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UPPER TRIANGULAR INVARIANTS

BY
H. E. A. CAMPBELL

ABSTRACT. We modify the construction of the mod 2 Dyer-Lashof (co)-algebra to obtain a (co)-algebra W which is (also) unstable over the Steenrod algebra A_*. W has canonical sub-coalgebras $W[k]$ such that the hom-dual $W[k]^*$ is isomorphic as an A-algebra to the ring of upper triangular invariants in $\mathbb{Z}/2\mathbb{Z}\,[x_1, \ldots, x_k]$.

Introduction. Let $P_k = \mathbb{Z}/2\mathbb{Z}\,[x_1, \ldots, x_k]$ denote the polynomial algebra over $\mathbb{Z}/2\mathbb{Z}$ on k generators of degree one; we write $|x_i| = 1$. Let $G_k = G\ell_k(\mathbb{Z}/2\mathbb{Z})$ denote the group of kxk matrices of determinant one with entries from $\mathbb{Z}/2\mathbb{Z}$ acting on P_k as a group of algebra automorphisms, that is, if $C = [c_{ij}] \in G_k$ then $Cx_i = \sum_{j=1}^{k} c_{ji}x_j$ and C is extended as an algebra map to all of P_k. Let T_k denote the 2-Sylow subgroup of G_k consisting of the upper triangular matrices with "ones" on the main diagonal. We denote the invariants with respect to the actions of these two groups by $P_k{}^{G_k}$ and $P_k{}^{T_k}$. It is well known that $P_k{}^{G_k}$ (which is called the Dickson algebra) can be obtained as the dual of $R[k]$, a canonical subcoalgebra of the Dyer-Lashof algebra R (see section one). In this paper we modify the construction of the Dyer-Lashof algebra by killing only those monomials suffering from negative excess to obtain a Hopf algebra W with subcoalgebras $W[k]$ such that $W[k]^* \cong P_k{}^{T_k}$ as algebras over the Steenrod algebra. We use these facts to obtain a description of the action of the Milnor primitives Sq^{Δ_r} on $P_k{}^{T_k}$; this description uses the known action of the Sq^{Δ_r} on $P_k{}^{G_k}$ which we reproduce here from [1] for completeness. In addition, we provide an invariant-theoretic interpretation of the dual basis for $P_k{}^{G_k}$ coming from $R[k]$.

Recall that $P_k{}^{G_k} = \mathbb{Z}/2\mathbb{Z}[a_{1,k}, \ldots, a_{k,k}]$ with $|a_{i,k}| = 2^k - 2^{k-i}$ and that $P_k{}^{T_k} = \mathbb{Z}/2\mathbb{Z}[v_1, \ldots, v_k]$ with $|v_i| = 2^{i-1}$; see for example, Dickson [3], Mùi [4] or Wilkerson [7] (but note that our notation is different $a_{i,k} = Q_{k,k-i} = C_{k,k-i}$).

We note that for experts this paper is more or less an observation; the technical details in connection with the Dyer-Lashof algebra have long since been worked out. Our presentation here relies heavily on J. P. May [2] and C. Wilkerson [7] and our inspiration is due, in part, to the work of W. M. Singer [5, 6]. The author is grateful to Paul Selick for expert and excellent advice.

Recollections of the past and the construction and basic properties of W. Let F be the free associative algebra on symbols $\{f^s | s \geq 0\}$, with $|f^s| = s$. Given a sequence

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$I = (i_1, \ldots, i_k)$ of non-negative integers we define the *length*, *degree* and *excess* of I by $\ell(I) = k$, $d(I) = i_1 + \ldots + i_k$ and $e(I) = i_1 - i_2 - \ldots - i_k$, respectively. The sequence I determines an element $f^I = f^{i_1} \ldots f^{i_k} \epsilon F$, and we define the degree of f^I denoted $|f^I|$ as $d(I)$. Let L denote the two-sided ideal of F generated by the elements f^I of negative excess, and define W to be the quotient algebra F/L; let e^I denote the image of f^I in W.

