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Existence of singular rotationally symmetric gradient Ricci solitons in higher dimensions

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Abstract. By using fixed point argument, we give a proof for the existence of singular rotationally symmetric steady and expanding gradient Ricci solitons in higher dimensions with metric $g = \frac{da^2}{h(a^2)} + a^2 g_{S^n}$ for some function h where g_{S^n} is the standard metric on the unit sphere S^n in $\mathbb{R}^n$ for any $n \geq 2$. More precisely, for any $\lambda \geq 0$ and $c_0 > 0$, we prove that there exist infinitely many solutions $h \in C^2((0,\infty);\mathbb{R}^+)$ for the equation $2r^2 h(r)h_{rr}(r) = (n-1)h(r)(h(r)-1) + rh_r(r)(rh_r(r) - \lambda r - (n-1))$, $h(r) > 0$, in $(0,\infty)$ satisfying $\lim_{r\to 0} r^{\sqrt{n}-1} h(r) = c_0$ and prove the higher-order asymptotic behavior of the global singular solutions near the origin. We also find conditions for the existence of unique global singular solution of such equation in terms of its asymptotic behavior near the origin.

1 Introduction

Ricci flow is an important technique in geometry and has a lot of applications in geometry [10, 12, 14, 15]. For example, recently, Perelman [14, 15] used Ricci flow to prove the Poincaré conjecture. In the study of Ricci flow, one is interested to study the Ricci solitons which are self-similar solutions of Ricci flow. On the other hand, by a limiting argument, the behavior of the Ricci flow near the singular time is usually similar to the behavior of Ricci solitons.

Hence, in order to understand Ricci flow, it is important to study the Ricci solitons. In [3], Brendle used singular rotationally symmetric steady solitons to construct barrier functions which plays an important role in the proof there that confirms a conjecture of Perelman on three-dimensional ancient κ solution to the Ricci flow. We refer the reader to the papers by Alexakis, Chen, and Fournodavlos [1], Brendle [2], Bryant [4], Cao and Zhou [5, 6], Feldman, Ilmanen, and Knopf [8], Hsu [9], Li and Wang [11], Munteanu and Sesum [13], Petersen and Wylie [16], and so forth and the book [7] by Chow et al. for some recent results on Ricci solitons.

We say that a Riemannian metric $g = (g_{ij})$ on a Riemannian manifold M is a gradient Ricci soliton if there exist a smooth function f on M and a constant $\lambda \in \mathbb{R}$ such that the Ricci curvature R_{ij} of the metric g satisfies

$$R_{ij} = \nabla_i \nabla_j f - \lambda g_{ij} \quad \text{on } M. \tag{1.1}$$

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The gradient soliton is called an expanding gradient Ricci soliton if $\lambda > 0$. It is called a steady gradient Ricci soliton if $\lambda = 0$ and it is called a shrinking gradient Ricci soliton if $\lambda < 0$.

Existence of rotationally symmetric steady and expanding three-dimensional gradient Ricci solitons were proved by Bryant [4] using the phase method and by Hsu [9] using fixed-point argument. On the other hand, as observed by Bryant [4] for $n = 2$ and Chow et al. (cf. Lemma 1.21 and Section 4 of Chapter 1 of [7]) for $n \geq 2$, for any $n \geq 2$, if (M, g) is an $(n+1)$-dimensional rotational symmetric gradient Ricci soliton which satisfies (1.1) for some smooth function f and constant $\lambda \in \mathbb{R}$ with

$$g = dt^2 + a(t)^2 g_{S^n}, \tag{1.2}$$

where g_{S^n} is the standard metric on the unit sphere S^n in $\mathbb{R}^n$, then the Ricci curvature of g is given by

$$\mathrm{Ric}(g) = -\frac{na_{tt}(t)}{a(t)}\,dt^2 + \left(n - 1 - a(t)a_{tt}(t) - (n-1)a_t(t)^2\right) g_{S^n} \tag{1.3}$$

and

$$\mathrm{Hess}(f) = f_{tt}(t)\,dt^2 + a(t)a_t(t)f_t(t)\,g_{S^n}. \tag{1.4}$$

Hence, by (1.1), (1.3), and (1.4) (cf. [1, 4, 7]), we get

$$-na(t)a_{tt}(t) = a(t)(f_{tt}(t) - \lambda) \tag{1.5}$$

and

$$n - 1 - a(t)a_{tt}(t) - (n-1)a_t(t)^2 = a(t)a_t(t)f_t(t) - \lambda a(t)^2. \tag{1.6}$$

