \leq n$ is given by

$$\sigma^n_i(v_1, \ldots, v_n) = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n);$$

– the negative connection map $\gamma^n_{i,0} \colon I_m^{\otimes n} \to I_m^{\otimes n-1}$ for $1 \leq i \leq n-1$ is given by

$$\gamma^n_{i,0}(v_1, \ldots, v_n) = (v_1, \ldots, v_{i-1}, \max(v_i, v_{i+1}), v_{i+2}, \ldots, v_n);$$

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– the positive connection map $\gamma^n_{i,1}\colon I_m^{\otimes n} \to I_m^{\otimes n-1}$ for $1 \leq i \leq n-1$ is given by

$$\gamma^n_{i,1}(v_1, \dots, v_n) = (v_1, \dots, v_{i-1}, \min(v_i, v_{i+1}), v_{i+2}, \dots, v_n).$$

It is straightforward to verify that these maps satisfy cubical identities. This defines a functor $\square \to \mathsf{Graph}$ which sends $[1]^n$ to $I_m^{\otimes n}$. Left Kan extension along the Yoneda embedding gives an adjunction $\mathsf{cSet} \rightleftarrows \mathsf{Graph}$:

$$\begin{tikzcd} \square \arrow[r] \arrow[d] & \mathsf{Graph} \arrow[dl, shift left] \\ \mathsf{cSet} \arrow[ur, shift left] & \end{tikzcd}$$

Definition 3.1. For $m \geq 1$,

(i) the *m-realization* functor $|{-}|_m\colon \mathsf{cSet} \to \mathsf{Graph}$ is the left Kan extension of the functor $\square \to \mathsf{Graph}$ which sends $[1]^n$ to $I_m^{\otimes n}$.
(ii) the *m-nerve* functor $\mathrm{N}_m\colon \mathsf{Graph} \to \mathsf{cSet}$ is the right adjoint of the m-realization functor defined by

$$(\mathrm{N}_m G)_n := \mathsf{Graph}(I_m^{\otimes n}, G).$$

Remark 3.2. The 1-nerve of a graph G is constructed in [BBdLL06] as the *cubical set associated to G*, denoted $M_*(G)$. The geometric realization of this cubical set is referred to as the *cell complex associated to G*, denoted X_G.

For a cubical set $X \in \mathsf{cSet}$, the graph $|X|_m$ may be explicitly described as the colimit

$$|X|_m := \operatorname{colim}(G_{m,0} \hookrightarrow G_{m,1} \hookrightarrow G_{m,2} \hookrightarrow \cdots)$$

where:

– $G_{m,0}$ is the discrete graph whose vertices are X_0;
– $G_{m,n+1}$ is obtained from $G_{m,n}$ via the following pushout (where $(X_n)_{\mathsf{nd}}$ is the subset of X_n consisting of non-degenerate cubes)

$$\begin{tikzcd} \coprod\limits_{x \in (X_n)_{\mathsf{nd}}} \partial I_m^{\otimes n} \arrow[r] \arrow[d] & G_{m,n} \arrow[d] \\ \coprod\limits_{x \in (X_n)_{\mathsf{nd}}} I_m^{\otimes n} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & G_{m,n+1} \end{tikzcd}$$

in which $\partial I_m^{\otimes n}$ is the subgraph of $I_m^{\otimes n}$ defined by

$$\partial I_m^{\otimes n} := \{(v_1, \dots, v_n) \in I_m^{\otimes n} \mid v_i = 0 \text{ or } m \text{ for some } i = 0, \dots, n\}$$

where the edge set is discrete if $m = n = 1$ and full otherwise.

Example 3.3.

(i) We describe the graph $|\sqcap^2_{2,1}|_1$. The cubical set $\sqcap^2_{2,1}$ has four 0-cubes; thus, the graph $G_{1,0}$ is the discrete graph with four vertices. The open box $\sqcap^2_{2,1}$ has three non-degenerate 1-cubes. The pushout constructed to obtain $G_{1,1}$ glues three copies of I_1 to $G_{1,0}$ (Figure 3). As $\sqcap^2_{2,1}$ contains only degenerate cubes above dimension 1, constructing $G_{1,1}$ completes the construction of the graph $|\sqcap^2_{2,1}|_1$.
(ii) To construct $|\sqcap^2_{2,1}|_3$, we instead glue three copies of I_3 (Figure 4). Observe that this process adds new vertices to the graph.

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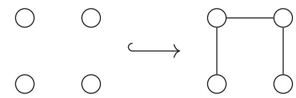

FIGURE 3. The embedding of $G_{1,0} \hookrightarrow G_{1,1}$ for $|\sqcap^2_{2,1}|_1$.

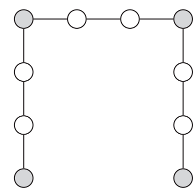

FIGURE 4. The graph $|\sqcap^2_{2,1}|_3$. The image of the embedding $G_{3,0} \hookrightarrow |\sqcap^2_{2,1}|_3$ is shaded.

For a graph G, let $l^*, r^*\colon \mathrm{N}_m G \to \mathrm{N}_{m+1} G$ denote the cubical maps obtained by precomposition with the surjections $l^{\otimes n}, r^{\otimes n}\colon I_{m+1}^{\otimes n} \to I_m^{\otimes n}$ (Figure 5). We think of these maps as inclusions of n-cubes of size m into n-cubes of size $m+1$ (Figure 6). We write $c^*\colon \mathrm{N}_m G \to \mathrm{N}_{m+2} G$ for the composite $l^* r^* = r^* l^*$.

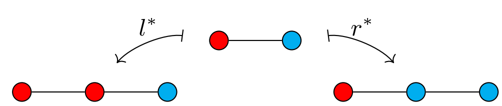

FIGURE 5 (colour online). For a 1-cube $f\colon I_1 \to G$ of $\mathrm{N}_1 G$, the map $l^*\colon \mathrm{N}_1 G \to \mathrm{N}_2 G$ sends f to the 1-cube $fl\colon I_2 \to G$ of $\mathrm{N}_2 G$, whereas $r^*\colon \mathrm{N}_1 G \to \mathrm{N}_2 G$ sends f to the 1-cube $fr\colon I_2 \to G$ of $\mathrm{N}_2 G$.

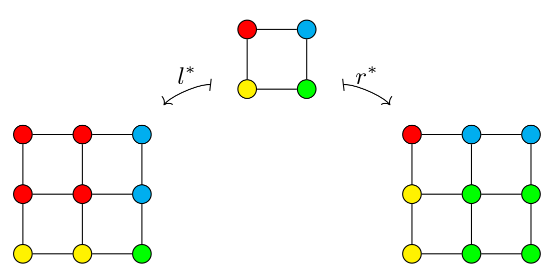

FIGURE 6 (colour online). For a 2-cube $g\colon I_1^{\otimes 2} \to G$ of $\mathrm{N}_1 G$, the map $l^*\colon \mathrm{N}_1 G \to \mathrm{N}_2 G$ sends g to the 2-cube $gl^{\otimes 2}\colon I_2^{\otimes 2} \to G$ of $\mathrm{N}_2 G$, whereas $r^*\colon \mathrm{N}_1 G \to \mathrm{N}_2 G$ sends g to the 2-cube $gr^{\otimes 2}\colon I_2^{\otimes 2} \to G$ of $\mathrm{N}_2 G$.

Remark 3.4. While the maps $l^{\otimes n}, r^{\otimes n}$ have sections $I_m^{\otimes n} \to I_{m+1}^{\otimes n}$, these maps do not commute with face maps, hence do not give retractions $\mathrm{N}_{m+1} G \to \mathrm{N}_m G$. To demonstrate this, we show the map $l^*\colon \mathrm{N}_1 I_2 \to \mathrm{N}_2 I_2$ does not have a retraction. The identity map $\mathrm{id}_{I_2}\colon I_2 \to I_2$ gives a 1-cube of $\mathrm{N}_2 I_2$ whose faces are the 0-cubes 0 and 2. Observe that a retraction of l^* must send the 0-cubes 0 and 2 to 0 and 2, respectively. There is no map $f\colon I_1 \to I_2$ such that $f\partial_{1,0} = 0$ and $f\partial_{1,1} = 2$. That is, there is no 1-cube of $\mathrm{N}_1 I_2$ which $\mathrm{id}_{I_2} \in (\mathrm{N}_2 I_2)_1$ can be mapped to. Thus, the map l^* does not have a retraction.

For a cubical set X, we analogously have maps $l_*, r_*\colon |X|_{m+1} \to |X|_m$. We write $c_*\colon |X|_{m+2} \to |X|_m$ for the composite $l_* r_* = r_* l_*$.

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We define the *nerve* functor $\mathrm{N}\colon \mathsf{Graph} \to \mathsf{cSet}$ by

$$(\mathrm{N}G)_n := \{f\colon I_\infty^{\otimes n} \to G \mid f \text{ is stable in all directions } (i,\varepsilon)\}.$$

Cubical operators of $\mathrm{N}G$ are given as follows.

– The map $\partial_{i,\varepsilon}^n\colon (\mathrm{N}G)_n \to (\mathrm{N}G)_{n-1}$ for $i = 1,\dots,n$ and $\varepsilon = 0,1$ is given by $f\partial_{i,\varepsilon}^n\colon I_\infty^{\otimes n-1} \to G$ defined by

$$f\partial_{i,\varepsilon}^n(v_1,\dots,v_{n-1}) = f(v_1,\dots,v_{i-1},(2\varepsilon-1)M,v_i,\dots,v_{n-1})$$

where M is such that f is stable in direction (i,ε).

– The map $\sigma_i^n\colon (\mathrm{N}G)_n \to (\mathrm{N}G)_{n+1}$ for $i = 1,\dots,n$ is given by $f\sigma_i^n\colon I_\infty^{\otimes n+1} \to G$ defined by

$$f\sigma_i^n(v_1,\dots,v_{n+1}) = f(v_1,\dots,v_{i-1},v_{i+1},\dots,v_{n+1}).$$

– The map $\gamma_{i,0}^n\colon (\mathrm{N}G)_n \to (\mathrm{N}G)_{n+1}$ for $i = 1,\dots,n-1$ is given by $f\gamma_{i,0}^n\colon I_\infty^{\otimes n+1} \to G$ defined by

$$f\gamma_{i,0}^n(v_1,\dots,v_{n+1}) = f(v_1,\dots,v_{i-1},\max(v_i,v_{i+1}),v_{i+1},\dots,v_{n+1}).$$

– The map $\gamma_{i,1}^n\colon (\mathrm{N}G)_n \to (\mathrm{N}G)_{n+1}$ for $i = 1,\dots,n-1$ is given by $f\gamma_{i,0}^n\colon I_\infty^{\otimes n+1} \to G$ defined by

$$f\gamma_{i,0}^n(v_1,\dots,v_{n+1}) = f(v_1,\dots,v_{i-1},\min(v_i,v_{i+1}),v_{i+1},\dots,v_{n+1}).$$

One verifies these maps satisfy cubical identities, thus $\mathrm{N}G$ is a cubical set. A straightforward computation gives the following statement.

Proposition 3.5. *We have an isomorphism*

$$\mathrm{N}G \cong \operatorname{colim}(\mathrm{N}_1G \xrightarrow{l^*} \mathrm{N}_2G \xrightarrow{r^*} \mathrm{N}_3G \xrightarrow{l^*} \mathrm{N}_4G \xrightarrow{r^*} \cdots)$$

natural in G.

The nerve and realization functors satisfy the following categorical properties.

Proposition 3.6. *For cubical sets X, Y and $m \geq 1$, we have an isomorphism*

$$|X \otimes Y|_m \cong |X|_m \otimes |Y|_m$$

natural in X and Y.

Proof. The composite functors

$$|-\otimes-|_m, |-|_m \otimes |-|_m\colon \mathsf{cSet} \times \mathsf{cSet} \to \mathsf{Graph}$$

preserve all colimits. As cSet is a presheaf category, every cubical set is a colimit of representable presheaves. Thus, it suffices to show these composites are naturally isomorphic on pairs $(\Box^a, \Box^b)$ for $a, b \geq 0$. We compute

$$\begin{aligned}
|\Box^a \otimes \Box^b|_m &\cong |\Box^{a+b}|_m \\
&\cong I_m^{\otimes a+b} \\
&\cong I_m^{\otimes a} \otimes I_m^{\otimes b} \\
&\cong |\Box^a|_m \otimes |\Box^b|_m.
\end{aligned}$$

□

Corollary 3.7. *Let X be a cubical set and G be a graph. For $m \geq 1$, we have isomorphisms*

$$\hom_L(X, \mathrm{N}_mG) \cong \mathrm{N}_m(\hom^{\otimes}(|X|_m, G)) \cong \hom_R(X, \mathrm{N}_mG)$$

natural in X and G.

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Proof. The square

$$\begin{tikzcd} \mathsf{cSet} \arrow[r, "-\otimes X"] \arrow[d, "|-|_m"'] & \mathsf{cSet} \arrow[d, "|-|_m"] \\ \mathsf{Graph} \arrow[r, "-\otimes |X|_m"] & \mathsf{Graph} \end{tikzcd}$$

commutes up to natural isomorphism by Proposition 3.6, thus the corresponding square of right adjoints

$$\begin{tikzcd} \mathsf{Graph} \arrow[r, "\hom^{\otimes}(|X|_m, -)"] \arrow[d, "\mathrm{N}_m"'] & \mathsf{Graph} \arrow[d, "\mathrm{N}_m"] \\ \mathsf{cSet} \arrow[r, "\hom_L(X, -)"] & \mathsf{cSet} \end{tikzcd}$$

commutes up to natural isomorphism. For naturality in X, the required square commutes by faithfulness of the Yoneda embedding $\mathsf{cSet} \to \mathsf{Set}^{\mathsf{cSet}^{\mathrm{op}}}$. A similar argument involving $X \otimes -$ constructs the isomorphism involving $\hom_R(X, -)$. $\square$

PROPOSITION 3.8. *The nerve functor* $\mathrm{N} \colon \mathsf{Graph} \to \mathsf{cSet}$ *preserves finite limits.*

Proof. By Proposition 3.5, the nerve of G is a filtered colimit. This then follows as filtered colimits commute with finite limits and $\mathrm{N}_m \colon \mathsf{Graph} \to \mathsf{cSet}$ is a right adjoint for all $m \geq 1$. $\square$

We prove that the nerve functors preserve filtered colimits, which we use to give an analogue of Corollary 3.7 for the nerve functor $\mathrm{N} \colon \mathsf{Graph} \to \mathsf{cSet}$.