Let K' denote the two-sided ideal of F generated by the *Adem relations*: if $r > 2s$,

$$f^r f^s + \sum \binom{j-s-1}{2j-r} f^{r+s-j} f^j,$$

and let K denote the image of K' in W. It is frequently convenient to use lower notation for the elements of W, that is, we define $e^i x = e_{i-|x|}x$, for example $e^2 e^1 = e^2 e_1 = e_1 e_1$. In this notation $|e_I| = i_1 + 2i_2 + \ldots + 2^{k-1} i_k$, and the set $\{e_I | i_j \geq 0, j = 1, \ldots, k\}$ is a $\mathbb{Z}/2\mathbb{Z}$-basis for W. The Adem relations in lower notation are: for $r > s$,

$$e_r e_s + \sum \binom{j-s-1}{r-j-1} e_{r+2s-2j} e_j.$$

Let W_n denote the subspace of W spanned by $\{e_I \| e_I| = n\}$, note that $W_0 = \mathbb{Z}/2\mathbb{Z}[e_0]$; let $W[k]$ denote the subspace of W spanned by $\{e_I | \ell(I) = k\}$ with $W[0] = \mathbb{Z}/2\mathbb{Z}$. We have $W = \oplus_{n \geq 0} W_n = \oplus_{k \geq 0} W[k]$. An element e_I (or I itself) is said to be *admissible* if $i_1 \leq i_2 \leq \ldots \leq i_k$.

The *Dyer-Lashof algebra* R, is defined to be the quotient algebra F modulo the two-sided ideal generated by both the Adem relations and the monomials of negative excess; here we have $R \cong W/K$. The image of e_I in R under the natural map $\phi : W \twoheadrightarrow R$ is denoted Q_I. The set $\{Q_I | I \text{ is admissible}\}$ is a $\mathbb{Z}/2\mathbb{Z}$-basic for R. R is a Hopf algebra under the coproduct defined on generators by $\psi(Q_i) = \Sigma Q_{i-j} \otimes Q_j$ and if $R[k]$ denotes $\phi(W[k])$ then $R[k]$ is a connected subcoalgebra and $R = \oplus_{k \geq 0} R[k]$ as a coalgebra. In fact, R is a component coalgebra; πR is the free monoid generated by Q_0 and $R[k]$ is the component of $Q_0{}^k = Q_0 \ldots Q_0$ (k times), $k \geq 0$. The product in R sends $R[k] \otimes R[\ell]$ to $R[k + \ell]$ and the elements Q_i, $i \geq 0$ are all indecomposable.

The (opposite of the) Steenrod algebra A_* acts on R via the *Nishida relations*:

$$Sq^r_* Q_i x = \sum_s \binom{|x| + i - r}{r - 2s} Q_{i-r+2s} Sq^s_* x.$$

Then R and $R[k]$ are unstable A_*-coalgebras. The set

$$\Big\{ \overbrace{Q_0 \ldots Q_0}^{k-i} \; \overbrace{Q_1 \ldots Q_1}^{i}, \qquad 1 \leq i \leq k \Big\}$$

is a basis for the primitives $PR[k]$ of $R[k]$. Consequently, the dual $R[k]^*$ is a polynomial algebra on $Q_0 \ldots Q_0 Q_1 \ldots Q_1{}^*$ and $R[k]^* \cong P_k{}^{G_k}$ as an A-algebra under the map $Q_0 \ldots Q_0 Q_1 \ldots Q_1{}^* \rightarrow a_{i,k}$, (see [7], IV, p. 430); we denote the dual of the inverse isomorphism $\zeta_* : P_k{}^{G_k} \rightarrow R[k]$ for later use. For all of the above, see May's paper in [2].

Now $P_k^{T_k} = \mathbb{Z}/2\mathbb{Z}[v_1, \ldots, v_k]$; to determine the action of the Steenrod algebra on $P_k^{T_k}$ we recall the work of C. Wilkerson [7]. Let Y_k denote the vector space $(P_k)_1$ with basis $\{x_1, \ldots, x_k\}$ and let

$$f_k(x) = \prod_{y \in Y_k} (x + y) = x^{2^k} + a_{1,k}x^{2^{k-1}} + \ldots + a_{k-1,k}x + a_{k,k},$$

where $P_k^{G_k} = \mathbb{Z}/2\mathbb{Z}[a_{1,k}, \ldots, a_{k,k}]$.

