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The Fixed Point Locus of the Verschiebung on $\mathcal{M}_X(2,0)$ for Genus-2 Curves X in Charateristic 2

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Abstract. We prove that for every ordinary genus-2 curve X over a finite field κ of characteristic 2 with $\mathrm{Aut}(X/\kappa) = \mathbb{Z}/2\mathbb{Z} \times S_3$ there exist $\mathrm{SL}(2, \kappa[[s]])$-representations of $\pi_1(X)$ such that the image of $\pi_1(\overline{X})$ is infinite. This result produces a family of examples similar to Y. Laszlo's counterexample to A. J. de Jong's question regarding the finiteness of the geometric monodromy of representations of the fundamental group.

1 Introduction

It was conjectured by A. J. de Jong in [9, Conjecture 2.3] that given a finite field $\mathbb{F}$ of characteristic l and a normal variety Y over a finite field κ of characteristic $p \neq l$, every representation $\rho\colon \pi_1(Y) \to \mathrm{GL}(r, \mathbb{F}((s)))$ has a finite geometric monodromy. This conjecture was proved by de Jong in the GL_2-case [9], by G. Böckle and K. Khare in the GL_n-case under some mild condition [3], and by D. Gaitsgory modulo the theory of $\mathbb{F}((s))$-sheaves [6]. Then a natural question comes up. If the hypothesis $l \neq p$ is dropped, and, moreover, Y is proper over κ, does the conjecture remain true? Note that when Y/κ is not proper, a counterexample has already been given in [9]

In [12], Y. Laszlo gave a negative answer to the above question. He showed that there exists a non-trivial family of rank-2 bundles fixed by the square of Frobenius over a specific genus-2 curve $C_0/\mathbb{F}_2$. From this he deduced the existence of the desired representations of $\pi_1(C_0 \otimes \mathbb{F}_{2^d})$. Recently, H. Esnault and A. Langer [4] have employed Laszlo's example to improve the statement of a p-curvature conjecture in characteristic p.

It is suspected by de Jong that the representations with an infinite geometric monodromy are rare. Thus one would like to understand the underlying mechanics of Laszlo's example and to obtain such representations in other characteristics.

In this note, we give a geometric interpretation of Laszlo's example based on the study of the action of the automorphism group of the curve; this interpretation allows us to produce a family of similar examples. Meanwhile, our method also provides some indication in characteristics 3 and 5, though it does not directly provide examples.

Now we give a brief summary of our results. In [12], Laszlo deduced representations from a non-trivial family of bundles. We show that the converse also holds.

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This equivalence is well known to the experts. Recall that the geometric Frobenius map of a scheme Y over a finite field κ of characteristic p is defined to be the d-th power of the absolute Frobenius map of Y, where $d = [\kappa : \mathbb{F}_p]$.

Theorem 1.1 *Let Y be a projective smooth geometrically connected curve over a finite field κ and $\mathcal{M}_Y(r,0)$ be the coarse moduli space of rank-r semistable bundles over $\overline{Y}$ with trivialized determinant. Denote by*

$$V : \mathcal{M}_Y(r,0) \dashrightarrow \mathcal{M}_Y(r,0)$$

the rational map defined by $[E] \mapsto [F_{geo}^* E]$ with respect to the geometric Frobenius map F_{geo} of Y over κ. Then the following are equivalent:

(i) *There exists a finite extension $\widetilde{\kappa}$ of κ and a representation $\rho\colon \pi_1(Y \otimes_\kappa \widetilde{\kappa}) \to \mathrm{SL}(r, \widetilde{\kappa}[[s]])$ such that $\rho|_{\pi_1(\overline{Y})} \bmod s$ is absolutely irreducible and $\#\rho(\pi_1(\overline{Y})) = \infty$.*
(ii) *There exists some $N \in \mathbb{N}$ such that the fixed point locus $\mathrm{Fix}(V^N)$ is of positive dimension and contains a stable point in a connected component, where $\mathrm{Fix}(V^N) = \{x \in \mathcal{M}_Y(r,0) | V^N(x) = x\}$.*

Because of the above equivalence, the question of looking for representations is converted to studying the fixed point locus $\mathrm{Fix}(V^N)$. In [12], the expression of V_{C_0} for C_0 was applied to locate a projective line $\triangle$ in $\mathcal{M}_{C_0}(2,0)$ such that $(V_{C_0}^2)|_\triangle$ is the identity map. Here our observation is that $\triangle$ is the fixed point locus of the G-action on $\mathcal{M}_{C_0}(2,0)$, where $G = \mathrm{Aut}(C_0 \otimes \mathbb{F}_{2^2}/\mathbb{F}_{2^2}) = \mathbb{Z}/2\mathbb{Z} \times S_3$. Indeed, this property is common to all genus-2 ordinary curves in characteristic 2 with a G-action.