PROPOSITION 3.9. *For* $m \geq 0$*, the functors* $\mathrm{N}_m, \mathrm{N} \colon \mathsf{Graph} \to \mathsf{cSet}$ *preserve filtered colimits.*

Proof. For N_m, it suffices to show filtered colimits are preserved componentwise, i.e. that

$$\mathsf{Graph}(I_m^{\otimes n}, -) \colon \mathsf{Graph} \to \mathsf{Set}$$

preserves filtered colimits. This follows from Corollary 1.7.

For N, this follows since N_m preserves filtered colimits and colimits commute with colimits. $\square$

For a cubical set X, define a functor $P_X \colon \mathsf{Graph} \to \mathsf{Graph}$ by

$$P_X G := \operatorname{colim} \left(\hom^{\otimes}(|X|_1, G) \xrightarrow{(l_*)^*} \hom^{\otimes}(|X|_2, G) \xrightarrow{(r_*)^*} \hom^{\otimes}(|X|_3, G) \xrightarrow{(l_*)^*} \cdots \right).$$

As an example, the path graph PG of a graph is exactly $P_{\square^1} G$.

PROPOSITION 3.10. *Let* X *be a cubical set with finitely many non-degenerate cubes. We have isomorphisms*

$$\mathrm{N}(P_X G) \cong \hom_L(X, \mathrm{N}G) \cong \hom_R(X, \mathrm{N}G)$$

natural in X *and* G*.*

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Proof. By Proposition 3.5, the left term $\mathrm{N}(P_X G)$ is the colimit

$$\mathrm{N}(P_X G) \cong \operatorname{colim} \left(\begin{tikzcd} \mathrm{N}_1 \hom^{\otimes}(|X|_1, G) \arrow[r, hook, "l^*"] \arrow[d, hook, "(l_*)^*"] & \mathrm{N}_2 \hom^{\otimes}(|X|_1, G) \arrow[r, hook, "r^*"] \arrow[d, hook, "(l_*)^*"] & \cdots \\ \mathrm{N}_1 \hom^{\otimes}(|X|_2, G) \arrow[r, hook, "l^*"] \arrow[d, hook, "(r_*)^*"] & \mathrm{N}_2 \hom^{\otimes}(|X|_2, G) \arrow[r, hook, "r^*"] \arrow[d, hook, "(r_*)^*"] & \cdots \\ \cdots & \cdots & \cdots \end{tikzcd} \right),$$

where the vertical maps are monomorphisms since $l_*, r_* \colon |X|_{m+1} \to |X|_m$ are epimorphisms for all $m \geq 1$. Computing this colimit componentwise in Set, this colimit is naturally isomorphic to the colimit

$$\mathrm{N}(P_X G) \cong \operatorname{colim}(\mathrm{N}_1 \hom^{\otimes}(|X|_1, G) \to \mathrm{N}_2 \hom^{\otimes}(|X|_2, G) \to \cdots)$$

along the diagonal. Applying Corollary 3.7, we may write

$$\mathrm{N}(P_X G) \cong \operatorname{colim}(\hom_L(X, \mathrm{N}_1 G) \to \hom_L(X, \mathrm{N}_2 G) \to \cdots).$$

As X has finitely many non-degenerate cubes, the functor $\hom_L(X, -)$ preserves filtered colimits. Thus,

$$\mathrm{N}(P_X G) \cong \hom_L(X, \mathrm{N}G).$$

An analogous proof applies in the case of $\hom_R(X, \mathrm{N}G)$. □

We show that the nerve functors 'detect' concatenation of paths. That is, they contain all possible composition squares.

Proposition 3.11. *Let $f \colon (I_\infty, I_{\geq M}) \to (G, v)$ and $g \colon (I_\infty, I_{\geq N}) \to (G, v)$ be based graph maps for some $M, N \geq 0$ such that $f\partial_{1,1} = g\partial_{1,0}$. The concatenation $f \cdot g$ of f followed by g is a concatenation of f and g in the $(M+N)$-nerve $\mathrm{N}_{(M+N)}G$.*

Proof. The horizontal concatenation of $f\gamma_{1,0} \colon I_M^{\otimes 2} \to G$ on the left and $g\sigma_2 \colon I_N^{\otimes 2} \to G$ on the right gives a square $\eta \colon I_{M+N}^{\otimes 2} \to G$ such that

$$\begin{aligned} \eta\partial_{1,0} &= f, & \eta\partial_{1,1} &= g\partial_{1,1}\sigma_1, \\ \eta\partial_{2,0} &= f \cdot g, & \eta\partial_{2,1} &= g. \end{aligned}$$

This is exactly a composition square witnessing $f \cdot g$ as a composition of f and g in $\mathrm{N}_{(M+N)}G$. □

As well, the nerve functors reflect isomorphisms.

Proposition 3.12. *The nerve functors reflect isomorphisms. That is, given a graph map $f \colon G \to H$, if either*

(i) *$\mathrm{N}_m f \colon \mathrm{N}_m G \to \mathrm{N}_m H$ is an isomorphism for some $m \geq 1$, or*
(ii) *$\mathrm{N}f \colon \mathrm{N}G \to \mathrm{N}H$ is an isomorphism,*

then f is.

Proof. We prove that $\mathrm{N} \colon \mathsf{Graph} \to \mathsf{cSet}$ reflects isomorphisms as the case for $\mathrm{N}_m \colon \mathsf{Graph} \to \mathsf{cSet}$ is analogous.

The inverse of $\mathrm{N}f$ is, in particular, an inverse on 0-cubes $(\mathrm{N}H)_0 \to (\mathrm{N}G)_0$, i.e. an inverse of f on vertices $H_V \to G_V$. It suffices to show this map $g \colon H_V \to G_V$ is a graph map.

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An edge in H gives a map $e\colon I_1 \to H$. This gives a 1-cube $\overline{e}\colon \square^1 \to \mathrm{N}G$ by the inclusion $\mathrm{N}_1 H \hookrightarrow \mathrm{N}H$. As $\mathrm{N}f\colon \mathrm{N}G \to \mathrm{N}H$ is an isomorphism, there is a unique 1-cube $\overline{p}\colon \square^1 \to \mathrm{N}G$ such that $\mathrm{N}f(\overline{p}) = \overline{e}$. This corresponds to a map $p\colon I_\infty \to G$ such that p stabilizes in both directions and $fp = e$.

As f is injective on vertices and e is a path of length 1 (i.e. it consists of two vertices and one edge), we deduce that p is a path of length 1. That is, p is an edge, hence g is a graph map. $\square$

Recall that a functor which reflects isomorphisms also reflects any (co)limits which it preserves. Thus, we have the following corollary.

COROLLARY 3.13.

(i) *For $m \geq 1$, the m-nerve $\mathrm{N}_m\colon \mathsf{Graph} \to \mathsf{cSet}$ reflects all limits.*
(ii) *The nerve functor $\mathrm{N}\colon \mathsf{Graph} \to \mathsf{cSet}$ reflects finite limits.*

4. Main result

Statement

Our main theorem is the following result.

THEOREM 4.1. *For any graph G,*

(i) *the nerve $\mathrm{N}G$ of G is a Kan complex;*
(ii) *the natural inclusion $\mathrm{N}_1 G \hookrightarrow \mathrm{N}G$ is anodyne.*

Before proving Theorem 4.1, we explain the proof strategy and establish some auxiliary lemmas.

To show that the nerve of any graph is a Kan complex, we construct a map $\Phi\colon I_{3m}^{\otimes n} \to |\sqcap^n_{i,\varepsilon}|_m$ so that the triangle

$$\begin{tikzcd} {|\sqcap^n_{i,\varepsilon}|_{3m}} \arrow[r, "c^m_*"] \arrow[d, hook] & {|\sqcap^n_{i,\varepsilon}|_m} \\ I_{3m}^{\otimes n} \arrow[ur, dotted, "\Phi"] & \end{tikzcd}$$

commutes. We show that every map $\sqcap^n_{i,\varepsilon} \to \mathrm{N}G$ must factor through some m-nerve $\sqcap^n_{i,\varepsilon} \to \mathrm{N}_m G \to \mathrm{N}G$. The map Φ gives a filler for the composite $\sqcap^n_{i,\varepsilon} \to \mathrm{N}_m G \to \mathrm{N}_{3m} G$, thus a filler in $\mathrm{N}G$.

To show that the map $\mathrm{N}_1 G \hookrightarrow \mathrm{N}G$ is anodyne, we show the maps $l^*, r^*\colon \mathrm{N}_m G \hookrightarrow \mathrm{N}_{m+1} G$ are anodyne. This is done by an explicit construction establishing l^* and r^* as a transfinite composition of pushouts along coproducts of open box inclusions. This implies $\mathrm{N}_1 G \hookrightarrow \mathrm{N}G$ is anodyne as the nerve of G is a transfinite composition of l^* and r^*.

Proof of part (i)

CONSTRUCTION 4.2. Fix $m, n \geq 1$, $i \in \{1, \ldots, n\}$, and $\varepsilon \in \{0, 1\}$. We construct the map $\Phi\colon I_{3m}^{\otimes n} \to |\sqcap^n_{i,\varepsilon}|_m$. For $n = 1$, we compute $|\sqcap^1_{i,\varepsilon}|_m \cong I_0$. In this case, the map Φ is immediate. Thus, we assume $n \geq 2$.

Define a map $\varphi\colon I_{3m} \to I_{3m}$ by

$$\varphi v := \begin{cases} m & v \leq m \\ i & m \leq v \leq 2m \\ 2m & v \geq 2m. \end{cases}$$

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It is straightforward to verify this definition gives a graph map. From this, we have a map $\varphi^{\otimes n-1}\colon I_{3m}^{\otimes n-1} \to I_{3m}^{\otimes n-1}$ which sends $(v_1, \dots, v_{n-1})$ to $(\varphi v_1, \dots, \varphi v_{n-1})$.

For $v = (v_1, \dots, v_{n-1}) \in I_{3m}^{\otimes n-1}$, let $d(v)$ denote the node distance between v and $\varphi^{\otimes n-1}v$. That is,

$$d(v) := \sum_{i=1}^{n-1} |v_i - \varphi v_i|.$$

Observe this gives a graph map $d\colon I_{3m}^{\otimes n-1} \to I_\infty$ (Figure 7).

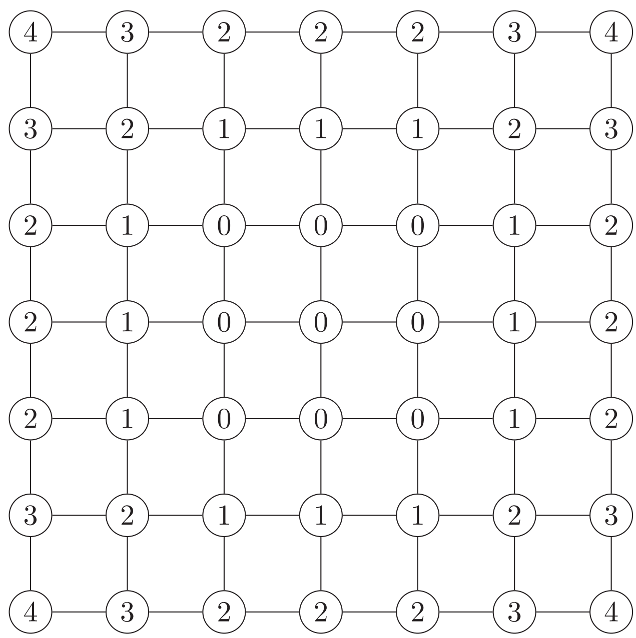

Figure 7. Vertices of $I_6^{\otimes 2}$ labeled by their image under $d\colon I_6^{\otimes 2} \to I_\infty$.

For $t \geq 0$, we have a graph map $\beta[t]\colon I_\infty \to I_t$ defined by

$$\beta[t](v) := \begin{cases} 0 & v \leq 0 \\ v & 0 \leq v \leq t \\ t & v \geq t. \end{cases}$$

One thinks of $\beta[t]$ as bounding the graph I_∞ between 0 and t. Recall that the map $c^m\colon I_{3m} \to I_m$ is given by

$$c^m v := \begin{cases} 0 & v \leq m \\ v - m & m \leq v \leq 2m \\ m & v \geq 2m. \end{cases}$$

For a vertex $(v_1, \dots, v_n) \in I_{3m}^{\otimes n}$, we write $\sigma_i v$ for the vertex $(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n) \in I_{3m}^{\otimes n-1}$. As well, let $\alpha_\varepsilon^{v_i}$ denote the value

$$\alpha_\varepsilon^{v_i} := \varepsilon m + (1 - 2\varepsilon)(c^m v_i).$$

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We may also write this as

$$\alpha_\varepsilon^{v_i} = \begin{cases} c^m v_i & \varepsilon = 0 \\ m - c^m v_i & \varepsilon = 1. \end{cases}$$

We define $\Phi\colon I_{3m}^{\otimes n} \to |\sqcap_{i,\varepsilon}^n|_m$ by

$$\begin{aligned} &\Phi(v_1, \dots, v_n) \\ &\quad := (c^m v_1, \dots, c^m v_{i-1}, (1-\varepsilon)m + (2\varepsilon - 1)(\beta[\alpha_{(1-\varepsilon)}^{v_i}](d(\sigma_i v) - \alpha_\varepsilon^{v_i})), c^m v_{i+1}, \dots, c^m v_n). \end{aligned}$$

That is,

$$\begin{aligned} &\Phi(v_1, \dots, v_n) \\ &\quad = \begin{cases} (c^m v_1, \dots, c^m v_{i-1}, m - \beta[m - c^m v_i](d(\sigma_i v) - c^m v_i), c^m v_{i+1}, \dots, c^m v_n) & \text{if } \varepsilon = 0 \\ (c^m v_1, \dots, c^m v_{i-1}, \beta[c^m v_i](d(\sigma_i v) + c^m v_i - m), c^m v_{i+1}, \dots, c^m v_n) & \text{if } \varepsilon = 1. \end{cases} \end{aligned}$$

We first show this formula is well defined, i.e. that this tuple lies in the subgraph $|\sqcap_{i,\varepsilon}^n|_m$ of $I_m^{\otimes n}$. Observe that if $c^m v_k = 0$ or m for some $k \neq i$ in $\{1, \dots, n\}$ then this tuple indeed lies in $|\sqcap_{i,\varepsilon}^n|_m$. For $k < i$, this tuple lies on the $(k, 0)$- or $(k, 1)$-face. For $k > i$, this tuple lies on the $(k+1, 0)$- or $(k+1, 1)$-face. Otherwise, if $0 < c^m v_k < m$ for all $k \neq i$ then $d(\sigma_i v) = 0$ since $v_k = \varphi v_k$ for all $k \neq i$. From this, it follows that

$$\beta[\alpha_{1-\varepsilon}^{v_i}](d(\sigma_i v) - \alpha_\varepsilon^{v_i}) = \beta[\alpha_{1-\varepsilon}^{v_i}](-\alpha_\varepsilon^{v_i}) = 0,$$

thus $\Phi(v_1, \dots, v_n)$ lies on the $(i, 1-\varepsilon)$-face, i.e. the face opposite the missing face.