LEMMA 1. *$Sq^r f_k(x) = (Sq^{r-2^{k-1}} a_{1,k}) f_k(x)$, for $r \neq 2^k$ or 0, and $Sq^{2^k} f_k(x) = f_k(x)^2$.*

$$Sq^{\Delta_i} f_k(x) = 0, \; i \leq k,$$

and

$$Sq^{\Delta_k} f_k(x) = a_{k,k} f_k(x).$$

PROOF. See propositions 2.1 and 2.2 in [7] (recall $C_{k,k-1} = a_{1,k}$). □

THEOREM 2.

$$Sq^{2^r} r_{k+1} = \begin{cases} v_{k+1}v_k + v_{k+1}v_{k-1}^2 + \ldots + v_{k+1}v_1^{2^{k-1}}, & r = k-1 \\ v_{k+1}^2, & r = k \\ 0, & r \neq k, k-1 \end{cases}$$

$$Sq^{\Delta_i} v_{k+1} = \begin{cases} 0, & i < k \\ v_{k+1} \ldots v_1 = a_{k+1,k+1}, & i = k. \end{cases}$$

PROOF. Mùi has observed that $f_k(x_{k+1}) = v_{k+1}$, (see [4], 3.2, 3.4, p. 328). Since $|v_{k+1}| = 2^k$, it is immediate that $Sq^{2^k} v_{k+1} = v_{k+1}^2$ and that $Sq^{2^r} v_{k+1} = 0$ for $r > k$. It follows from the lemma that $Sq^{2^r} v_{k+1} = 0$ if $r < k - 1$, and that $Sq^{2^{k-1}} v_{k+1} = a_{1,k} v_{k+1}$. Now $a_{1,k} = v_k = v_{k-1}^2 + \ldots + v_1^{2^{k-1}}$ (use proposition 13(b) of [7]). The statements concerning the Milnor elements also follow directly, noting that $a_{k,k} = v_k \ldots v_1$. □

Thus $P_k^{T_k}$ is an A-algebra and consequently $P_k^{T_k}{}_*$ is an A_*-coalgebra with primitives $(v_i)_*$ and commutative coproduct, that is, if $v^I = v_1^{i_1} \ldots v_k^{i_k}$ then

$$\psi(v_*^I) = \sum_{I'+I''=I} v_*^{I'} \otimes v_*^{I''}$$

since $v^{I'} v^{I''} = v^I$ whenever $I' + I'' = I$. We define a map of vector spaces $\rho : P_k^{T_k}{}_* \to W[k]$ by $\rho(v_*^I) = e_I$; it is clear that ρ is an isomorphism of vector spaces. We give $W[k]$ the induced commutative coproduct

$$\psi(e_I) = \sum_{I'+I''=I} e_{I'} \otimes e_{I''}$$

with primitives $\rho((v_i)_*) = e_{\Delta_{i,k}}$ where $\Delta_{i,k} = (0, \ldots, 0, 1, 0, \ldots, 0)$, the 1 in the i-th spot from the left. Furthermore, we give $W[k]$ the induced A_*-action. This construction gives $W = \oplus_{k \geq 0} W[k]$ the structure of a Hopf algebra (it is easy to check that ψ is an algebra map; the augmentation on $W_0 = \mathbb{Z}/2\mathbb{Z}[e_0]$ is $\epsilon(e_0^k) = 1$, $k \geq 0$), and of an unstable A_*-coalgebra (the inclusions $P_k^{T_k} \to P_{k+1}^{T_{k+1}}$ are maps of A_*-algebras) and $W[k]$ is an unstable A_*-subcoalgebra.

Note that the inclusion $\phi^* : P_k^{G_k} \to P_k^{T_k}$ is a map of A-algebras; consequently the map $\phi_* : P_k^{T_k}{}_* \to P_k^{G_k}{}_*$ is a map of coalgebras, so we obtain a commutative diagram of A_*-coalgebras

$$\begin{array}{ccc} P_k^{T_k}{}_* & \underset{\cong}{\overset{\rho}{\to}} & W[k] \\ \phi_* \downarrow & & \downarrow \\ P_k^{G_k}{}_* & \underset{\cong}{\overset{\zeta_*}{\to}} & R[k]. \end{array}$$