Theorem 1.2 *Let X be a projective smooth ordinary curve of genus 2 over a finite field κ of characteristic 2 with $\mathrm{Aut}(X/\kappa) = \mathbb{Z}/2\mathbb{Z} \times S_3 \doteq G$. Let*

$$V : \mathcal{M}_X(2,0) \dashrightarrow \mathcal{M}_X(2,0)$$

be the rational map defined by taking a pullback of bundles with respect to the geometric Frobenius map of X over κ. Then the fixed point locus of the G-action on $\mathcal{M}_X(2,0)$ is a projective line, denoted by $\triangle_X$. And $V|_{\triangle_X} = \mathrm{id}_{\triangle_X}$.

Combining this with Theorem 1.1, for every curve in Theorem 1.2 there exist representations of the fundamental group with an infinite geometric monodromy.

A large part of the proof of Theorem 1.2 can be applied to other characteristics, particularly the application of a group action in locating a sublocus in the moduli space. However, when considering whether the restriction of the Verschiebung to the sublocus is reduced to a linear map, the condition regarding the existence of a single base point on the sublocus is sufficient only in charateristic 2. In other characteristics, more is required to ensure that the restriction of the Verschiebung is the identity.

This note is organized as follows. In Section 2, we establish equivalences among different categories under consideration. In Section 3, we prove Theorem 1.2. In Section 4, we discuss the case of characteristic $p > 2$.

2 Representations and Frobenius-periodic Vector Bundles

In this section we establish equivalences among the categories of Frobenius-periodic vector bundles, smooth étale sheaves and representations.

Notation κ is a finite field of order $q = p^d$, $S = \operatorname{Spec} \kappa[[s]]$, $\mathcal{S} = \operatorname{Spf} \kappa[[s]]$, and $S_n = \operatorname{Spec} \kappa[[s]]/(s^n)$ for $n \in \mathbb{Z}^+$; Y is a noetherian κ-scheme and F_Y is the absolute Frobenius of Y. By *vector bundle*, we mean a locally free sheaf of finite rank.

2.1 Preliminaries

Our definition of smooth étale $\kappa[[s]]$-sheaves is similar to that of a lisse l-adic sheaf in [15, Chap.V, §1]. When Y is connected, there is an equivalence between the category of locally free smooth $\kappa[[s]]$-sheaves over Y_{et} and the category of continuous $\pi_1(Y)$-modules that are free $\kappa[[s]]$-modules of finite rank, denoted by $\mathcal{C}_{1,Y_{\mathrm{et}}} \rightleftharpoons \mathcal{C}_{2,\pi_1(Y)}$.

Definition 2.1 A vector bundle $\mathcal{F}$ over $Y \times_\kappa S$ (resp. $Y \times_\kappa S_n$) is said to be *Frobenius-periodic* if there exists an isomorphism $\xi \colon \mathcal{F} \to (F_Y^d \times \mathrm{id}_S)^* \mathcal{F}$ (resp. $\xi \colon \mathcal{F} \to (F_Y^d \times \mathrm{id}_{S_n})^* \mathcal{F}$), denoted by $(\mathcal{F}, \xi)$. A *Frobenius-periodic vector bundle* over $Y \times_\kappa \mathcal{S}$ is a projective system $(\mathcal{F}, \xi) = ((\mathcal{F}_n, \xi_n))_{n \in \mathbb{Z}^+}$ of sheaves over $|Y_{\mathrm{zar}}|$ such that for each n, $\mathcal{F}_n$ is a Frobenius-periodic vector bundle over $Y \times_\kappa S_n$, the given map $\mathcal{F}_{n+1} \to \mathcal{F}_n$ is compatible with ξ_n's and is isomorphic to the natural map $\mathcal{F}_{n+1} \to \mathcal{F}_{n+1} \otimes_{\kappa[[s]]} \kappa[[s]]/(s^n)$.

Definition 2.2 Given $(\mathcal{F}_n, \xi_n)$ over $Y \times_\kappa S_n$, for any morphism

$$Z \xrightarrow{f} Y, \quad \big((f \times \mathrm{id}_{S_n}\big)^* \mathcal{F}_n, (f \times \mathrm{id}_{S_n})^* \xi_n)$$

can be viewed as a Frobenius-periodic vector bundle over $Z \times_\kappa S_n$, denoted by $f^*(\mathcal{F}_n, \xi_n)$ or $(f^* \mathcal{F}_n, f^* \xi_n)$.

A section $s \in \Gamma(Y \times_\kappa S_n, \mathcal{F}_n)$ is said to be *fixed by* ξ if $\xi_n(s) = (F_Y^d \times \mathrm{id}_{S_n})^* s \doteq 1 \otimes s$; $(\mathcal{F}_n, \xi_n)$ is said to be *trivializable* if $\mathcal{F}_n$ has a global basis fixed by ξ_n; it is said to be *étale trivializable* if there exists : $Y_n \xrightarrow{f} Y$ a finite étale morphism such that $f^*(\mathcal{F}_n, \xi_n)$ is trivializable, in this case we also say that Y_n/Y trivializes $(\mathcal{F}_n, \xi_n)$.