To see this formula gives a graph map, suppose $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ are connected vertices in $I_{3m}^{\otimes n}$. By definition, there exists $k = 1, \dots, n$ so that v_k and w_k are connected in I_{3m} and $v_j = w_j$ for all $j \neq k$. We first consider the case where $k \neq i$. Observe that if $c^m v_k = c^m w_k$ then this is immediate. If $c^m v_k \neq c^m w_k$ then $\sigma_i v = \varphi(\sigma_i v)$. This gives that $d(\sigma_i v) = d(\sigma_i w)$, hence $\Phi(v_1, \dots, v_n)$ and $\Phi(w_1, \dots, w_n)$ are equal on all components except the kth component. That is, they are connected. In the case where $k = i$, we have that dv and dw differ by at most 1. This implies $(1-e)m + (2\varepsilon - 1)(\beta[\alpha_{1-\varepsilon}^{v_i}](dv - \alpha_\varepsilon^{v_i}))$ and $(1-e)m + (2\varepsilon - 1)(\beta[\alpha_{1-\varepsilon}^{v_i}](dw - \alpha_\varepsilon^{v_i}))$ differ by at most 1, thus $\Phi(v_1, \dots, v_n)$ and $\Phi(w_1, \dots, w_n)$ are connected.

Example 4.3. We look at the map $\Phi\colon I_{3m}^{\otimes n} \to |\sqcap_{i,\varepsilon}^n|_m$ in the case of $n = 3$, $m = 2$, and $(i, \varepsilon) = (3, 1)$ (Figure 8).

We may write the map $\Phi\colon I_6^{\otimes 3} \to |\sqcap_{3,1}^3|_2$ as

$$\Phi(v_1, v_2, v_3) = \begin{cases} (c^2 v_1, c^2 v_2, 0) & \text{if } v_3 \leq 2 \\ (c^2 v_1, c^2 v_2, \beta[1](d(v_1, v_2) - 1)) & \text{if } v_3 = 3 \\ (c^2 v_1, c^2 v_2, \beta[2](d(v_1, v_2))) & \text{if } v_3 \geq 4. \end{cases}$$

If $v_3 \leq 2$ then (v_1, v_2, v_3) is contained in the $\partial_{3,0}$-face of $|\sqcap_{3,1}^3|_2$, which is the face opposite the missing face. For the cross-section where $v_3 = 3$, if $d(v_1, v_2) \geq 2$ then (v_1, v_2, v_3) is sent to a vertex in the $v_3 = 1$ cross-section of $|\sqcap_{3,1}^3|_2$. For $v_3 \geq 3$, if $d(v_1, v_2) = 1$ then (v_1, v_2, v_3) is sent to a vertex in the $v_3 = 1$ cross-section. If $d(v_1, v_2) \geq 2$ then (v_1, v_2, v_3) is sent to a vertex in the $v_3 = 2$ cross-section. In all three cross-sections, if $d(v_1, v_2) = 0$ then (v_1, v_2, v_3) is contained in the face opposite the missing face of $|\sqcap_{3,1}^3|_2$ (Figure 9).

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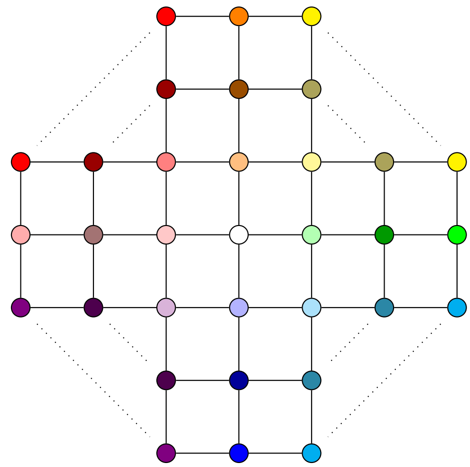
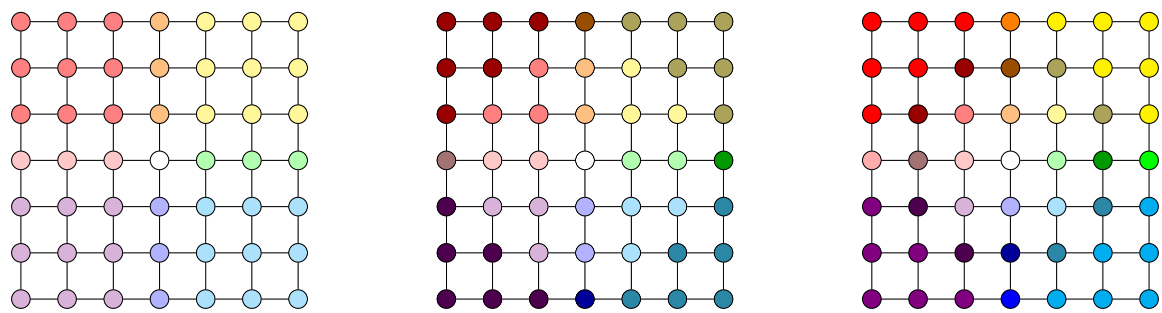


Figure 8 (colour online). The graph $|\sqcap^3_{3,1}|_2$ as a net. Vertices connected by a dotted line are identical.



(a) $v_3 \leqslant 2$. (b) $v_3 = 3$ (c) $v_3 \geqslant 4$

Figure 9 (colour online). The map $\Phi\colon I_6^{\otimes 3} \to |\sqcap^3_{3,1}|_2$ split into cross-sections.

Lemma 4.4. *The diagram*

$$\begin{tikzcd} {|\sqcap^n_{i,\varepsilon}|_{3m}} \arrow[r, "(c_*)^m"] \arrow[d, hook] & {|\sqcap^n_{i,\varepsilon}|_m} \\ I_{3m}^{\otimes n} \arrow[ur, "\Phi"] & \end{tikzcd}$$

commutes.

Proof. For $n = 1$, we have $|\sqcap^1_{i,\varepsilon}|_m \cong I_0$. The diagram then commutes as I_0 is terminal in Graph.

For $n \geq 2$, fix $(v_1, \ldots, v_n) \in |\sqcap^n_{i,\varepsilon}|_{3m}$. It suffices to show

$$(1-\varepsilon)m + (2\varepsilon - 1)(\beta[\alpha^{v_i}_{1-\varepsilon}](d(\sigma_i v) - \alpha^{v_i}_{\varepsilon})) = c^m v_i.$$

If $v_i = (1-\varepsilon)(3m)$ then $d(\sigma_i v) = 0$. Thus,

$$\begin{aligned} (1-\varepsilon)m + (2\varepsilon - 1)(\beta[\alpha^{v_i}_{1-\varepsilon}](d(\sigma_i v) - \alpha^{v_i}_{\varepsilon})) &= (1-\varepsilon)m + (2\varepsilon - 1)(\beta[\alpha^{v_i}_{1-\varepsilon}](-\alpha^{v_i}_{\varepsilon})) \\ &= (1-\varepsilon)m + (2\varepsilon - 1)(0) \\ &= (1-\varepsilon)m \\ &= c^m((1-\varepsilon)(3m)) \\ &= c^m v_i. \end{aligned}$$

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Otherwise, if $v_i \neq (1-\varepsilon)(3m)$ then $d(\sigma_i v) \geq m$ (since $v_k = 0, 3m$ for some $k \neq i$). For $\varepsilon = 0$, we compute

$$\begin{aligned}(1-\varepsilon)m + (2\varepsilon - 1)(\beta[\alpha^{v_i}_{1-\varepsilon}](d(\sigma_i v) - \alpha^{v_i}_{\varepsilon})) &= m - \beta[\alpha^{v_i}_1](d(\sigma_i v) - \alpha^{v_i}_0) \\ &= m - \beta[m - c^m v_i](d(\sigma_i v) - c^m v_i) \\ &= m - (m - c^m v_i) \\ &= c^m v_i.\end{aligned}$$

For $\varepsilon = 1$, we compute

$$\begin{aligned}(1-\varepsilon)m + (2\varepsilon - 1)(\beta[\alpha^{v_i}_{1-\varepsilon}](d(\sigma_i v) - \alpha^{v_i}_{\varepsilon})) &= \beta[\alpha^{v_i}_0](d(\sigma_i v) - \alpha^{v_i}_1) \\ &= \beta[c^m v_i](d(\sigma_i v) + c^m v_i - m) \\ &= c^m v_i.\end{aligned}$$

□

THEOREM 4.5. *For any graph G, the nerve $\mathrm{N}G$ of G is a Kan complex.*

Proof. Fix a map $f\colon \sqcap^n_{i,\varepsilon} \to \mathrm{N}G$. We know that $\mathrm{N}G \cong \operatorname{colim}(\mathrm{N}_1 G \hookrightarrow \mathrm{N}_3 G \hookrightarrow \mathrm{N}_5 G \hookrightarrow \cdots)$ by Proposition 3.5. Recall that in a presheaf category, any map from a representable presheaf to a colimit must factor through some component of the colimit cone by the Yoneda lemma. Thus, for any $k \geq 0$ and $x\colon \square^k \to \sqcap^n_{i,\varepsilon}$, the map $fx\colon \square^k \to \mathrm{N}G$ factors through an inclusion $\mathrm{N}_m G \hookrightarrow \mathrm{N}G$ for some $m \geq 1$. As $\sqcap^n_{i,\varepsilon}$ has only finitely many non-degenerate cubes, f factors through the natural inclusion $\mathrm{N}_m G \hookrightarrow \mathrm{N}G$ as a map $g\colon \sqcap^n_{i,\varepsilon} \to \mathrm{N}_m G$ for some $m \geq 0$.

By adjointness, g corresponds to a map $\overline{g}\colon |\sqcap^n_{i,\varepsilon}|_m \to G$. Lemma 4.4 shows that $\overline{g}\Phi\colon I^{\otimes n}_{3m} \to G$ is a lift of the composite map $\overline{g}(c^*)^m\colon |\sqcap^n_{i,\varepsilon}|_{3m} \to G$:

$$\begin{tikzcd} {|\sqcap^n_{i,\varepsilon}|_{3m}} \arrow[r, "(c_*)^m"] \arrow[d, hook] & {|\sqcap^n_{i,\varepsilon}|_m} \arrow[r, "\overline{g}"] & G \\ {|\square^n|_{3m}} \arrow[ur, "\Phi"'] & & \end{tikzcd}$$

By adjointness, this gives a filler $\square^n \to \mathrm{N}_{3m} G$ for the map $(c^*)^m g\colon \sqcap^n_{i,\varepsilon} \to \mathrm{N}_{3m} G$. Post-composing with the natural inclusion $\mathrm{N}_{3m} G \hookrightarrow \mathrm{N}G$, this gives a filler $\square^n \to \mathrm{N}G$ of f. □

Via Theorem 4.5, we may speak of the homotopy groups of $\mathrm{N}G$. We prove that the A-homotopy groups of G are exactly the cubical homotopy groups of $\mathrm{N}G$.

THEOREM 4.6. *We have an isomorphism $A_n(G, v) \cong \pi_n(\mathrm{N}G, v)$ natural in (G, v).*

Proof. It is straightforward to verify that a relative graph map $(I^{\otimes n}_\infty, I^{\otimes n}_{\geq M}) \to (G, v)$ is exactly a relative cubical map $(\square^n, \partial\square^n) \to (\mathrm{N}G, v)$ and that a relative homotopy between two such relative graph maps is exactly a relative homotopy between two such relative cubical maps. This gives a set bijection $A_n(G, v) \to \pi_n(\mathrm{N}G, v)$. Proposition 3.11 shows this map is a group homomorphism. □

Remark 4.7. Theorem 4.6 shows why the functor $\mathrm{N}\colon \mathsf{Graph} \to \mathsf{cSet}$ fails to preserve arbitrary limits, even though, by Proposition 3.8, it preserves finite ones. To see that, we first note that

$$A_1\left(\prod_{k \geq 5}(C_k, 0)\right) \cong \bigoplus_{k \geq 5} \mathbb{Z},$$

since a map $I_\infty \to \prod_{k \geq 5} C_k$ that stabilizes outside of a finite interval must necessarily be null-homotopic in all but finitely many C_k. That is, $A_1\colon \mathsf{Graph}_* \to \mathsf{Grp}$ does not preserve infinite products. Theorem 4.6 shows that $A_1 \cong \pi_1 \circ \mathrm{N}$, and $\pi_1 \colon \mathsf{cSet}_* \to \mathsf{Grp}$ preserves infinite products by [CK23, Proposition 4.1]. Thus, we conclude that

$$\mathrm{N}\left(\prod_{k \geq 5} (C_k, 0)\right) \ncong \prod_{k \geq 5} (\mathrm{N}C_k, 0).$$

Proof of part (ii)
Now we show that the inclusions $l^*, r^* \colon \mathrm{N}_m G \hookrightarrow \mathrm{N}_{m+1} G$ are anodyne. We first explain the intuition for why this statement holds.

The 1-nerve $\mathrm{N}_1 G$ of a graph G contains, as 1-cubes, all paths of length 1 in G (i.e. paths with two vertices and one edge). Consider the image of the embedding $l^* \colon \mathrm{N}_1 G \to \mathrm{N}_2 G$ as a cubical subset $X \subseteq \mathrm{N}_2 G$ of the 2-nerve of G. A 1-cube of $\mathrm{N}_2 G$ is exactly a path of length 2 in G; a 1-cube of X is a path of length 1 regarded as a path of length 2 whose first two vertices are the same. Given a path $f \colon I_2 \to G$ of length 2 in G, we may define a 3×3 square $g \colon I_2^{\otimes 2} \to G$ in G by

$$g(v_1, v_2) := \begin{cases} f(l(v_1)) & v_2 < 2 \\ f(v_1) & v_2 = 2. \end{cases}$$

Observe that the $\partial_{1,0}$-, $\partial_{1,1}$-, and $\partial_{2,0}$-faces of g are 1-cubes of X, i.e. they are paths of length 2 whose first two vertices are the same, whereas the $\partial_{2,1}$-face of g is f. That is, we have shown the restriction $g|_{|\sqcap^2_{2,1}|_2} \colon |\sqcap^2_{2,1}|_2 \to G$ of g to the open box corresponds to a map $\sqcap^2_{2,1} \to \mathrm{N}_2 G$ whose image is contained in X (Figure 10).