It follows that the A_*-action on $W[k]$ must be given via the Nishida relations:

$$Sq_*^r e_i x = \sum \binom{|x| + i = r}{r - 2s} e_{i-r+2s} Sq_*^s x.$$

Applications. The Dickson algebra $P_k^{G_k} = \mathbb{Z}/2\mathbb{Z}[a_{1,k}, \ldots, a_{k,k}]$ has two obvious bases, the basis of monomials $\{a^R = a_{1,k}^{r_1} \ldots a_{k,k}^{r_k} | r_j \geq 0\}$ and the dual basis $Q_I{}^*$ coming from the basis $\{Q_I | I \text{ is admissible}\}$ for $R[k]$. The construction above provides an invariant-theoretic description of the Q_I^*basis. A duality argument shows that if I is admissible then the T_k-invariant v^I determines a unique G_k-invariant, namely $A_I^* = v^I + \Sigma v^J$, the sum being taken over all J such that $Q_J = Q_I$ + others, after applying Adem relations. Moreover, given a non-increasing sequence J one determines the G_k-invariants in which v^J appears as a term by applying Adem relations, that is, if $Q_J = Q_{I_1} + \ldots + Q_{I_\ell}$ for increasing I_j, $j = 1, \ldots, \ell$, then v^J appears as a term only in the G_k-invariants $Q_{I_1}^*, \ldots, Q_{I_\ell}^*$. For example, the one term Adem relations are $e_{\ell+1}e_\ell$, $\ell \geq 0$ and $e_{2\ell+1}e_0, e_{2\ell+2}e_1, \ldots, e_{4\ell-1}r_{2\ell-2}$, $\ell \geq 1$. Consequently, if J has consecutive entries of the form $\ell + 1, \ell$ for $\ell \geq 0$ or $2\ell + m + 1, m$ for $\ell \geq 1$, $0 \leq m \leq 2\ell - 2$ then v^J cannot appear as a term in any G_k-invariant polynomial. The author has not been able to prove these purely invariant-theoretic facts in any other way.

We now want to compute the action of the Milnor primitives Sq^{Δ_r}, which are inductively defined as $Sq^{\Delta_r} = [Sq^{2^{r-1}}, Sq^{\Delta_{r-1}}]$ with $Sq^{\Delta_1} = Sq^1$, on $P_k^{T_k}$. We first reproduce the description of the action of the Sq^{Δ_r} on $P_k^{G_k}$ from [1] (we include the proof for completeness). It is well-known that $Sq^{\Delta_j} a_{i,k} = \delta_i^{k-j} a_{k,k}$, $j < k$, where

$$\delta_j^{k-j} = \begin{cases} 0, & k - j \neq i \\ 1, & k - j = i \end{cases}$$

and that $Sq^{\Delta_k} a_{i,k} = a_{i,k} a_{k,k}$ (see, for example, [7], Corollary 2.3(b), p. 425).

THEOREM 3: $Sq^{\Delta_{k+s}} a_{i,k} = Q_I^*$ *for*

$$I = (\overbrace{1, \ldots, 1}^{k-i}, \overbrace{2, \ldots, 2, 2^{s+1}}^{i}), \qquad s \geq 0,\ 1 \leq j \leq k,\ k \geq 1.$$

Note that $Q_I^* = a_{1,k}^{2^{r+1}-2} a_{i,k} a_{k,k}$ + *others, where* $r = k + s$.

PROOF. Write the I of the theorem as $I(s)$, and induct on s. When $s = 0$ we have

$$Sq^{\Delta_k}a_{i,k} = a_{i,k}a_{k,k} = Q^*_{(0,\ldots,0,1)}Q^*_{(1,\ldots,1)} = Q^*_{(1,\ldots,1,2,\ldots,2)} = Q^*_{I(0)}.$$

So assume $Sq^{\Delta_{k+s}}a_{i,k} = Q^*_{I(s)}$ and write $k + s = r$; now

$$Sq^{\Delta_{r+1}}a_{j,k} = Sq^{2^r}Sq^{\Delta_r}a_{j,k} + Sq^{\Delta_r}Sq^{2^r}a_{j,k} = Sq^{2^r}Q^*_{I(s)}$$

by induction. So we have to show that $Sq^{2^r}Q^*_{I(s)} = Q^*_{I(s+1)}$. By duality, it is sufficient to show that $Q_{I(s+1)}$ is the only element mapped to $Q_{I(s)}$ under $Sq^{2^r}{}_*$ and since we are in $R[k]$ we need only consider admissibles. So suppose that $J = (j_1, \ldots, j_k)$ is admissible and that $Sq^{2^r}Q_J = Q_{I(s)}$. Write x_ℓ for $Q_{(j_\ell,\ldots,j_1)}$ so that $x_1 = Q_J$ and note that $|Q_J| = |Q_{I(s+1)}| = 2^{r+1} - 1 + 2^k - 2^{k-i}$. Repeated applications of the Nishida relations yield

$$Sq^{2^r}_* Q_J = \Sigma\, c_1 \ldots c_k Q_{j_1-2^r+2r_2}Q_{j_2-r_2+2r_3} \ldots Q_{j_k-r_k},$$

where

$$c_i = \binom{|x_{i+1}| + j_i - r_i}{r_i - 2r_{i+1}}, \qquad i = 1, \ldots, k$$

with $r_1 = 2^r$ and $r_{k+1} = 0 = |x_{k+1}|$.