Remark 2.3 Given $(\mathcal{F}_n, \xi_n)$ over $Y \times_\kappa S_n$, an étale sheaf can be defined as follows:

$$(U \xrightarrow{f} Y) \in \mathrm{Et}(Y) \longmapsto \big\{ s \in \Gamma(U, f^* \mathcal{F}) | f^*(\xi)(s) = 1 \otimes s \big\}.$$

We will see in Lemma 2.4 that it is a locally free smooth $\kappa[[s]]/(s^n)$-sheaf.

Recall from [5, Appendix I] that a covering space of Y is a finite étale morphism $f \colon Z \to Y$, and it is Galois if $\# \operatorname{Aut}(Z/Y) = \deg(f)$.

Lemma 2.4 *Given $(\mathcal{F}, \xi) = ((\mathcal{F}_n, \xi_n))_{n \in \mathbb{Z}^+}$ over $Y \times_\kappa \mathcal{S}$, then there exists a family of covering spaces $Y \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots \leftarrow Y_n \leftarrow \cdots$ such that Y_n/Y trivializes $(\mathcal{F}_n, \xi_n)$.*

Proof Prove by induction on n. Case $n = 1$ is proved in [11, Proposition 1.2].

Induction step Assume that there is a covering space $Y_n \to Y$ that factors through Y_{n-1} and trivializes $(\mathcal{F}_n, \xi_n)$. Let $\{e_1^n, \dots, e_r^n\}$ be a basis of $\mathcal{F}_{n+1}|_{U \times_\kappa S_{n+1}}$ for an affine open subscheme $U \subset Y_n$ s.t. it extends to a global basis of $\mathcal{F}_n|_{Y_n \times_\kappa S_n}$ fixed by ξ_n, *i.e.*,

$$\xi_{n+1}\{e_1^n, \dots, e_r^n\} = \{(F_{Y_n}^d \times \mathrm{id}_{S_{n+1}})^* e_1^n, \dots, (F_{Y_n}^d \times \mathrm{id}_{S_{n+1}})^* e_r^n\}(I_{(r)} + s^n D_n),$$

for some $D_n \in \mathrm{Mat}(r \times r, \mathcal{O}_{Y_n}(U))$. Finding a basis $\{e_1^{n+1}, \dots, e_r^{n+1}\}$ fixed by ξ_{n+1} and of the form $\{e_1^n, \dots, e_r^n\}(I_{(r)} + s^n \triangle_{n+1})$, it is equivalent to finding $\triangle_{n+1} = (m_{ij})$ such that

$$D_n + \triangle_{n+1} = \triangle_{n+1}^{(q)},$$

where $\triangle_{n+1}^{(q)} = (m_{ij}^q)$. Then define

$$U_{n+1} = \mathrm{Spec}\, \mathcal{O}_{Y_n}(U)[m_{11}, \dots, m_{rr}]/(D_n + \triangle_{n+1} - \triangle_{n+1}^{(q)}).$$

Clearly $U_{n+1} \to U$ is a covering space and trivializes $(\mathcal{F}_{n+1}, \xi_{n+1})$. Therefore, the étale sheaf associated with $(\mathcal{F}_{n+1}, \xi_{n+1})$ is locally free and smooth. By [15, Chap.V, §1], there exists a covering space $Y_{n+1} \to Y_n \to Y$ that trivializes $(\mathcal{F}_{n+1}, \xi_{n+1})$. ■

Remark 2.5 Actually, for an affine open covering $\{U\}$ of Y_n, the local covering spaces $\{U_{n+1} \to U\}$ can be built up canonically to a covering space $Y_{n+1} \to Y_n$.

The trivial line bundle with a non-trivializable Frobenius structure may be trivialized by field extension. To avoid such cases, we give the following definition.

Definition 2.6 $(\mathcal{F}, \xi)$ over $Y \times_\kappa S$ (resp. $Y \times_\kappa S_n$) is said to be *strictly Frobenius-periodic* if $(\det(\mathcal{F}), \det(\xi))$ is trivializable, denoted by $(\mathcal{F}, \xi, \det = 1)$; $((\mathcal{F}_n, \xi_n))_{n \in \mathbb{Z}^+}$ over $Y \times_\kappa \mathcal{S}$ is said to be *strictly Frobenius-periodic* if every $(\mathcal{F}_n, \xi_n)$ is.