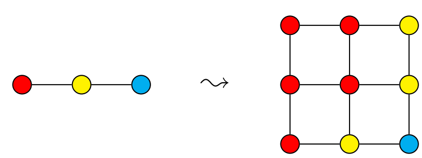

Figure 10 (colour online). The square $g \colon I_2^{\otimes 2} \to G$ constructed from the path $f \colon I_2 \to G$.

Let $Y \subseteq \mathrm{N}_2 G$ denote the cubical subset generated by X and g, i.e. the cubical subset containing X, the 2-cube $g \in (\mathrm{N}_2 G)_2$, and all faces, degeneracies, and connections of g. The square

$$\begin{tikzcd} \sqcap^2_{2,1} \arrow[r, "g|_{|\sqcap^2_{2,1}|_2}"] \arrow[d, hook] & X \arrow[d, hook] \\ \square^2 \arrow[r, "g"] \arrow[ur, phantom, "\ulcorner", very near end] & Y \end{tikzcd}$$

is a pushout by definition of Y. This gives exactly that the inclusion $X \hookrightarrow Y$ is anodyne. Following this approach, one may construct an anodyne inclusion $X \hookrightarrow X_n$ such that X_n contains all n-cubes of $\mathrm{N}_2 G$. With this, the colimit of the sequence $X \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$ is exactly $\mathrm{N}_2 G$ and the inclusion $X \hookrightarrow \mathrm{N}_2 G$ is anodyne by closure under transfinite composition.

For $n > 0$ and $j \in \{0, \ldots, n\}$, the maps $l, r \colon I_{m+1} \to I_m$ yield maps

$$(\mathrm{id}^{\otimes j}_{I_{m+1}} \otimes l^{\otimes n-j}), (\mathrm{id}^{\otimes j}_{I_{m+1}} \otimes r^{\otimes n-j}) \colon I^{\otimes n}_{m+1} \to I^{\otimes j}_{m+1} \otimes I^{\otimes n-j}_m.$$

For $j \in \{0, \ldots, n-1\}$, we define maps $\lambda^n_{m,j}, \rho^n_{m,j} \colon I^{\otimes n}_{m+1} \otimes I_1 \to I^{\otimes j+1}_{m+1} \otimes I^{\otimes n-j-1}_m$ by

$$\lambda^n_{m,j}(v_1, \ldots, v_n, v_{n+1}) = \begin{cases} (v_1, \ldots, v_j, lv_{j+1}, lv_{j+2}, \ldots, lv_n) & v_{n+1} = 0 \\ (v_1, \ldots, v_{j+1}, lv_{j+2}, \ldots, lv_n) & v_{n+1} = 1, \end{cases}$$

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$$\rho^n_{m,j}(v_1,\dots,v_n,v_{n+1}) = \begin{cases} (v_1,\dots,v_j,rv_{j+1},rv_{j+2},\dots,rv_n) & v_{n+1}=0 \\ (v_1,\dots,v_{j+1},rv_{j+2},\dots,rv_n) & v_{n+1}=1. \end{cases}$$

That is, the restriction of $\lambda^n_{m,j}$ to $I^{\otimes n}_{m+1}\otimes\{0\}$ is the map $\mathrm{id}^{\otimes j}_{I_{m+1}}\otimes l^{\otimes n-j}$ and its restriction to $I^{\otimes n}_{m+1}\otimes\{1\}$ is $\mathrm{id}^{\otimes j+1}_{I_{m+1}}\otimes l^{\otimes n-j-1}$ (likewise for $\rho^n_{m,j}$).

The maps $l^m, r^m\colon I_{m+1}\to I_1$ denote application of the maps l, r a total of m times. That is,

$$l^m(v) = \begin{cases} 0 & v<m+1 \\ 1 & v=m+1, \end{cases}$$

$$r^m(v) = \begin{cases} 0 & v=0 \\ 1 & v>0. \end{cases}$$

We write $\bar\lambda^n_{m,j}\colon I^{\otimes n+1}_{m+1}\to I^{\otimes j+1}_{m+1}\otimes I^{\otimes n-j-1}_m$ for the composition of $\mathrm{id}^{\otimes n}_{I_{m+1}}\otimes l^m\colon I^{\otimes n+1}_{m+1}\to I^{\otimes n}_{m+1}\otimes I_1$ followed by $\lambda^n_{m,j}\colon I^{\otimes n}_{m+1}\otimes I_1\to I^{\otimes j+1}_{m+1}\otimes I^{\otimes n-j-1}_m$ (Figure 11), and we write $\bar\rho^n_{m,j}\colon I^{\otimes n+1}_{m+1}\to I^{\otimes j+1}_{m+1}\otimes I^{\otimes n-j-1}_m$ for the composition of $\mathrm{id}^{\otimes n}_{I_{m+1}}\otimes r^n\colon I^{\otimes n+1}_{m+1}\to I^{\otimes n}_{m+1}\otimes I_1$ followed $\rho^n_{m,j}\colon I^{\otimes n}_{m+1}\otimes I_1\to I^{\otimes j+1}_{m+1}\otimes I^{\otimes n-j-1}_m$, respectively (Figure 12).

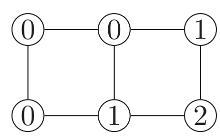
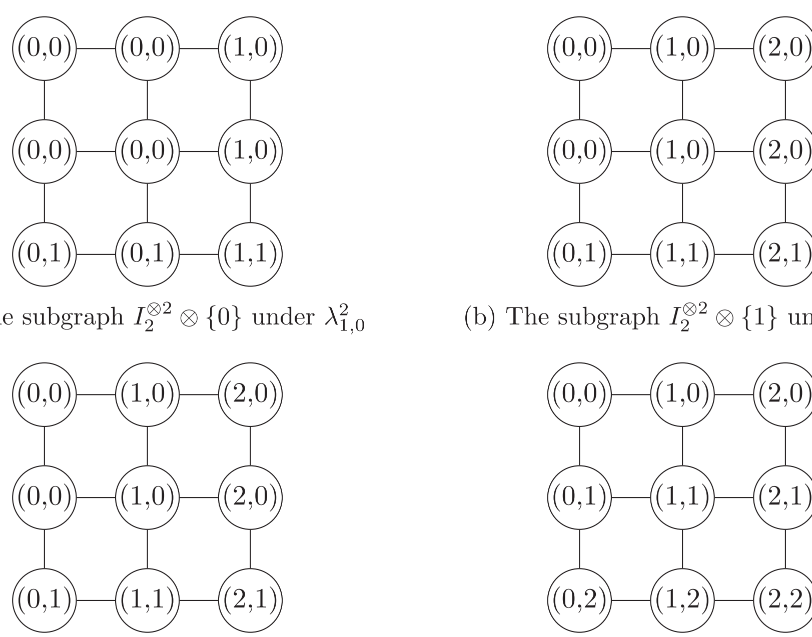


FIGURE 11. The graph $I_2\otimes I_1$ with vertices labeled by their image under $\lambda^n_{1,0}\colon I_2\otimes I_1\to I_2$.



(a) The subgraph $I_2^{\otimes 2}\otimes\{0\}$ under $\lambda^2_{1,0}$

(b) The subgraph $I_2^{\otimes 2}\otimes\{1\}$ under $\lambda^2_{1,1}$

(c) The subgraph $I_2^{\otimes 2}\otimes\{0\}$ under $\lambda^2_{1,1}$

(d) The subgraph $I_2^{\otimes 2}\otimes\{1\}$ under $\lambda^2_{1,1}$

FIGURE 12. Cross-sections of the graph $I_2^{\otimes 2}\otimes I_1$ with vertices labeled by their image under the maps $\lambda^2_{1,0}\colon I_2^{\otimes 2}\otimes I_1\to I_2\otimes I_1$ and $\lambda^2_{1,1}\colon I_2^{\otimes 2}\otimes I_1\to I_2^{\otimes 2}$.

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PROPOSITION 4.8. *Let $m, n > 0$ and $j \in \{0, \dots, n\}$. For $i = 1, \dots, n$ such that $i \neq j+1$ and $\varepsilon = 0, 1$,*

(i) $\bar{\lambda}^n_{m,j}\partial_{i,\varepsilon}\colon I^{\otimes n}_{m+1} \to I^{\otimes n}_{m+1}$ *factors through* $\bar{\lambda}^{n-1}_{m,j}\colon I^{\otimes n}_{m+1} \to I^{\otimes n-1}_{m+1}$;
(ii) $\bar{\rho}^n_{m,j}\partial_{i,\varepsilon}\colon I^{\otimes n}_{m+1} \to I^{\otimes n}_{m+1}$ *factors through* $\bar{\rho}^{n-1}_{m,j}\colon I^{\otimes n}_{m+1} \to I^{\otimes n-1}_{m+1}$.

Proof. We consider the result for $\bar{\lambda}^n_{m,j}$, as the result for $\bar{\rho}^n_{m,j}$ is analogous.

Fix $(v_1, \dots, v_n) \in I^{\otimes n}_{m+1}$. If $i < j+1$ then we have

$$\begin{aligned}
\bar{\lambda}^n_{m,j}\partial_{i,\varepsilon}(v_1, \dots, v_n) &= \bar{\lambda}^n_{m,j}(v_1, \dots, v_{i-1}, \varepsilon(m+1), v_i, \dots, v_n) \\
&= \lambda^n_{m,j}(v_1, \dots, v_{i-1}, \varepsilon(m+1), v_i, \dots, v_{n-1}, l^m v_n) \\
&= \begin{cases} (v_1, \dots, v_{i-1}, \varepsilon(m+1), v_i, \dots, v_j, lv_{j+1}, \dots, lv_{n-1}) & \text{if } l^m v_n = 0 \\ (v_1, \dots, v_{i-1}, \varepsilon(m+1), v_i, \dots, v_{j+1}, lv_{j+2}, \dots, lv_{n-1}) & \text{if } l^m v_n = 1 \end{cases} \\
&= \partial^n_{i,\varepsilon}\lambda^n_{m,j}(v_1, \dots, v_{n-1}, l^m v_n) \\
&= \partial^n_{i,\varepsilon}\bar{\lambda}^n_{m,j}(v_1, \dots, v_n).
\end{aligned}$$

Thus, $\bar{\lambda}^n_{m,j}\partial_{i,\varepsilon} = \partial_{i,\varepsilon}\bar{\lambda}^{n-1}_{m,j}$.

Otherwise, we have $i > j+1$. Consider the embedding $\iota\colon I^{\otimes n-1}_{m+1} \to I^{\otimes n}_{m+1}$ given by

$$\iota(v_1, \dots, v_{n-1}) = (v_1, \dots, v_{i-1}, \varepsilon m, v_i, \dots, v_{n-1}).$$

With this, we may write

$$\begin{aligned}
\bar{\lambda}^n_{m,j}\partial^{n+1}_{i,0}(v_1, \dots, v_n) &= \bar{\lambda}^n_{m,j}(v_1, \dots, v_{i-1}, \varepsilon(m+1), v_i, \dots, v_n) \\
&= \lambda^n_{m,j}(v_1, \dots, v_{i-1}, \varepsilon(m+1), v_i, \dots, v_{n-1}, l^m v_n) \\
&= \begin{cases} (v_1, \dots, v_j, lv_{j+1}, \dots, lv_{i-1}, l(\varepsilon(m+1)), lv_i, \dots, lv_{n-1}) & \text{if } l^m v_n = 0 \\ (v_1, \dots, v_{j+1}, lv_{j+2}, \dots, lv_{i-1}, l(\varepsilon(m+1)), lv_i, \dots, lv_{n-1}) & \text{if } l^m v_n = 1 \end{cases} \\
&= \begin{cases} (v_1, \dots, v_j, lv_{j+1}, \dots, lv_{i-1}, \varepsilon m, lv_i, \dots, lv_{n-1}) & \text{if } l^m v_n = 0 \\ (v_1, \dots, v_{j+1}, lv_{j+2}, \dots, lv_{i-1}, \varepsilon m, lv_i, \dots, lv_{n-1}) & \text{if } l^m v_n = 1 \end{cases} \\
&= \iota\lambda^n_{m,j}(v_1, \dots, v_{n-1}, l^m v_n) \\
&= \iota\bar{\lambda}^n_{m,j}(v_1, \dots, v_n).
\end{aligned}$$

Thus, $\bar{\lambda}^n_{m,j}\partial_{i,\varepsilon} = \iota\bar{\lambda}^{n-1}_{m,j}$. □

PROPOSITION 4.9. *Let $m, n > 0$ and $j \in \{0, \dots, n\}$. Then*

(i) $\bar{\lambda}^n_{m,j}\partial_{j+1,0}$ *and* $\bar{\lambda}^n_{m,j}\partial_{j+1,1}$ *factor through* $\mathrm{id}^{\otimes j}_{I_{m+1}} \otimes l^{\otimes n-j}\colon I^{\otimes n}_{m+1} \to I^{\otimes j}_{m+1} \otimes I^{\otimes n-j}_m$;
(ii) $\bar{\rho}^n_{m,j}\partial_{j+1,0}$ *and* $\bar{\rho}^n_{m,j}\partial_{j+1,1}$ *factor through* $\mathrm{id}^{\otimes j}_{I_{m+1}} \otimes r^{\otimes n-j}\colon I^{\otimes n}_{m+1} \to I^{\otimes j}_{m+1} \otimes I^{\otimes n-j}_m$.

Proof. We show the result for $\bar{\lambda}^n_{m,j}$ as the result for $\bar{\rho}^n_{m,j}$ is analogous.

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Fix $(v_1, \dots, v_n) \in I_{m+1}^{\otimes n}$. We compute

$$
\begin{aligned}
\bar{\lambda}^n_{m,j}\partial_{j+1,0}(v_1,\dots,v_n) &= \bar{\lambda}^n_{m,j}(v_1,\dots,v_j,0,v_{j+1},\dots,v_n) \\
&= \lambda^n_{m,j}(v_1,\dots,v_j,0,v_{j+1},\dots,l^m v_n) \\
&= \begin{cases} (v_1,\dots,v_j,l0,lv_{j+1},\dots,lv_{n-1}) & \text{if } l^m v_n = 0 \\ (v_1,\dots,v_j,0,lv_{j+1},\dots,lv_{n-1}) & \text{if } l^m v_n = 1 \end{cases} \\
&= (v_1,\dots,v_j,0,lv_{j+1},\dots,lv_{n-1}) \\
&= \partial_{j+1,0}\sigma_n(\mathrm{id}^{\otimes j}_{I_{m+1}} \otimes l^{\otimes n-j})(v_1,\dots,v_n),
\end{aligned}
$$

thus $\bar{\lambda}^n_{m,j}\partial_{j+1,0} = \partial_{j+1,0}\sigma_n(\mathrm{id}^{\otimes j}_{I_{m+1}} \otimes l^{\otimes n-j})$.