Let's note right away that, for $c_1 \equiv \ldots \equiv c_k \equiv 1 \pmod 2$, we must have $2r_{\ell+1} \leq r_\ell$, for $\ell = 1, \ldots k - 1$ with $r_1 = 2^r$ so that $r_\ell \leq 2^{r-\ell+1}$ for $\ell = 1, \ldots, k$. We also require $j_1 \leq j_2 \leq \ldots \leq j_k$ and

$$j_\ell - r_\ell + 2r_{\ell+1} = \begin{cases} 1, & 1 \leq \ell \leq k - i \quad, \\ 2, & k - i + 1 \leq \ell \leq k - 1, \end{cases}$$

and $j_k - r_k = 2^{s+1}$. Hence

$$c_k = \binom{2^{s+1}}{r_k}$$

so that $r_k = 0,\ 2^{s+1}$ for $c_k \equiv 1 \pmod 2$.

CASE (i). $r_k = 2^{s+1}$. Then $j_k = 2^{s+2}$ and r_{k-1} is forced to be 2^{s+2} for $c_{k-1} \equiv 1 \pmod 2$. The same argument forces $r_\ell = 2^{r-\ell+1}$ for $1 \leq \ell \leq k - 2$, so that

$$j_1 = 1, \ldots, j_{k-1} = 1, j_{k-i+1} = 2, \ldots, j_{k-1} = 2, j_k = 2^{s+2}$$

and $J = I(s + 1)$.

CASE (ii). $r_k = 0$. Then $j_k = 2^{s+1}$ *for* $c_k \equiv 1 \pmod 2$. Now, if $c_1 \equiv 1 \pmod 2$, we must have $|x_2| + j_1 \geq 2^r$ but since J is admissable,

$$\begin{aligned} j_1 + |x_2| &\leq 2^{s+1} + 2^{s+1} + \ldots + 2^{k-2}(2^{s+1}) \\ &= 2^{s+1} + 2^{s+1} + 2^{s+2} + \ldots + 2^{r-1} = 2^r, \end{aligned}$$

so that $j_1 + |x_2| = 2^r$ and $j_\ell = 2^{s+1}$ for $1 \leq \ell \leq k$. But such a Q_J can never give $Q_{I(s)}$ since, for example, we need $j_{k-1} - r_{k-1} = 2$ and thus

$$c_{k-1} = \binom{2^{s+1}+2}{2^{s+1}-2} \equiv 0 \pmod 2.$$

Note that the above arguments work with minor changes required for $i = 1$ or $i = k$. □

This theorem also computes the action of the Sq^{Δ_r} on $P_k^{T_k}$ since $a_{1,k} = v_k + v_{k-1}^2 + \ldots + v_1^{2^{k-2}}$ so that $Sq^{\Delta_r} a_{1,k} = Sq^{\Delta_r} v_k$ since Sq^{Δ_r} is a derivation. Consequently, combining theorems 2 and 3 we have

THEOREM 4.

$$Sq^{\Delta_r} v_k = \begin{cases} 0 & , \quad r < k-1 \\ a_{k,k} & , \quad r = k-1 \\ a_{1,k} a_{k,k} & , \quad r = k \\ Q^*_{(1,\ldots,1,2^{s+1})}, & \quad r = k+s,\ s > 0. \end{cases}$$

□

Of course, we have that

$$Q^*_{(1,\ldots,1,2^{s+1})} = v_1 \ldots v_{k-1} v_k^{2^{s+1}} + \Sigma\, v^J,$$

the sum over all J such that $Q_J = Q_{(1,\ldots,1,2^{s+1})}$ + others, after applying Adem relations.

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DEPARTMENT OF MATHEMATICS AND STATISTICS
QUEEN'S UNIVERSITY
KINGSTON, ONTARIO
CANADA K7L 3N6