Proposition 2.7

(i) *Let $\mathcal{C}_{1,Y_{\mathrm{et}}}$ be the category of locally free smooth $\kappa[[s]]$-sheaves over Y_{et} and let $\mathcal{C}_{3,Y_{\mathrm{zar}}}$ be the category of Frobenius-periodic vector bundles over $Y \times_\kappa \mathcal{S}$. Then there is an equivalence $\mathcal{C}_{1,Y_{\mathrm{et}}} \rightleftharpoons \mathcal{C}_{3,Y_{\mathrm{zar}}}$.*

(ii) *Asssume that Y is connected. Let $\mathcal{C}^{\mathrm{sl}}_{2,\pi_1(Y)}$ be the full subcategory of $\mathcal{C}_{2,\pi_1(Y)}$ whose objects are SL-representations of $\pi_1(Y)$ and $\mathcal{C}^{\mathrm{str}}_{3,Y_{\mathrm{zar}}}$ be the full subcategory of $\mathcal{C}_{3,Y_{\mathrm{zar}}}$ whose objects are strictly Frobenius-periodic vector bundles over $Y \times_\kappa \mathcal{S}$. Then there is an equivalence*

$$\mathcal{C}^{\mathrm{sl}}_{2,\pi_1(Y)} \rightleftharpoons \mathcal{C}^{\mathrm{str}}_{3,Y_{\mathrm{zar}}}, \quad \rho \leftrightarrow (\mathcal{F}_\rho, \xi_\rho, \det = 1) \quad \textit{or} \quad \rho_{(\mathcal{F},\xi)} \leftrightarrow (\mathcal{F}, \xi, \det = 1).$$

Proof (i) We can assume that Y is connected. The functor $\mathcal{C}_{3,Y_{\mathrm{zar}}} \to \mathcal{C}_{1,Y_{\mathrm{et}}}$ is clear from Remark 2.3 and Lemma 2.4. The functor $\mathcal{C}_{1,Y_{\mathrm{et}}} \to \mathcal{C}_{3,Y_{\mathrm{zar}}}$ is the composition $\mathcal{C}_{1,Y_{\mathrm{et}}} \to \mathcal{C}_{2,\pi_1(Y)} \to \mathcal{C}_{3,Y_{\mathrm{zar}}}$. The proof follows from Galois descent theory; see [16, §12, Thereom 1].

(ii) We only need to show that $(\mathcal{F}_n, \xi_n, \det = 1)$ induces a SL-representation. Let $Y_n \xrightarrow{f_n} Y$ be a Galois covering space that trivializes $(\mathcal{F}_n, \xi_n)$ by Lemma 2.4. Then the induced representation is the composition

$$\pi_1(Y, \overline{y}) \longrightarrow \mathrm{Gal}(Y_n/Y) \longrightarrow \mathrm{GL}\big(r, \kappa[[s]]/(s^n)\big),$$

with a basis $\{e^n_1, \ldots, e^n_r\}$ of $f_n^* \mathcal{F}_n$ preserved by $f_n^*\xi_n$; the latter is defined by $g \mapsto M_{g^{-1}}$, where $(g^{-1} \times \mathrm{id}_{S_n})^* \{e^n_1, \ldots, e^n_r\} = \{e^n_1, \ldots, e^n_r\} M_{g^{-1}}$. Thus $M_{g^{-1}} \in \mathrm{SL}(r, \kappa[[s]]/(s^n))$. ∎

2.2 The Equivalence over a Projective Base

Now we turn to the case where Y is a projective smooth geometrically connected scheme over κ. Let $\overline{Y} = Y \times_\kappa \operatorname{Spec} \overline{\kappa}$. In this case, the categories of vector bundles over $Y \times_\kappa \mathcal{S}$ and over $Y \times_\kappa S$ are equivalent by Grothendieck's existence theorem. Given $(\mathcal{F}, \xi)$ over $Y \times_\kappa S$, we say that $(\mathcal{F}, \xi)$ is *constant* if it is isomorphic to the pullback of $\mathcal{F} \bmod s$. We refer to [7, 8] regarding the definition of geometrically slope-stable vector bundles and that of an absolutely irreducible representation.

Lemma 2.8

(i) [9, Lemma 3.15] *Let $\rho\colon H \to \mathrm{GL}(r, K[[s]])$ be a representation of a finite group H, where K is a field. If $\rho_0 = \rho \bmod s$ is absolutely irreducible, then $\rho \simeq \rho_0 \otimes_K K[[s]]$.*

(ii) [9, Lemma 2.7] *Let $1 \to \Gamma \to H \to \widehat{Z} \to 0$ be an exact sequence of profinite groups. Suppose that $\rho\colon H \to \mathrm{SL}(V)$ is a continuous representation such that $\rho|_\Gamma$ is absolutely irreducible. Then $\#\rho(\Gamma) < \infty \Leftrightarrow \#\rho(H) < \infty$.*

Lemma 2.9 *Given $(\mathcal{F}, \xi, \det = 1) \leftrightarrow \rho$, if $(\mathcal{F}, \xi)$ is constant, then $\#\rho(\pi_1(Y)) < \infty$. If $\rho \bmod s$ is absolutely irreducible, then $(\mathcal{F}, \xi)$ is constant $\Leftrightarrow \#\rho(\pi_1(Y)) < \infty$.*

Proof The proof follows from Lemmas 2.4, 2.8(i) and descent theory. ∎

Proposition 2.10 *Given $(\mathcal{F}, \xi, \det = 1) \leftrightarrow \rho$ as in Proposition 2.7. The following are equivalent:*

(i) *$\mathcal{F} \bmod s$ is geometrically slope-stable (g.s.s.).*

(ii) *$(\rho \bmod s)|_{\pi_1(\overline{Y})}$ is absolutely irreducible (a.i.).*

If these conditions hold, then $\mathcal{F}$ is non-constant if and only if $\#\rho(\pi_1(\overline{Y})) = \infty$.