Consider the map $f\colon I^{\otimes j}_{m+1} \otimes I^{\otimes n-j}_m \to I^{\otimes j+1}_{m+1} \otimes I^{\otimes n-j-1}_m$ defined by

$$
f(a_1,\dots,a_j,b_1,\dots,b_{n-j}) = \begin{cases} (a_1,\dots,a_j,m,b_1,\dots,b_{n-j-1}) & \text{if } l^{m-1}b_{n-j} = 0 \\ (a_1,\dots,a_j,m+1,b_1,\dots,b_{n-j-1}) & \text{if } l^{m-1}b_{n-j} = 1. \end{cases}
$$

It is straightforward to verify this is a graph map. With this, we may write

$$
\begin{aligned}
\bar{\lambda}^n_{m,j}\partial_{j+1,1}(v_1,\dots,v_n) &= \bar{\lambda}^n_{m,j}(v_1,\dots,v_j,m+1,v_{j+1},\dots,v_n) \\
&= \lambda^n_{m,j}(v_1,\dots,v_j,m+1,v_{j+1},\dots,l^m v_n) \\
&= \begin{cases} (v_1,\dots,v_j,l(m+1),lv_{j+1},\dots,lv_{n-1}) & \text{if } l^m v_n = 0 \\ (v_1,\dots,v_j,m+1,lv_{j+1},\dots,lv_{n-1}) & \text{if } l^m v_n = 1 \end{cases} \\
&= \begin{cases} (v_1,\dots,v_j,m,lv_{j+1},\dots,lv_{n-1}) & \text{if } l^m v_n = 0 \\ (v_1,\dots,v_j,m+1,lv_{j+1},\dots,lv_{n-1}) & \text{if } l^m v_n = 1 \end{cases} \\
&= f(v_1,\dots,v_j,lv_{j+1},\dots,lv_n) \\
&= f(\mathrm{id}^{\otimes j}_{I_{m+1}} \otimes l^{\otimes n-j})(v_1,\dots,v_n).
\end{aligned}
$$

Thus, $\bar{\lambda}^n_{m,j}\partial_{j+1,1} = f(\mathrm{id}^{\otimes j}_{I_{m+1}} \otimes l^{\otimes n-j})$. □

Lemma 4.10. *Let $m, n > 0$, $j \in \{0, \dots, n-1\}$, and G be a graph. Consider a subobject X of $\mathrm{N}_{m+1}G$ which contains:*

- *all n-cubes of $\mathrm{N}_{m+1}G$ which factor through $l^{\otimes n}\colon I^{\otimes n}_{m+1} \to I^{\otimes n}_m$;*
- *(if $n > 1$) for any $x\colon I^{\otimes h}_{m+1} \otimes I^{\otimes k-h}_m \to G$ where $k < n$ and $h \leq k$, the $(k+1)$-cube $x\bar{\lambda}^k_{m,h-1}\colon I^{\otimes k+1}_{m+1} \to G$;*
- *(if $j > 0$) for any $x\colon I^{\otimes j}_{m+1} \otimes I^{\otimes n-j}_m \to G$, the $(n+1)$-cube $x\bar{\lambda}^n_{m,j-1}\colon I^{\otimes n+1}_{m+1} \to G$ of $\mathrm{N}_{m+1}G$.*

Then, for any n-cube of $\mathrm{N}_{m+1}G$ which factors through $\mathrm{id}^{\otimes j+1}_{I_{m+1}} \otimes l^{\otimes n-j-1}\colon I^{\otimes n}_{m+1} \to I^{\otimes j+1}_{m+1} \otimes I^{\otimes n-j-1}_m$ as some $x\colon I^{\otimes j+1}_{m+1} \otimes I^{\otimes n-j-1}_m \to G$, the restriction of the $(n+1)$-cube $x\bar{\lambda}^n_{m,j}\colon I^{\otimes n+1}_{m+1} \to G$ to the open box $x\bar{\lambda}^n_{m,j}|_{\sqcap^{n+1}_{n+1,1}}\colon \sqcap^{n+1}_{n+1,1} \to \mathrm{N}_{m+1}G$ factors through the inclusion $X \hookrightarrow \mathrm{N}_{m+1}G$.

Proof. We first show that X contains all n-cubes which factor through the map $\mathrm{id}^{\otimes j}_{I_{m+1}} \otimes l^{\otimes n-j}\colon I^{\otimes n}_{m+1} \to I^{\otimes j}_{m+1} \otimes I^{\otimes n-j}_m$. If $j = 0$ then this follows by assumption. Otherwise, consider such an n-cube, which we write as $y(\mathrm{id}^{\otimes j}_{I_{m+1}} \otimes l^{\otimes n-j})\colon I^{\otimes n}_{m+1} \to G$ for some $y\colon I^{\otimes j}_{m+1} \otimes I^{\otimes n-j}_m \to G$. Recall $\bar{\lambda}^n_{m,j-1}\partial_{n+1,1} = \mathrm{id}^{\otimes j}_{I_{m+1}} \otimes l^{\otimes n-j}$. This gives that $y(\mathrm{id}^{\otimes j}_{I_{m+1}} \otimes l^{\otimes n-j}) =$

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$y\bar{\lambda}^n_{m,j-1}\partial_{n+1,1}$. By assumption, X contains $y\bar{\lambda}^n_{m,j-1}$. Thus, it contains all faces of $y\bar{\lambda}^n_{m,j-1}$, including y.

To see that each face of $x\bar{\lambda}^n_{m,j}|_{\sqcap^{n+1}_{n+1,1}}$ is contained in X, fix $(i,\varepsilon) \neq (n+1,1)$. For $i = n+1$ and $\varepsilon = 0$, this follows as $\bar{\lambda}^n_{m,j}\partial_{n+1,0} = \mathrm{id}^{\otimes j}_{I_{m+1}} \otimes l^{\otimes n-j}$. Otherwise, if $i \neq j+1$, this follows from Proposition 4.8. If $i = j+1$ then Proposition 4.9 gives that $\bar{\lambda}^n_{m,j}$ factors through $\mathrm{id}^{\otimes j}_{I_{m+1}} \otimes l^{\otimes n-j}$. $\square$

We have an analogous result for $\bar{\rho}^n_{m,j}$ as well.

Lemma 4.11. *Let $m, n > 0$, $j \in \{0, \dots, n-1\}$, and G be a graph. Consider a subobject X of $\mathrm{N}_{m+1}G$ which contains:*

- *all n-cubes of $\mathrm{N}_{m+1}G$ which factor through $r^{\otimes n}\colon I^{\otimes n}_{m+1} \to I^{\otimes n}_m$;*
- *(if $n > 1$) for any $x\colon I^{\otimes h}_{m+1} \otimes I^{\otimes k-h}_m \to G$ where $k < n$ and $h \leq k$, the $(k+1)$-cube $x\bar{\rho}^n_{m,h-1}\colon I^{\otimes n+1}_{m+1} \to G$;*
- *(if $j > 0$) for any $x\colon I^{\otimes j}_{m+1} \otimes I^{\otimes n-j}_m \to G$, the $(n+1)$-cube $x\bar{\rho}^n_{m,j-1}\colon I^{\otimes n+1}_{m+1} \to G$ of $\mathrm{N}_{m+1}G$.*

Then, for any n-cube of $\mathrm{N}_{m+1}G$ which factors through $\mathrm{id}^{\otimes j+1}_{I_{m+1}} \otimes r^{\otimes n-j-1}\colon I^{\otimes n}_{m+1} \to I^{\otimes j+1}_{m+1} \otimes I^{\otimes n-j-1}_m$ as some $x\colon I^{\otimes j+1}_{m+1} \otimes I^{\otimes n-j-1}_m \to G$, the restriction of the $(n+1)$-cube $x\bar{\rho}^n_{m,j}\colon I^{\otimes n+1}_{m+1} \to G$ to the open box $x\bar{\rho}^n_{m,j}|_{\sqcap^{n+1}_{n+1,1}}\colon \sqcap^{n+1}_{n+1,1} \to \mathrm{N}_{m+1}G$ factors through the inclusion $X \hookrightarrow \mathrm{N}_{m+1}G$.

Theorem 4.12. *Let $m > 0$ and G be a graph. The maps $l^*, r^*\colon \mathrm{N}_mG \to \mathrm{N}_{m+1}G$ are anodyne.*

Proof. We show that l^* is anodyne as the result for r^* is analogous. Let X_0 denote the image of N_mG in $\mathrm{N}_{m+1}G$ under the embedding $l^*\colon \mathrm{N}_mG \hookrightarrow \mathrm{N}_{m+1}G$. We show $\mathrm{N}_{m+1}G$ can be obtained from X_0 by a transfinite composition of pushouts along coproducts of open box inclusion. For $n > 0$ and $j \in \{1, \dots, n\}$, let $X_{n,j}$ be the subobject of $\mathrm{N}_{m+1}G$ generated by:

- X_0;
- (if $n > 1$) for any $x\colon I^{\otimes l}_{m+1} \otimes I^{\otimes k-l}_m \to G$ where $k < n$ and $l \leq k$, the $(k+1)$-cube $x\bar{\lambda}^k_{m,l-1}\colon I^{\otimes n+1}_{m+1} \to G$ of $\mathrm{N}_{m+1}G$;
- for any $x\colon I^{\otimes i}_{m+1} \otimes I^{\otimes n-i}_m \to G$ where $i \leq j$, the $(n+1)$-cube $x\bar{\lambda}^n_{m,i-1}\colon I^{\otimes n+1}_{m+1} \to G$ of $\mathrm{N}_{m+1}G$.

By construction, there is a sequence of embeddings

$$X_0 \hookrightarrow X_{1,1} \hookrightarrow X_{2,1} \hookrightarrow X_{2,2} \hookrightarrow X_{3,1} \hookrightarrow \cdots$$

Note that the subobject $X_{n,n}$ contains all n-cubes of $\mathrm{N}_{m+1}G$. This is because any n-cube $x\colon I^{\otimes n}_{m+1} \to G$ is the $\partial_{n+1,1}$-face of $x\bar{\lambda}^n_{m,n-1}\colon I^{\otimes n+1}_{m+1} \to G$. By construction, $X_{n,n}$ contains $x\bar{\lambda}^n_{m,n-1}$. Thus, it contains all faces of $x\bar{\lambda}^n_{m,n-1}$, including x. With this, we have

$$\mathrm{N}_{m+1}G \cong \operatorname{colim}(X_0 \hookrightarrow X_{1,1} \hookrightarrow X_{2,1} \hookrightarrow X_{2,2} \hookrightarrow \cdots).$$

It remains to show $X_{n,j+1}$ is a pushout of $X_{n,j}$ along a coproduct of open box inclusions and $X_{n+1,1}$ is a pushout of $X_{n,n}$ along a coproduct of open box inclusions.

Fix $n > 0$ and $j \in \{1, \dots, n-1\}$. Let $S_{n,j+1}$ be the set of n-cubes $I^{\otimes n}_{m+1} \to G$ which factor through the map $\mathrm{id}^{\otimes j+1}_{I_{m+1}} \otimes l^{\otimes n-j-1}\colon I^{\otimes n}_{m+1} \to I^{\otimes j+1}_{m+1} \otimes I^{\otimes n-j-1}_m$ and are not contained in $X_{n,j}$. We write an element of $S_{n,j+1}$ as $x(\mathrm{id}^{\otimes j+1}_{I_{m+1}} \otimes l^{\otimes n-j-1})$ for some $x\colon I^{\otimes j+1}_{m+1} \otimes I^{\otimes n-j-1}_m \to G$. By construction, $X_{n,j+1}$ contains all n-cubes of $S_{n,j+1}$ as such an n-cube $x(\mathrm{id}^{\otimes j+1}_{I_{m+1}} \otimes l^{\otimes n-j-1})$ is the

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$\partial_{n+1,1}$-face of $x\bar{\lambda}^n_{m,j}$ (which is contained in $X_{n,j+1}$ by construction). This gives a map

$$\coprod_{x(\mathrm{id}^{\otimes j+1}_{I_{m+1}} \otimes l^{\otimes n-j-1}) \in S_{n,j+1}} \square^{n+1} \xrightarrow{x\bar{\lambda}^n_{m,j}} X_{n,j+1}.$$

For each $x(\mathrm{id}^{\otimes j+1}_{I_{m+1}} \otimes l^{\otimes n-j-1}) \in S_{n,j+1}$, the restriction of the map $x\bar{\lambda}^n_{m,j}$ to the open box $\sqcap^{n+1}_{n+1,1}$ factors through $X_{n,j}$ by Lemma 4.10:

$$\coprod_{x(\mathrm{id}^{\otimes j+1}_{I_{m+1}} \otimes l^{\otimes n-j-1}) \in S_{n,j+1}} \sqcap^{n+1}_{n+1,1} \xrightarrow{x\bar{\lambda}^n_{m,j}|_{\sqcap}} X_{n,j}.$$

As $x(\mathrm{id}^{\otimes j+1}_{I_{m+1}} \otimes l^{\otimes n-j-1})$ is not in $X_{n,j}$, the $(n+1)$-cube $x\bar{\lambda}^n_{m,j}$ is also not in $X_{n,j}$ (as one of its faces is $x(\mathrm{id}^{\otimes j+1}_{I_{m+1}} \otimes l^{\otimes n-j-1})$). The generating cubes of $X_{n,j+1}$ which are not contained in $X_{n,j}$ are exactly those of the form $x\bar{\lambda}^n_{m,j}$ for some $x \in S_{n,j+1}$. Thus, we may write $X_{n,j+1}$ as the following pushout.

$$\begin{tikzcd}
\coprod\limits_{x(\mathrm{id}^{\otimes j+1}_{I_{m+1}} \otimes l^{\otimes n-j-1}) \in S_{n,j+1}} \sqcap^{n+1}_{n+1,1} \arrow[r, "x\bar{\lambda}^n_{m,j}|_{\sqcap}"] \arrow[d, hook] & X_{n,j} \arrow[d, hook] \\
\coprod\limits_{x(\mathrm{id}^{\otimes j+1}_{I_{m+1}} \otimes l^{\otimes n-j-1}) \in S_{n,j+1}} \square^{n+1} \arrow[r, "x\bar{\lambda}^n_{m,j}"'] \arrow[ur, phantom, "\ulcorner", very near end] & X_{n,j+1}
\end{tikzcd}$$

Thus, the map $X_{n,j} \to X_{n,j+1}$ is a pushout along a coproduct of open box inclusions.