Proof Let $\mathcal{F}_0 = \mathcal{F} \bmod s$ and $\rho_0 = \rho \bmod s$. (g.s.s.) $\Longrightarrow$ (a.i.). The reducibility of $\rho_0|_{\pi_1(\overline{Y})} \otimes \overline{\kappa}$ implies the existence of a proper subbundle of $\mathcal{F}_0 \otimes \overline{\kappa}$ with slope 0.

(a.i.) $\Longrightarrow$ (g.s.s.): As $\mathcal{F}_0$ is étale trivialized, it is geometrically slope-semistable. Since a subbundle with slope 0 of a trivial bundle is trivial, the existence of a proper subbundle of $\mathcal{F}_0 \otimes \overline{\kappa}$ with slope 0 implies the reducibility of $\rho_0|_{\pi_1(\overline{Y})} \otimes \overline{\kappa}$.

Since the absolute irreducibility of $(\rho \bmod s)|_{\pi_1(\overline{Y})}$ implies the same property for $\rho|_{\pi_1(\overline{Y})}$ and $\rho \bmod s$ then the second equivalence follows from Lemmas 2.9 and 2.8(ii). ∎

Proof of Theorem 1.1

(i) $\Rightarrow$ (ii) By Proposition 2.7, there exists a strictly Frobenius-periodic rank-r vector bundle $(\mathcal{F}, \xi, \det = 1)$ over $(Y \otimes_\kappa \widetilde{\kappa}) \times_{\widetilde{\kappa}} \operatorname{Spec} \widetilde{\kappa}[[s]]$. Locally, $(\mathcal{F}, \xi, \det = 1)$ is defined by transition matrices and linear maps. Let $A \subset \widetilde{\kappa}[[s]]$ be the finitely generated $\widetilde{\kappa}$-algebra generated by elements appearing in the matrices that define $(\mathcal{F}, \xi, \det = 1)$. Clearly, there exists canonically a strictly Frobenius-periodic bundle $(\mathcal{F}', \xi', \det = 1)$ over $(Y \otimes_\kappa \widetilde{\kappa}) \times_{\widetilde{\kappa}} \operatorname{Spec} A$ such that its pullback to $Y \times_\kappa \operatorname{Spec} \widetilde{\kappa}[[s]]$

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is exactly $(\mathcal{F}, \xi, \det = 1)$. $\mathcal{F}'$ can be viewed as a family of bundles over $\overline{Y}$ fixed by the geometric Frobenius map of $Y \otimes_\kappa \widetilde{\kappa}$ over $\widetilde{\kappa}$, *i.e.*, the N-th power of the geometric Frobenius map of Y over κ, where $N = [\widetilde{\kappa} : \kappa]$. Thus the image of the modular morphism $\operatorname{Spec} A \to \mathcal{M}_Y(r, 0)$ is in $\operatorname{Fix}(V^N)$. By Proposition 2.10, $\mathcal{F}'$ is a non-constant family and consists mostly of stable bundles, thus $\operatorname{Fix}(V^N)$ has the required properties.

(ii)⇒(i) It follows from the construction of $\mathcal{M}_{X_t}(2, 0)$ as a GIT quotient as shown in [12, Corollary 3.2 & Lemma 3.3]. ■

From now on, in order to obtain representations with an infinite geometric monodromy, we turn to the study of the fixed point locus of the Verschiebung.

3 Proof of Theorem 1.2

In this section, let $G = \mathbb{Z}/2\mathbb{Z} \times S_3$. Let X be a projective smooth ordinary curve of genus 2 over a field κ of characteristic 2 with $\operatorname{Aut}(X/\kappa) = G$. Except in the proof of Theorem 1.2, κ can be infinite. Let $X(1)$ be the scheme deduced from X by the extension of scalars $a \mapsto a^2$ and let $F_{X/\kappa} \colon X \to X(1)$ be the relative Frobenius map. Note that the G-action on X induces a G-action on $X(1)$ that is compatible with $F_{X/\kappa}$.

3.1 *G*-action

In this subsection, we study the fixed point locus of the G-action on the Kummer surface Km_X of X and on the coarse moduli space $\mathcal{M}_X(2, 0)$ of rank-2 semistable bundles with trivialized determinant over $\overline{X} = X \otimes_\kappa \overline{\kappa}$.

Let $\pi_X \colon X \to |K_X| = \mathbb{P}^1$ be the canonical morphism of X. As X is ordinary, the double covering π_X has three ramification points according to Fact 3.1.

Facts 3.1 *Let Y be a projective smooth curve of genus 2 over an algebraically closed field of characteristic $p > 0$. Assume that $\mathcal{L} \in \operatorname{Pic}^0(Y)$. Then $\mathcal{L}$ is of the form $\mathcal{O}_X(P - Q)$, where P, Q are closed points of Y. Moreover, if $\mathcal{L}^2 = \mathcal{O}_Y$, then $\mathcal{L}$ is of the form $\mathcal{O}_Y(R_1 - R_2)$, where $R_1, R_2 \in Y$ are ramification points of the canonical morphism $\pi_Y \colon Y \to |K_Y| = \mathbb{P}^1$.*

We can assume that the image of the ramification points of π_X are $\{0, 1, \infty\}$. The $\mathbb{Z}/2\mathbb{Z}$-action on X is generated by the hyperelliptic involution of π_X, denoted by ι; the S_3-action on X induces an action on the canonical linear system $|K_X|$ and hence can be identified as the permutation group of the branch points $\{0, 1, \infty\}$. Let $\tau_{01} = (01)(\infty)$ and $\sigma = (01\infty)$. Note that σ fixes four points on $\overline{X}$.

The G-action on X induces a G-action on the Jacobian J_X and thus on the Kummer surface Km_X of X. We can actually figure out the fixed points of G on Km_X.

Lemma 3.2 *The set of the fixed points $(\mathrm{Km}_X)^G$ of the G-action on Km_X consists of three points: $\mathcal{O}_X^{\oplus 2}$, $E_{1,X} = \mathcal{O}_X(Q - \tau_{01}(Q)) \oplus \mathcal{O}_X(\tau_{01}(Q) - Q)$, and $E_{2,X} = \mathcal{O}_X(Q - \iota \circ \tau_{01}(Q)) \oplus \mathcal{O}_X(\iota \circ \tau_{01}(Q) - Q)$, where $Q \in \overline{X}$ is a fixed point of σ.*

Proof It suffices to find all line bundles $\mathcal{L} \in \operatorname{Pic}^0(\overline{X})$ such that $g^*\mathcal{L} \simeq \mathcal{L}$ or $\mathcal{L}^{-1}$ for $g = \tau_{01}, \tau_{0\infty}$ and σ. By Fact 3.1, $\mathcal{L} \simeq \mathcal{O}_X(Q_1 - Q_2)$ for $Q_1, Q_2 \in \overline{X}$. The lemma is proved by a case-by-case analysis according to the three types of points: (I) the three ramification points, (II) the four fixed points of σ, (III) all the others. ■

Clearly $E_{1,X}$ and $E_{2,X}$ are independent of the choice of the fixed point of σ. Take a fixed point Q_1 of σ on $\overline{X}(1)$; we similarly define $E_{1,X(1)}$ and $E_{2,X(1)}$. We have the following lemma.

Lemma 3.3 *For $j = 1, 2$, $F^*_{X/\kappa} E_{j,X(1)} = E_{j,X}$.*

Proof Let $Q = F^{-1}_{X/\kappa}(Q_1)$. Then Q is a fixed point of σ because $Q_1 \in \overline{X}(1)$ is a fixed point of σ and $F_{X/\kappa}$ preserves the G-action. Since

$$F^*_{X/\kappa}\big[\mathcal{O}_{X(1)}\big(Q_1 - \tau_{01}(Q_1)\big)\big] = \mathcal{O}_X\big(2Q - 2\tau_{01}(Q)\big)$$
$$F^*_{X/\kappa}\big[\mathcal{O}_{X(1)}\big(Q_1 - \iota \circ \tau_{01}(Q_1)\big)\big] = \mathcal{O}_X\big(2Q - 2\iota \circ \tau_{01}(Q)\big),$$

it suffices to prove that $3Q \sim 3\tau_{01}(Q) \sim 3\iota \circ \tau_{01}(Q)$. This follows from that $Q, \tau_{01}(Q), \iota(Q), \iota \circ \tau_{01}(Q)$ are the ramification points with index 3 of the quotient $X \to X/\langle\sigma\rangle \simeq \mathbb{P}^1$. ■

By [13], $\mathcal{M}_X(2,0)$ is isomorphic to $|2\Theta| \simeq \mathbb{P}^3_\kappa$ and the Kummer surface Km_X is a quartic hypersurface. To find the fixed point locus $(\mathcal{M}_X(2,0))^G$, we need the following lemma.

Lemma 3.4 *Let H be a subgroup of $\mathrm{Aut}(\mathbb{P}^n_k/k)$, where k is a field of characteristic p. Assume that H is generated by elements with order of the form p^r. Let $P_1, P_2 \in \mathbb{P}^n_k$ be fixed by H, then the projective line $\overline{P_1P_2}$ is fixed by H.*

Proof Identify points P_1, P_2 with vectors $v_1, v_2 \in k^{n+1}$. Let $h \in H$ have order p^r and $\widetilde{h} \in \mathrm{GL}(n+1,k)$ be a preimage of h. Then $\widetilde{h}^{p^r} = \mu I_{n+1}$. By assumption, $\widetilde{h}(v_1) = \mu_1 v_1$ and $\widetilde{h}(v_2) = \mu_2 v_2$. Thus $\mu_1^{p^r} = \mu_2^{p^r}$ implies $\mu_1 = \mu_2$; therefore, h fixes the line $\overline{P_1P_2}$. ■