We now show the map $X_{n,n} \to X_{n+1,1}$ is a pushout along a coproduct of open box inclusions. Let $S_{n+1,1}$ be the set of $(n+1)$-cubes which factor through the map $\mathrm{id}_{I_{m+1}} \otimes l^{\otimes n} \colon I^{\otimes n+1}_{m+1} \to I_{m+1} \otimes I^{\otimes n}_m$ and are not contained in $X_{n,n}$. Similar to before, the generating cubes of $X_{n+1,1}$ which are not contained in $X_{n,n}$ are exactly those of the form $x\bar{\lambda}^{n+1}_{m,0}$ for some $x \in S_{n+1,1}$. Thus, Lemma 4.10 similarly shows that $X_{n+1,1}$ may be written as the following pushout.

$$\begin{tikzcd}
\coprod\limits_{x(\mathrm{id}_{I_{m+1}} \otimes l^{\otimes n}) \in S_{n+1,1}} \sqcap^{n+1}_{n+1,0} \arrow[r, "x\bar{\lambda}^{n+1}_{m,0}|_{\sqcap}"] \arrow[d, hook] & X_{n,n} \arrow[d, hook] \\
\coprod\limits_{x(\mathrm{id}_{I_{m+1}} \otimes l^{\otimes n}) \in S_{n+1,1}} \square^{n+1} \arrow[r, "x\bar{\lambda}^{n+1}_{m,0}"'] \arrow[ur, phantom, "\ulcorner", very near end] & X_{n+1,1}
\end{tikzcd}$$

Thus, the map $X_{n,n} \to X_{n+1,1}$ is a pushout along a coproduct of open box inclusions. $\square$

From this, we conclude that the inclusion of the 1-nerve of a graph into its nerve is anodyne.

COROLLARY 4.13. *The natural map* $\mathrm{N}_1 G \to \mathrm{N}G$ *is anodyne.*

Proof. By Proposition 3.5, $\mathrm{N}G$ is a transfinite composition of the maps $c^* \colon \mathrm{N}_m G \to \mathrm{N}_{m+2} G$ for $m > 0$. By Theorem 4.12, each c^* is anodyne. Thus, each component of the colimit cone $\mathrm{N}_m \to \mathrm{N}G$ is anodyne. $\square$

This gives us our main theorem.

Proof of Theorem 4.1. The first result is proven in Theorem 4.5. The second result is proven in Corollary 4.13. $\square$

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5. Consequences

Proof of the conjecture of Babson, Barcelo, de Longueville, and Laubenbacher
Using our main result, we obtain a proof of [BBdLL06, Theorem 5.2] which does not rely on the cubical approximation hypothesis [BBdLL06, Proposition 5.1].

Theorem 5.1. *There is a natural group isomorphism $A_n(G,v) \cong \pi_n(|\mathrm{N}_1\mathrm{G}|, v)$.*

Proof. We have

$$\begin{aligned} A_n(G,v) &\cong \pi_n(\mathrm{N}G, v) && \text{by Theorem 4.6} \\ &\cong \pi_n(|\mathrm{N}|\mathrm{G}, v) && \text{by Theorem 2.26} \\ &\cong \pi_n(|\mathrm{N}|_1\mathrm{G}, v) && \text{by Theorem 4.1, Corollary 2.17.} \end{aligned}$$

□

Discrete homology of graphs
In this subsection, we prove the discrete Hurewicz theorem, relating discrete homotopy and homology groups. To do so, we begin with a quick review of cubical homology (cf. [Mas80, BHS11, BGJW21]).

First, we recall the standard definition of homology (with integral coefficients) of a chain complex. A (bounded) *chain complex* (over $\mathbb{Z}$) consists of a collection $\{C_n \mid n \geq 0\}$ of abelian groups and, for $n \geq 1$, a group homomorphism $\partial_n \colon C_n \to C_{n-1}$ such that $\partial_{n-1}\partial_n = 0$. A map $f \colon C \to D$ of chain complexes consists of maps $f_n \colon C_n \to D_n$ for $n \geq 0$ which make the respective squares commute. We write Ch for the category of chain complexes. Define a functor $H_* \colon \mathsf{Ch} \to \mathsf{Ab}^{\mathbb{N}}$ by taking a chain complex C to a sequence of homology groups given by

$$H_nC := \begin{cases} C_0/\operatorname{im}\partial_0 & n = 0 \\ \ker\partial_n/\operatorname{im}\partial_{n+1} & n > 0. \end{cases}$$

We refer to H_nC as the *nth homology* of C with integer coefficients. One verifies that a map of chain complexes $C \to D$ induces maps $H_nC \to H_nD$ between their nth homologies.

Next, we explain how to construct a chain complex out of a cubical set via a construction analogous to the normalized complex in simplicial singular homology. We construct a functor $N \colon \mathsf{cSet}_* \to \mathsf{Ch}$ by

$$(NX)_n := F_*X_n/DX_n$$

where F_*X_n is the free abelian group on the pointed set X_n and DX_n is the subgroup of F_*X_n generated by degenerate cubes, i.e. those that lie in the image of a degeneracy or a connection $X_{n-1} \to X_n$. The chain differentials $\partial \colon (NX)_n \to (NX)_{n-1}$ are given by

$$\partial(x) := \sum_{\substack{i=1,\ldots,n \\ \varepsilon=0,1}} (-1)^{i+\varepsilon} x\partial_{i,\varepsilon}.$$

Definition 5.2. The *reduced cubical homology with integer coefficients* functor is the functor $\widetilde{H}_* \colon \mathsf{cSet}_* \to \mathsf{Ab}^{\mathbb{N}}$ given by the composite

$$\mathsf{cSet}_* \xrightarrow{N} \mathsf{Ch} \xrightarrow{H_*} \mathsf{Ab}^{\mathbb{N}}.$$

We contrast the definition of *reduced* homology with that of *unreduced homology*, which is defined on the category of non-pointed cubical sets by using non-pointed free abelian groups.

Unlike in the construction of simplicial homology, we are forced to take the quotient of $F_*X_\bullet$ by the subcomplex of cubes in the image of a degeneracy map. We do, however, have flexibility

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regarding whether or not to quotient by the subcomplex of cubes in the image of a connection map, as the two complexes are quasi-isomorphic [BGJW21, Corollary 3.10].

Definition 5.3 (cf. [BCW14, § 2]). The *reduced discrete homology* functor $\widetilde{DH}_*\colon \mathsf{Graph}_* \to \mathsf{Ab}^{\mathbb{N}}$ is the composite of functors

$$\mathsf{Graph}_* \xrightarrow{\mathrm{N}_1} \mathsf{cSet}_* \xrightarrow{\widetilde{H}_*} \mathsf{Ab}^{\mathbb{N}}.$$

By the homotopy invariance of cubical homology [CKT23, Theorem 3.11], we have the following fact.

Proposition 5.4. *Let $f\colon (X,x) \to (Y,y)$ be a pointed cubical map. If f is a weak equivalence then $H_*f\colon H_*(X,x) \to H_*(Y,y)$ is an isomorphism of graded abelian groups.*

From this, we deduce that the discrete homology of a graph is the same as the cubical homology of its nerve.

Corollary 5.5. *For a pointed graph (G,v), the natural map $\mathrm{N}_1 G \to \mathrm{N}G$ induces an isomorphism*

$$\widetilde{H}_*(\mathrm{N}_1 G, v) \cong \widetilde{H}_*(\mathrm{N}G, v).$$

of graded abelian groups.

Proof. This follows from Proposition 5.4 and Theorem 4.1. □

For any pointed Kan complex (X,x), using the unit of the adjunction

$$F_* : \mathsf{cSet}_* \rightleftarrows \mathsf{Ab}^{\square^{\mathrm{op}}} : U_*$$

between pointed cubical sets and cubical abelian groups, we may construct a natural map $\pi_n(X,x) \to \widetilde{H}_*(X,x)$, since $\pi_n(U_*F_*(X,x)) \cong \widetilde{H}_n(X,x)$ by [CKT23, Theorem 4.11]. This is the *Hurewicz homomorphism*; cf. [CKT23, Definition 4.14].

We then have the classical theorem of Hurewicz, phrased in the language of cubical sets.

Theorem 5.6 [CKT23, Theorem 4.16]. *Let $n \geq 2$ and (X,x) be a pointed connected Kan complex. Suppose $\pi_i(X,x) = 0$ for all $i \in \{1,\ldots,n-1\}$, i.e. X is n-connected. Then the Hurewicz homomorphism $\pi_n(X,x) \to \widetilde{H}_n(X,x)$ is an isomorphism.*

Definition 5.7. For any pointed connected graph (G,v) and $n \geq 2$, we therefore obtain the *discrete Hurewicz homomorphism* $A_n(G,v) \to \widetilde{DH}_n(G,v)$ as the composite

$$\begin{array}{lll} A_n(G,v) & \cong \pi_n(\mathrm{N}G, v) & \text{by Theorem 4.6} \\ & \to \widetilde{H}_n(\mathrm{N}G) & \text{the Hurewicz homomorphism} \\ & \cong \widetilde{H}_n(\mathrm{N}_1 G) & \text{by Corollary 5.5} \\ & = \widetilde{DH}_n(G,v) & \text{by Definition 5.3.} \end{array}$$

One may verify that this map recovers the homomorphism defined in [Lut21, § 5.2]. We then have the expected discrete analogue of the Hurewicz theorem.

Theorem 5.8 (Discrete Hurewicz theorem). *Let $n \geq 2$ and (G,v) be a pointed connected graph. Suppose $A_i(G,v) = 0$ for all $i \in \{1,\ldots,n-1\}$. Then the induced Hurewicz map $A_n(G,v) \to \widetilde{DH}_n(G,v)$ is an isomorphism.*

Proof. This is an immediate consequence of the definition of the discrete Hurewicz homomorphism and Theorem 5.6. □

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Fibration category of graphs

Via Theorem 4.1, we may view the nerve functor as a functor $\mathrm{N}\colon \mathsf{Graph} \to \mathsf{Kan}$ taking values in Kan complexes. From this, we induce a fibration category structure on the category of graphs.

Theorem 5.9. *The category* Graph *of graphs and graph maps carries a fibration category structure where:*

- *the weak equivalences are the weak homotopy equivalences, i.e. maps $f\colon G \to H$ such that, for all $v \in G$ and $n \geq 0$, the map $A_n f\colon A_n(G, v) \to A_n(H, f(v))$ is an isomorphism;*
- *the fibrations are maps $f\colon G \to H$ which are sent to fibrations $\mathrm{N}f\colon \mathrm{N}G \to \mathrm{N}H$ under the nerve functor $\mathrm{N}\colon \mathsf{Graph} \to \mathsf{cSet}$.*

Before proving this, we consider factorization of the diagonal map separately. Recall that, for any graph G, we have a commutative triangle

$$\begin{tikzcd} & PG \arrow[dr, "(\partial^*_{1,0}{,}\partial^*_{1,1})"] & \\ G \arrow[ur, hook] \arrow[rr, "(\mathrm{id}_G{,}\mathrm{id}_G)"] & & G \times G \end{tikzcd}$$

where $G \hookrightarrow PG$ sends a vertex v to the constant path on v.

Lemma 5.10. *For any graph G, applying $\mathrm{N}\colon \mathsf{Graph} \to \mathsf{cSet}$ to the diagram*

$$\begin{tikzcd} & PG \arrow[dr, "(\partial^*_{1,0}{,}\partial^*_{1,1})"] & \\ G \arrow[ur, hook] \arrow[rr, "(\mathrm{id}_G{,}\mathrm{id}_G)"] & & G \times G \end{tikzcd}$$

gives a factorization of the diagonal map $\mathrm{N}G \to \mathrm{N}G \times \mathrm{N}G$ as a weak equivalence followed by a fibration.

Proof. Applying naturality of the isomorphism in Proposition 3.10 to the maps $\square^1 \to \square^0$ and $\partial\square^1 \to \square^1$ gives that the diagram

$$\begin{tikzcd} \mathrm{N}G \arrow[r] \arrow[d, "\cong"'] & \mathrm{N}(PG) \arrow[r] \arrow[d, "\cong"] & \mathrm{N}(G \times G) \arrow[d, "\cong"] \\ \mathrm{N}G \arrow[r] & \hom_R(\square^1, \mathrm{N}G) \arrow[r] & \mathrm{N}G \times \mathrm{N}G \end{tikzcd}$$

commutes. Our earlier paper [CK23, Proposition 2.22] shows that the bottom composite provides a factorization of the diagonal map into a weak equivalence followed by a fibration. Thus, the top left map is a weak equivalence and the top right map is a fibration. □

Proof of Theorem 5.9. Both the two-out-of-six property for weak equivalences and closure of acyclic fibrations under isomorphisms follow from Theorem 2.10. Pullbacks exist as Graph is complete; they preserve (acyclic) fibrations by Proposition 3.8. Theorem 4.1 shows that the nerve of every graph is a Kan complex. Factorization of the diagonal map is shown in Lemma 5.10, and this gives the factorization of an arbitrary map via [Bro73, Factorization lemma]. □

The following proposition gives a useful criterion for verifying whether a map is a fibration.