Proposition 3.5 *The fixed point locus of the G-action on $\mathcal{M}_X(2,0)$ is a projective line, denoted by $\triangle_X$.*

Proof As $(\mathcal{M}_X(2,0))^G \cap \mathrm{Km}_X$ is a set of three points and Km_X is a hypersurface, by Lemma 3.4, $(\mathcal{M}_X(2,0))^G$ is a projective line. ■

3.2 Verschiebung

Let $X(n)$ be the scheme deduced from X by the extension of scalars $a \mapsto a^{2^n}$. Denote by F_n the relative Frobenius $F_n \colon X(n) \to X(n+1)$ and by V_n the Verschiebung

$$V_n \colon \mathcal{M}_{X(n+1)}(2,0) \dashrightarrow \mathcal{M}_{X(n)}(2,0), \quad [E] \mapsto [F_n^* E].$$

As $X(n)$ has the same properties as X, the results for X also hold for $X(n)$. Let $\triangle_{X(n)}$ be the projective line of $\mathcal{M}_{X(n)}(2,0)$ in Proposition 3.5. As F_n is compatible with the G-action, the pullback of a G-bundle is a G-bundle, hence $V_n(\triangle_{X(n+1)}) \subset \triangle_{X(n)}$.

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To reduce $V_n|_{\triangle_{X(n+1)}}: \triangle_{X(n+1)} \to \triangle_{X(n)}$ to a linear map, we point out a base point of V_n on $\triangle_{X(n+1)}$. Recall from [17] that there is a theta characteristic B_n of $X(n)$ defined as $0 \to \mathcal{O}_{X(n)} \to F_{n-1*}\mathcal{O}_{X(n-1)} \to B_n \to 0$. Consider the rank-2 bundle $F_{n*}B_n^{-1}$ over $X(n+1)$. Clearly $\det(F_{n*}B_n^{-1}) = \mathcal{O}_{X(n+1)}$. Because $0 \to B_n \to F_n^*(F_{n*}B_n^{-1}) \to B_n^{-1} \to 0$ and $\deg(B_n) = 1$, $F_{n*}B_n^{-1}$ is stable and $F_n^*(F_{n*}B_n^{-1})$ is unstable. Moreover, $F_{n*}B_n^{-1}$ has a G-action by construction, thus $[F_{n*}B_n^{-1}] \in \triangle_{X(n+1)}$. Therefore,

Lemma 3.6 *The restriction*

$$V_n|_{\triangle_{X(n+1)}}: \triangle_{X(n+1)} \dashrightarrow \triangle_{X(n)}$$

is a linear map.

Proof By [13, Proposition 6.1], V_n is defined by qradratic polynomials. Thus $V_n|_{\triangle_{X(n+1)}}$ is given by two quadratic polynomials $\{h_1, h_2\}$ in two variables. As $V_n|_{\triangle_{X(n+1)}}$ has a base point $[F_{n*}B_n^{-1}]$, h_1 and h_2 have a common linear factor, thus $V_n|_{\triangle_{X(n+1)}}$ is reduced to a linear map. ■

Proof of Theorem 1.2 Assume that $\#\kappa = 2^d$. Note that $X(d) = X$. As the rational map V is the composition $V_0 \circ V_1 \circ \cdots \circ V_{d-1}$ and

$$V_n|_{\triangle_{X(n+1)}}: \triangle_{X(n+1)} \dashrightarrow \triangle_{X(n)}$$

is linear for $0 \leq n \leq d-1$ by Lemma 3.6, thus $V|_{\triangle_X}$ is linear. Moreover, Lemma 3.3 holds for every $X(n)$, *i.e.*, there are semistable bundles $E_{1,X(n)}, E_{2,X(n)}$ such that $V_n([E_{j,X(n+1)}]) = [E_{j,X(n)}]$ for $j = 1, 2$. Thus $V|_{\triangle_X}$ has three distinct fixed points, *i.e.*, $[\mathcal{O}_X^{\oplus 2}]$, $[E_{1,X}]$, and $[E_{2,X}]$. In conclusion, $V|_{\triangle_X}$ is the identity map. ■

Remark 3.7 Actually, it can be shown that there exists a vector bundle $\mathcal{E}$ over $X \times_\kappa \Lambda$ with $\Lambda = \operatorname{Spec} \kappa[\lambda]$ such that (1) the modular morphism $i: \Lambda \to \triangle_X$ is an open immersion with the only missing point to be $[\mathcal{O}_X^{\oplus 2}]$; (2) there exists a morphism $(F_X^d \times \mathrm{id}_\Lambda)^*\mathcal{E} \to \mathcal{E}$ that is isomorphic if replacing Λ by an open subset; (3) all bundles $\mathcal{E}_\lambda$ are subbundles of $K_X^2 \oplus K_X$.

Remark 3.8 By [1], a curve in Theorem 1.2 is defined by the equation

$$y^2 + (x^2 + x)y + (t^2 + t)(x^5 + x) + t^2x^3 = 0, \quad t \neq 0, 1. \tag{3.1}$$

Note that the automorphism group $\mathrm{Aut}(C_0/\mathbb{F}_2)$ of the curve $C_0/\mathbb{F}_2$ in [12] is not $\mathbb{Z}/2\mathbb{Z} \times S_3$. Instead, we consider $C_0 \otimes_{\mathbb{F}_2} \mathbb{F}_{2^2}$ with $\mathrm{Aut}(C_0 \otimes_{\mathbb{F}_2} \mathbb{F}_{2^2}/\mathbb{F}_{2^2}) = \mathbb{Z}/2\mathbb{Z} \times S_3$. That is the case when $t \in \mathbb{F}_4 \backslash \mathbb{F}_2$ in equation (3.1).

4 Discussion of Characteristics 3 and 5

Let Y be a projective smooth genus-2 curve over a finite field κ of characteristic $p > 2$. Note that the canonical morphism $\pi_Y: Y \to |K_Y| = \mathbb{P}^1$ is ramified at 6 points, which implies that $\mathrm{Aut}(\overline{Y}/\overline{\kappa}) \subset \mathbb{Z}/2\mathbb{Z} \times S_6$. It is known that $\mathcal{M}_Y(2,0)$ is isomorphic to the linear system $|2\Theta| \simeq \mathbb{P}^3_\kappa$ over $\mathrm{Pic}^1(\overline{Y})$. Let

$$V\colon \mathcal{M}_Y(2,0) \dashrightarrow \mathcal{M}_Y(2,0)$$

be the rational map induced by the geometric Frobenius map of Y over κ. Let $Y(n)$, $F_n\colon Y(n) \to Y(n+1)$ and

$$V_n\colon \mathcal{M}_{Y(n+1)}(2,0) \dashrightarrow \mathcal{M}_{Y(n)}(2,0)$$

be the same as in Subsection 3.2. By [14, Proposition A.2], V_n's are given by polynomials of degree p. Assume that $\#\kappa = p^d$, then $Y(d) = Y$.

Recall from the proof of Theorem 1.2 that a large part can be applied to other characteristics. In particular, the following two facts are true.

Lemma 4.1 *If $V^m|_Z = \mathrm{id}_Z$ for a reduced subscheme $Z \subset \mathcal{M}_Y(2,0)$ of positive dimension and for some $m > 0$, then the closure $\overline{Z}$ of Z contains a base point of V_{d-1}.*

Lemma 4.2 *Assume that $p = 3$ or 5. Let $H \subset \mathrm{Aut}(Y/\kappa)$ be a subgroup generated by elements with order of the form p^n. Let*

$$V\colon \mathcal{M}_Y(2,0) \dashrightarrow \mathcal{M}_Y(2,0)$$

be the rational map given in the notations. Assume that the set $(\mathrm{Km}_Y)^H$ of fixed points of H on the Kummer surface Km_Y is finite. Given a semistable bundle $[E] \in \mathcal{M}_Y(2,0)^H$ satisfying that F_Y^*E is semistable and $[F_Y^*E] \neq [\mathcal{O}_Y^{\oplus 2}]$, where F_Y is the absolute Frobenius map of Y. Then the fixed point locus of the H-action on $\mathcal{M}_Y(2,0)$ is a projective line, denoted by $\triangle_Y$, and the restriction of V to $\triangle_Y$ is a rational map $V|_{\triangle_Y}\colon \mathbb{P}^1_\kappa \to \mathbb{P}^1_\kappa$.

The special property of characteristic 2 that is used in proving Theorem 1.2 is that the existence of a single base point on $\triangle_{Y(n+1)}$ is sufficient to lower the degree of the polynomials that define $(V_n)|_{\triangle_{Y(n+1)}}$ from 2 to 1. Similar cases may happen in other small characteristics. However, in large characteristic, as was proved in [10, Proposition 3.1] that every V_n has exactly 16 base points for characteristic $p > 2$, then the intersection number of $\triangle_{Y(n)}$ with the scheme-theoretic base locus $\mathcal{B}_n$ of V_{n-1} is required to check if $(V_{n-1})|_{\triangle_{Y(n)}}$ can be reduced to a linear map. To calculate $\triangle_{Y(n)} \cap \mathcal{B}_n$, more about V_n should be discovered.

If suitable conditions could be found to ensure that every $(V_n)|_{\triangle_{Y(n+1)}}$ is linear, then the map $V|_{\triangle_Y}$ in Lemma 4.2 is linear and non-constant; moreover, as $V|_{\triangle_Y}$ is defined over a finite field, there exists some N such that $(V|_{\triangle_Y})^N = \mathrm{id}$. Therefore, by Theorem 1.1, we would obtain representations of $\pi_1(Y \otimes_\kappa \widetilde{\kappa})$ with an infinite geometric monodromy.

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