PROPOSITION 5.11. *A graph map $f\colon G \to H$ is a fibration if and only if, for any commutative square*

$$\begin{tikzcd} {|\sqcap^n_{i,\varepsilon}|_m} \arrow[r] \arrow[d] & G \arrow[d, "f"] \\ I_m^{\otimes n} \arrow[r] & H \end{tikzcd}$$

there exist $k \geq 0$ and a lift $I_{m+2k}^{\otimes n} \to G$ of the following square.

$$\begin{tikzcd} {|\sqcap^n_{i,\varepsilon}|_{m+2k}} \arrow[r, "{c_*}^k"] \arrow[d] & {|\sqcap^n_{i,\varepsilon}|_m} \arrow[r] \arrow[d] & G \arrow[d, "f"] \\ I_{m+2k}^{\otimes n} \arrow[r, "{c_*}^k"'] \arrow[urr, dotted] & I_m^{\otimes n} \arrow[r] & H \end{tikzcd}$$

Proof. The map f is a fibration if and only if any commutative square

$$\begin{tikzcd} \sqcap^n_{i,\varepsilon} \arrow[r] \arrow[d] & \mathrm{N}G \arrow[d, "\mathrm{N}f"] \\ \square^n \arrow[r] & \mathrm{N}H \end{tikzcd}$$

admits a lift. As N is a sequential colimit, every such square and lift (if it exists) factors as

$$\begin{tikzcd} \sqcap^n_{i,\varepsilon} \arrow[r] \arrow[d] & \mathrm{N}_m G \arrow[r, "(c^*)^k"] \arrow[d, "\mathrm{N}_m f"] & \mathrm{N}_{m+2k} G \arrow[r] \arrow[d, "\mathrm{N}_{m+2k} f"] & \mathrm{N}G \arrow[d, "\mathrm{N}f"] \\ \square^n \arrow[r] \arrow[urr, dotted] & \mathrm{N}_m H \arrow[r, "(c^*)^k"] & \mathrm{N}_{m+2k} H \arrow[r] & \mathrm{N}H \end{tikzcd}$$

for some $m \geq 1$ and $k \geq 0$. The left two squares in this diagram correspond to a diagram

$$\begin{tikzcd} {|\sqcap^n_{i,\varepsilon}|_{m+2k}} \arrow[r, "(c_*)^k"] \arrow[d] & {|\sqcap^n_{i,\varepsilon}|_m} \arrow[r] \arrow[d] & G \arrow[d, "f"] \\ I_{m+2k}^{\otimes n} \arrow[r, "(c_*)^k"'] \arrow[urr, dotted] & I_m^{\otimes n} \arrow[r] & H \end{tikzcd}$$

in Graph by the realization-nerve adjunction. □

By definition, the nerve functor $\mathrm{N}\colon \mathsf{Graph} \to \mathsf{cSet}$ reflects (and preserves) fibrations. We show that it reflects weak equivalences as well.

PROPOSITION 5.12. *The nerve functor $\mathrm{N}\colon \mathsf{Graph} \to \mathsf{cSet}$ reflects weak equivalences. That is, given a map $f\colon G \to H$, if $\mathrm{N}f\colon \mathrm{N}G \to \mathrm{N}H$ is a weak equivalence then f is a weak equivalence.*

Proof. This follows from Theorem 4.6 and [CK23, Theorem 4.7]. □

As the nerve functor preserves finite limits, it is straightforward to show it is exact.

PROPOSITION 5.13. *The nerve functor $\mathrm{N}\colon \mathsf{Graph} \to \mathsf{cSet}$ is exact.*

Proof. The nerve functor preserves fibrations and acyclic fibrations by definition and by Proposition 5.12, respectively. Proposition 3.8 shows that it preserves all finite limits. □

A consequence of Proposition 5.13 is that *fibration sequences* induce a long exact sequence of homotopy groups.

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Theorem 5.14. *Let $f\colon (G,v) \to (H,w)$ be a fibration between pointed graphs and (F,v) be the fiber of f over w, i.e. the pullback*

$$\begin{tikzcd}
(F,v) \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & (G,v) \arrow[d, "f"] \\
(I_0,0) \arrow[r, "w"] & (H,w)
\end{tikzcd}$$

in Graph_*. Then, there is a long exact sequence

$$\begin{tikzcd}
\ldots \arrow[r] & A_n(F,v) \arrow[r, "A_n i"] & A_n(G,v) \arrow[r, "A_n f"] & A_n(H,w) \\
& A_{n-1}(F,v) \arrow[r, "A_{n-1} i"] & A_{n-1}(G,v) \arrow[r, "A_{n-1} f"] & A_{n-1}(H,w) \\
& & \cdots & \\
& A_1(F,v) \arrow[r, "A_1 i"] & A_1(G,v) \arrow[r, "A_1 f"] & A_1(H,w) \\
& A_0(F,v) \arrow[r, "A_0 i"] & A_0(G,v) \arrow[r, "A_0 f"] & A_0(H,w).
\end{tikzcd}$$

Proof. By Proposition 5.13, the map $\mathrm{N}f\colon (\mathrm{N}G,v) \to (\mathrm{N}H,w)$ is a fibration and its fiber is naturally isomorphic to $(\mathrm{N}F,v)$. The result then follows by applying [CK23, Corollary 4.6] and Theorem 4.6. □

There are two large classes of examples of fibrations: fiber bundles and m-fibrations.

Definition 5.15 (Babson). For $m \geq 1$, a graph map $f\colon G \to H$ is an *m-fiber bundle* with fiber F if, for any $u\colon I_1^{\otimes n} \to H$, the left map in the pullback diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & G \arrow[d, "f"] \\
I_1^{\otimes n} \arrow[r, "u"] & H
\end{tikzcd}$$

is a product projection $I_1^{\otimes n} \times F \to I_1^{\otimes n}$ from the categorical product of $I_1^{\otimes n}$ and F.

Definition 5.16. A fiber bundle $G \to H$ is *trivial* if it is a product projection $H \times F \to H$.

Proposition 5.17. *Any pullback of an m-fiber bundle is an m-fiber bundle.*

Proof. This follows from the two-pullback lemma. □

Proposition 5.18. *For $m \geq 1$, every $(m+1)$-fiber bundle is an m-fiber bundle.*

Proof. Fix an $(m+1)$-fiber bundle $f\colon G \to H$ and a map $u\colon I_m^{\otimes n} \to H$. The map $l^{\otimes n}\colon I_{m+1}^{\otimes n} \to I_m^{\otimes n}$ admits a section $i\colon I_m^{\otimes n} \to I_{m+1}^{\otimes n}$. By assumption, the left map in the pullback diagram

$$\begin{tikzcd}
P \arrow[rr] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & & G \arrow[d, "f"] \\
I_{m+1}^{\otimes n} \arrow[r, "l^{\otimes n}"] & I_m^{\otimes n} \arrow[r, "u"] & H
\end{tikzcd}$$

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is a product projection $I_{m+1}^{\otimes n} \times F \to I_{m+1}^{\otimes n}$. As the right map in the pullback square

$$\begin{tikzcd}
I_m^{\otimes n} \times F \arrow[r, "i \times \mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & I_{m+1}^{\otimes n} \times F \arrow[d] \\
I_m^{\otimes n} \arrow[r, "i"] & I_{m+1}^{\otimes n}
\end{tikzcd}$$

is a product projection, the left map is as well. Thus, the outer square in

$$\begin{tikzcd}
I_m^{\otimes n} \times F \arrow[r, "i \times \mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & I_{m+1}^{\otimes n} \times F \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & G \arrow[d, "f"] \\
I_m^{\otimes n} \arrow[r, "i"] & I_{m+1}^{\otimes n} \arrow[r, "u \circ l^{\otimes n}"] & H
\end{tikzcd}$$

is a pullback whose left map is a product projection. □

Recall that a graph G is a *retract* of a graph H if there is an inclusion $i\colon G \hookrightarrow H$ with a retraction $r\colon H \to G$, i.e. $ri = \mathrm{id}$.

Lemma 5.19. *If H is a retract of $I_1^{\otimes n}$ for some $n \geq 0$ then any 1-fiber bundle $G \to H$ is trivial.*

Proof. As H is a retract of $I_1^{\otimes n}$, we fix an inclusion $i\colon H \hookrightarrow I_1^{\otimes n}$ and retraction $r\colon I_1^{\otimes n} \to H$. Given a fiber bundle $f\colon G \to H$, the left map in the pullback diagram

$$\begin{tikzcd}
F \times I_1^{\otimes n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & G \arrow[d, "f"] \\
I_1^{\otimes n} \arrow[r, "r"] & H
\end{tikzcd}$$

is a product projection. By universal property of the pullback, the outer square in

$$\begin{tikzcd}
G \arrow[drr, bend left, "\mathrm{id}"] \arrow[dr, dotted] \arrow[ddr, bend right, "if"'] & & \\
& F \times I_1^{\otimes n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & H \arrow[d, "f"] \\
& I_1^{\otimes n} \arrow[r, "r"] & H
\end{tikzcd}$$

induces a map $G \to F \times I_1^{\otimes n}$. By the two-pullback lemma, the left square in

$$\begin{tikzcd}
G \arrow[r, dotted] \arrow[d, "f"'] & F \times I_1^{\otimes n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & G \arrow[d, "f"] \\
H \arrow[r, hook, "i"] & I_1^{\otimes n} \arrow[r, "r"] & H
\end{tikzcd}$$

is a pullback. Therefore, the outer square in

$$\begin{tikzcd}
G \arrow[r, dotted] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near start] & F \times I_1^{\otimes n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & F \arrow[d] \\
H \arrow[r, hook, "i"] & I_1^{\otimes n} \arrow[r] & I_0
\end{tikzcd}$$

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is a pullback, which proves that G is a product of H and F, and f is the projection onto the first component. $\square$

Theorem 5.20. *For any $m \geq 1$, a 1-fiber bundle is an m-fiber bundle.*

Proof. Fix a 1-fiber bundle $f\colon G \to H$ and a map $u\colon I_m^{\otimes n} \to H$. By Proposition 5.17, the left map in the pullback diagram

$$\begin{tikzcd} u^*G \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & G \arrow[d, "f"] \\ I_m^{\otimes n} \arrow[r, "u"] & H \end{tikzcd}$$

is a 1-fiber bundle. By Lemma 5.19, it suffices to show $I_m^{\otimes n}$ is a retract of $I_1^{\otimes k}$ for some $k \geq 0$.

Define an embedding $i\colon I_m \hookrightarrow I_1^{\otimes m}$ by sending a vertex $k \in I_m$ to the tuple $(x_1, \ldots, x_m)$ where $x_i = 1$ if $i \leq k$ and $x_i = 0$ if $i > k$. This map has a retraction $r\colon I_1^{\otimes m} \to I_m$ which sends a tuple $(x_1, \ldots, x_n)$ to its sum $\sum_{i=1}^{n} x_i$. Thus, the map $i^{\otimes n}\colon I_m^{\otimes n} \to I_1^{\otimes mn}$ has a retraction $r^{\otimes n}\colon I_1^{\otimes mn} \to I_m^{\otimes n}$. $\square$

In particular, if $f\colon X \to Y$ is an m-fiber bundle for some $m \geq 1$ then it is a fiber bundle for all $m \geq 1$. In the remainder of this section, we refer to such an f as simply a *fiber bundle*.

Theorem 5.21. *Every fiber bundle is a fibration.*

Proof. Fix a fiber bundle $f\colon G \to H$. We apply Proposition 5.11 and consider a lifting problem

$$\begin{tikzcd} {|\sqcap^n_{i,\varepsilon}|_m} \arrow[r] \arrow[d, hook] & G \arrow[d, "f"] \\ I_m^{\otimes n} \arrow[r] & H \end{tikzcd}$$

Taking a pullback of the right and bottom map gives a factorization of this square as

$$\begin{tikzcd} {|\sqcap^n_{i,\varepsilon}|_m} \arrow[r, dotted] \arrow[d, hook] & I_1^{\otimes n} \times F \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & G \arrow[d, "f"] \\ I_m^{\otimes n} \arrow[r, "\mathrm{id}"] & I_m^{\otimes n} \arrow[r] & H \end{tikzcd}$$

The middle map is a fibration since it is a product projection. Hence, there exists $k \geq 0$ such that the outer square in

$$\begin{tikzcd} {|\sqcap^n_{i,\varepsilon}|_{m+2k}} \arrow[r, "c_*^k"] \arrow[d, hook] & {|\sqcap^n_{i,\varepsilon}|_m} \arrow[r, dotted] \arrow[d, hook] & I_1^{\otimes n} \times F \arrow[d] \\ I_{m+2k}^{\otimes n} \arrow[r, "c_*^k"'] \arrow[urr, dotted] & I_m^{\otimes n} \arrow[r, "\mathrm{id}"] & I_m^{\otimes n} \end{tikzcd}$$

admits a lift. Post-composing this lift with the map $I_1^{\otimes n} \times F \to G$ gives a lift of the outer square in

$$\begin{tikzcd} {|\sqcap^n_{i,\varepsilon}|_{m+2k}} \arrow[r, "c_*^k"] \arrow[d, hook] & {|\sqcap^n_{i,\varepsilon}|_m} \arrow[r] \arrow[d, hook] & G \arrow[d, "f"] \\ I_{m+2k}^{\otimes n} \arrow[r, "c_*^k"'] \arrow[urr, dotted] & I_m^{\otimes n} \arrow[r] & H \end{tikzcd}$$

$\square$

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Corollary 5.22. *A pointed fiber bundle* $f\colon (G,v) \to (H,w)$ *with fiber* (F,v) *induces a long exact sequence*

$$\begin{tikzcd} \ldots \arrow[r] & A_n(F,v) \arrow[r, "A_n i"] & A_n(G,v) \arrow[r, "A_n f"] & A_n(H,w) \arrow[dll, rounded corners] \\ & A_{n-1}(F,v) \arrow[r, "A_{n-1} i"] & A_{n-1}(G,v) \arrow[r, "A_{n-1} f"] & A_{n-1}(H,w) \arrow[dl, rounded corners] \\ & & \cdots \arrow[dl, rounded corners] & \\ & A_1(F,v) \arrow[r, "A_1 i"] & A_1(G,v) \arrow[r, "A_1 f"] & A_1(H,w) \arrow[dll, rounded corners] \\ & A_0(F,v) \arrow[r, "A_0 i"] & A_0(G,v) \arrow[r, "A_0 f"] & A_0(H,w). \end{tikzcd}$$

of A-homotopy groups.

Proof. This follows from Theorems 5.21 and 5.14. $\square$

The second class of maps which are fibrations is the class of m-fibrations.

Definition 5.23. For $m \geq 1$, a graph map $f\colon G \to H$ is an *m-fibration* if $\mathrm{N}_m f\colon \mathrm{N}_m G \to \mathrm{N}_m H$ is a Kan fibration.

Proposition 5.24. *Let* $m, k \geq 1$.

(i) *The inclusion* $\{0, km\} \to I_{km}$ *is in the saturation of* $\{\varnothing \to I_0, \{0,m\} \to I_m\}$.
(ii) *For* $\varepsilon = 0,1$, *the end-point inclusion* $\{\varepsilon km\} \to I_{km}$ *is in the saturation of* $\{\{\varepsilon m\} \to I_m\}$.
(iii) *The inclusion* $\partial I_{km}^{\otimes n} \to I_{km}^{\otimes n}$ *is in the saturation of* $\{\partial I_m^{\otimes j} \to I_m^{\otimes j} \mid j \leq n\}$.
(iv) *The inclusion* $|\sqcap^n_{i,\varepsilon}|_{km} \to I_{km}^{\otimes n}$ *is in the saturation of* $\{|\sqcap^j_{i,\varepsilon}|_m \to I_{km}^{\otimes j} \mid j \leq n\}$.

Proof. (i) By induction, if $\partial I_{km} \to I_m$ lies in the saturation then the bottom map in the pushout

$$\begin{tikzcd} \{0, km\} \sqcup \{0, m\} \arrow[r] \arrow[d, "{[f,g]}"'] \arrow[dr, phantom, "\ulcorner", very near end] & I_{km} \sqcup I_m \arrow[d] \\ \{0, km, (k+1)m\} \arrow[r] & I_{(k+1)m} \end{tikzcd}$$

is in the saturation as a pushout of the coproduct of maps in the saturation, where $[f, g]$ is defined by

$$\begin{aligned} f(0) &= 0, & f(km) &= km, \\ g(0) &= km, & g(m) &= (k+1)m. \end{aligned}$$

The inclusion $\{0, (k+1)m\} \to \{0, km, (k+1)m\}$ is a pushout along $\varnothing \to I_0$, thus the composite $\{0, (k+1)m\} \to I_{(k+1)m}$ lies in the saturation.

(ii) By induction, if $\{\varepsilon m\} \to I_{km}$ lies in the saturation then the bottom map in the pushout

$$\begin{tikzcd} I_0 \arrow[r, "\varepsilon km"] \arrow[d, "(1-\varepsilon)m"'] \arrow[dr, phantom, "\ulcorner", very near end] & I_{km} \arrow[d] \\ I_m \arrow[r] & I_{(k+1)m} \end{tikzcd}$$

lies in the saturation. Pre-composing with the end-point inclusion $\{\varepsilon m\} \to I_m$ gives the endpoint inclusion $\{\varepsilon(k+1)m\} \to I_{(k+1)m}$.

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(iii) The inclusion $\partial I_{km}^{\otimes n} \to I_{km}^{\otimes n}$ may be written as a pushout product

$$(\partial I_{km}^{\otimes n} \to I_{km}^{\otimes n}) = (\{0, km\} \to I_{km})^{\hat{\otimes} n}.$$

Noting the equality

$$\{\varnothing \to I_0, \{0, m\} \to I_m\}^{\hat{\otimes} n} = \{\partial I_m^{\otimes j} \to I_m^{\otimes j} \mid j \leq n\},$$

this follows from (i) by [HSS00, Proposition 5.3.4].

(iv) The inclusion $|\sqcap_{i,\varepsilon}^n|_{km} \to I_{km}^{\otimes n}$ may be written as a pushout product

$$(|\sqcap_{i,\varepsilon}^n|_{km} \to I_m^{\otimes n}) = (\{0, km\} \to I_{km})^{\hat{\otimes}(i-1)} \hat{\otimes} (\{\varepsilon km\} \to I_{km}) \hat{\otimes} (\{0, km\} \to I_{km})^{\hat{\otimes} n-i}.$$

Noting the equality

$$\begin{aligned} &\{\varnothing \to I_0, \{0, m\} \to I_m\}^{\hat{\otimes} i-1} \hat{\otimes} \{\{\varepsilon m\} \to I_m\} \hat{\otimes} \{\varnothing \to I_0, \{0, m\} \to I_m\}^{\hat{\otimes} n-i} \\ &\quad = \{|\sqcap_{i,\varepsilon}^j|_m \to I_m^{\otimes j} \mid j \leq n\}, \end{aligned}$$

this follows from (i) and (ii) by [HSS00, Proposition 5.3.4]. □

Theorem 5.25. *Let $k, m \geq 1$. Every m-fibration is both a km-fibration and a fibration.*

Proof. Let $f \colon G \to H$ be an m-fibration. We have that f is a km-fibration by Proposition 5.24. To see f is a fibration, we apply Proposition 5.11. For a commutative square

$$\begin{tikzcd} {|\sqcap_{i,\varepsilon}^n|_t} \arrow[r] \arrow[d] & G \arrow[d] \\ I_t^{\otimes n} \arrow[r] & H \end{tikzcd}$$

let $k \geq 0$ be the smallest non-negative integer such that $t + k$ is a multiple of m. As f is a $(t + k)$-fibration, the required lift exists. □

Proposition 5.13 also shows that the nerve functor preserves loop spaces.

Proposition 5.26. *The square*

$$\begin{tikzcd} \mathsf{Graph}_* \arrow[r, "\mathrm{N}"] \arrow[d, "\Omega"'] & \mathsf{Kan}_* \arrow[d, "\Omega"] \\ \mathsf{Graph}_* \arrow[r, "\mathrm{N}"] & \mathsf{Kan}_* \end{tikzcd}$$

commutes up to natural isomorphism.

Proof. Fix a pointed graph (G, v). By Proposition 1.22, the square

$$\begin{tikzcd} \Omega(G, v) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & PG \arrow[d, "{(\partial_{1,0}^*, \partial_{1,1}^*)}"] \\ I_0 \arrow[r, "{(v,v)}"] & G \times G \end{tikzcd}$$

is a pullback. Proposition 3.8 shows that the nerve functor preserves this pullback, thus the square

$$\begin{tikzcd} \mathrm{N}(\Omega(G, v)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{N}(PG) \arrow[d, "{(\partial_{1,0}^*, \partial_{1,1}^*)}"] \\ \mathrm{N}I_0 \arrow[r, "{(v,v)}"] & \mathrm{N}(G \times G) \end{tikzcd}$$

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is a pullback. Proposition 3.8 also shows the nerve preserves products and the terminal object. Lemma 5.10 implies that $\mathrm{N}(PG) \cong \hom_R(\Box^1, \mathrm{N}G)$, hence the square

$$\begin{tikzcd} \mathrm{N}(\Omega(G,v)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \hom_R(\Box^1, \mathrm{N}G) \arrow[d, "(\partial^*_{1,0}, \partial^*_{1,1})"] \\ \Box^0 \arrow[r, "(v,v)"] & \mathrm{N}G \times \mathrm{N}G \end{tikzcd}$$

is a pullback. That is, $\mathrm{N}(\Omega(G,v)) \cong \Omega(\mathrm{N}G, v)$. □

By Proposition 2.9, the category Graph_* of pointed graphs has a fibration category structure as well. Thus, the loop graph functor $\Omega \colon \mathsf{Graph}_* \to \mathsf{Graph}_*$ is a functor between fibration categories.

THEOREM 5.27. *The loop graph functor* $\Omega \colon \mathsf{Graph}_* \to \mathsf{Graph}_*$ *is exact.*

Proof. By Proposition 5.13, Theorem 2.27, the composite $\Omega(\mathrm{N}-,-) \colon \mathsf{Graph}_* \to \mathsf{cSet}_*$ is exact. Applying Proposition 5.26, this gives that the composite $\mathrm{N}(\Omega(-,-)) \colon \mathsf{Graph}_* \to \mathsf{cSet}_*$ is exact. The nerve functor reflects fibrations and acyclic fibrations as, by definition, it creates them. It reflects finite limits by Corollary 3.13. From this, it follows that $\Omega \colon \mathsf{Graph}_* \to \mathsf{Graph}_*$ preserves fibrations, acyclic fibrations, and finite limits. □

Cubical enrichment of the category of graphs

Recall (e.g. from [Rie14, Definition 3.4.1]) that a functor $F \colon \mathcal{C} \to \mathcal{D}$ between monoidal categories $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}})$ is *lax monoidal* if there exist natural transformations

$$Fc \otimes_{\mathcal{D}} Fc' \to F(c \otimes c') \quad \text{and} \quad I_{\mathcal{D}} \to FI_{\mathcal{C}}$$

subject to the associativity and unitality conditions, which we omit here.

LEMMA 5.28.

(i) *For* $m \geq 1$, *the functor* $\mathrm{N}_m \colon \mathsf{Graph} \to \mathsf{cSet}$ *is lax monoidal.*
(ii) *The functor* $\mathrm{N} \colon \mathsf{Graph} \to \mathsf{cSet}$ *is lax monoidal.*

Proof. By Proposition 3.6, the functors $|-|_m \colon \mathsf{cSet} \to \mathsf{Graph}$ are strong monoidal, and hence, in particular, oplax monoidal. Thus, their right adjoints $\mathrm{N}_m \colon \mathsf{Graph} \to \mathsf{cSet}$ are lax monoidal, proving (i).

Clearly, $\mathrm{N}I_0 \cong \Box^0$. As the geometric product preserves colimits in each variable and colimits commute with colimits, the cubical set $\mathrm{N}G \otimes \mathrm{N}H$ (for graphs G and H) is the colimit of the following diagram.

$$\mathrm{N}G \otimes \mathrm{N}H = \operatorname{colim} \left(\begin{tikzcd} \mathrm{N}_1G \otimes \mathrm{N}_1H \arrow[r, hook] \arrow[d, hook] & \mathrm{N}_2G \otimes \mathrm{N}_1H \arrow[r, hook] \arrow[d, hook] & \mathrm{N}_3G \otimes \mathrm{N}_1H \arrow[r, hook] \arrow[d, hook] & \cdots \\ \mathrm{N}_1G \otimes \mathrm{N}_2H \arrow[r, hook] \arrow[d, hook] & \mathrm{N}_2G \otimes \mathrm{N}_2H \arrow[r, hook] \arrow[d, hook] & \mathrm{N}_3G \otimes \mathrm{N}_2H \arrow[r, hook] \arrow[d, hook] & \cdots \\ \mathrm{N}_1G \otimes \mathrm{N}_3H \arrow[r, hook] \arrow[d, hook] & \mathrm{N}_2G \otimes \mathrm{N}_3H \arrow[r, hook] \arrow[d, hook] & \mathrm{N}_3G \otimes \mathrm{N}_3H \arrow[r, hook] \arrow[d, hook] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{tikzcd} \right)$$

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Computing this colimit componentwise in Set, one verifies that the colimit of the diagonal

$$\operatorname{colim}(\mathrm{N}_1 G \otimes \mathrm{N}_1 H \hookrightarrow \mathrm{N}_2 G \otimes \mathrm{N}_2 H \hookrightarrow \mathrm{N}_3 G \otimes \mathrm{N}_3 H \hookrightarrow \cdots)$$

computes the same cubical set. As $\mathrm{N}(G \otimes H)$ is the colimit

$$\mathrm{N}(G \otimes H) = \operatorname{colim}(\mathrm{N}_1(G \otimes H) \hookrightarrow \mathrm{N}_2(G \otimes H) \hookrightarrow \mathrm{N}_3(G \otimes H) \hookrightarrow \cdots),$$

the lax monoidal maps $\mathrm{N}_m G \otimes \mathrm{N}_m H \to \mathrm{N}_m(G \otimes H)$ induce a map on colimits $\mathrm{N}G \otimes \mathrm{N}H \to \mathrm{N}(G \otimes H)$ which satisfies the required associativity and unitality conditions. $\square$

Remark 5.29. An alternative proof of (ii) can be given using [DKLS24, Proposition 1.24], which gives an explicit description of the geometric product of cubical sets. Explicitly, the n-cubes of $\mathrm{N}G \otimes \mathrm{N}H$ are in bijective correspondence with pairs of cubes

$$(\square^k \to \mathrm{N}G,\ \square^l \to \mathrm{N}H),$$

where $k + l = n$. By definition of N, each such pair corresponds in turn to pair of maps

$$(I_\infty^{\otimes k} \to G,\ I_\infty^{\otimes l} \to H)$$

that stabilize in all directions. Taking products of these maps, we obtain a map $I_\infty^{\otimes n} \to G \otimes H$ that stabilizes in all directions.

Corollary 5.30. *The nerve functor* $\mathrm{N} \colon \mathsf{Graph} \to \mathsf{cSet}$ *preserves homotopy equivalences.*

We describe the notion of enriched categories informally, with a reference to [Rie14, Definition 3.3.1] in lieu of a fully formal statement.

Definition 5.31. For a monoidal category $(\mathcal{V}, \otimes, 1)$, a $(\mathcal{V}, \otimes, I)$-*enriched category* $\mathcal{C}$ consists of

- a class of objects $\operatorname{ob}\mathcal{C}$;
- for $X, Y \in \operatorname{ob}\mathcal{C}$, a *morphism* object $\mathcal{C}(X, Y) \in \mathcal{V}$;
- for $X, Y, Z \in \operatorname{ob}\mathcal{C}$, a *composition* morphism $\circ \colon \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ in $\mathcal{V}$;
- for $X \in \operatorname{ob}\mathcal{C}$, an *identity* morphism $1 \to \mathcal{C}(X, X)$ in $\mathcal{V}$,

subject to appropriate associativity and unitality axioms (cf. [Rie14, Definition 3.3.1]).

Example 5.32. Any locally small category is a $(\mathsf{Set}, \times, \{*\})$-enriched category, where the objects, morphism sets, composition function, and identity morphisms are as usually defined.

Example 5.33. Any closed monoidal category is enriched over itself. In particular, $\mathsf{Graph}_\mathsf{G}$ is a $(\mathsf{Graph}, \otimes, I_0)$-enriched category where:

- $\operatorname{ob}\mathsf{Graph}_\mathsf{G}$ is the collection of all graphs;
- for graphs G, H, the morphism graph is $\hom^\otimes(G, H)$;
- for graphs G, H, J, the composition morphism is the graph map given by composition of graph maps regarded as vertices,

$$\hom^\otimes(H, J) \otimes \hom^\otimes(G, H) \to \hom^\otimes(G, J);$$

- the identity morphism is the identity map on G as a vertex $\mathrm{id}_G \colon I_0 \to \hom^\otimes(G, G)$.

Example 5.34. Since enrichment can be transferred along lax monoidal functors [Rie14, Lemma 3.4.3], Lemma 5.28 implies there is a $(\mathsf{cSet}, \otimes, \square^0)$-enriched category $\mathsf{Graph}_\square$ of graphs where $\mathrm{N}(\hom^\otimes(G, H))$ is the morphism cubical set. Composition and identity morphisms are defined analogously.

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Definition 5.35. A $(\mathsf{cSet}, \otimes, \square^0)$-enriched category $\mathcal{C}$ is *locally Kan* if, for all $X, Y \in \operatorname{ob}\mathcal{C}$, the cubical set $\mathcal{C}(X,Y)$ is a Kan complex.

Theorem 5.36. *The* $(\mathsf{cSet}, \otimes, \square^0)$*-enriched category* $\mathsf{Graph}_\square$ *of graphs is locally Kan.*

Proof. This follows from Theorem 4.1. □

By the results of [KV20], we have established a presentation of the $(\infty, 1)$-category of graphs (localized at A-homotopy equivalences). In subsequent work, we use this presentation to show that A-homotopy equivalences are not part of a model structure on the category of graphs, as well as to study homotopy limits in discrete homotopy theory.

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Conflicts of interest

None.

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