By eliminating f from (1.5) and (1.6), we get that $a(t)$ satisfies

$$\begin{aligned} a(t)^2 a_t(t)a_{ttt}(t) =& a(t)a_t(t)^2 a_{tt}(t) + a(t)^2 a_{tt}(t)^2 - (n-1)a(t)a_{tt}(t) \\ & - \lambda a(t)^3 a_{tt}(t) - (n-1)a_t(t)^2 + (n-1)a_t(t)^4. \end{aligned} \tag{1.7}$$

Note that we can express g as

$$g = \frac{da^2}{h(a^2)} + a^2 g_{S^n}, \tag{1.8}$$

where $h(r)$, $r = a^2 \geq 0$, and $a = a(t)$ satisfies

$$a_t(t) = \sqrt{h(a(t)^2)}. \tag{1.9}$$

Then, by (1.7) and a direct computation, h satisfies

$$2r^2 h(r)h_{rr}(r) = (n-1)h(r)(h(r)-1) + rh_r(r)(rh_r(r) - \lambda r - (n-1)), \quad h(r) > 0. \tag{1.10}$$

We are now interested in rotational symmetric gradient Ricci soliton which blows up at $r = 0$ at the rate

$$\lim_{r \to 0} r^\alpha h(r) = c_0 \tag{1.11}$$

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for some constants $\alpha > 0$ and $c_0 > 0$. Let

$$w(r) = r^{\alpha} h(r) \quad \forall r > 0. \tag{1.12}$$

By (1.10), (1.12), and a direct computation, w satisfies

$$\begin{aligned} 2r^2 w(r) w_{rr}(r) = {} & 2\alpha r w(r) w_r(r) + (n-1)(\alpha - 1) r^{\alpha} w(r) + \alpha \lambda r^{\alpha+1} w(r) \\ & - (n-1) r^{\alpha+1} w_r(r) - \lambda r^{\alpha+2} w_r(r) + r^2 w_r(r)^2 \\ & - (\alpha^2 + 2\alpha - (n-1)) w(r)^2. \end{aligned} \tag{1.13}$$

Unless stated otherwise, we now let $\alpha = \sqrt{n} - 1 > 0$ for the rest of the paper. Then $\alpha^2 + 2\alpha - (n-1) = 0$. Hence, by (1.13), w satisfies

$$\begin{aligned} 2r^2 w(r) w_{rr}(r) = {} & 2\alpha r w(r) w_r(r) + (n-1)(\alpha - 1) r^{\alpha} w(r) + \alpha \lambda r^{\alpha+1} w(r) \\ & - (n-1) r^{\alpha+1} w_r(r) - \lambda r^{\alpha+2} w_r(r) + r^2 w_r(r)^2 \end{aligned} \tag{1.14}$$

with $\alpha = \sqrt{n} - 1 > 0$. We also impose the condition

$$\lim_{t \to 0^+} a(t) = 0. \tag{1.15}$$

Then, by (1.9) and (1.15),

$$t = \int_0^{a(t)} \frac{d\rho}{\sqrt{h(\rho^2)}}. \tag{1.16}$$

In the paper [4], Bryant by using power series expansion around the singular point at the origin gave the local existence of singular solution of (1.10) near the origin which blows up at the rate (1.11) for the case $n = 2$. On the other hand, by using phase plane analysis of the functions

$$W = \frac{1}{f_t(t) + n \frac{a_t(t)}{a(t)}}, \quad X = \sqrt{n} W \frac{a_t(t)}{a(t)}, \quad Y = \frac{\sqrt{(n-1)} W}{a(t)},$$

Alexakis, Chen, and Fournodavlos [1] gave a sketch of proof for the local existence of singular solution $(a(t), f(t))$, of (1.5) and (1.6), near the origin and its asymptotic behavior as $t \to 0^+$ for the case $n \geq 2$. When $\lambda = 0$, the existence of global solution $(a(t), f(t))$, of (1.5) and (1.6), in $(0, \infty)$ is also mentioned without detailed proof in [1].

In this paper, we will use fixed-point argument for the function w given by (1.12) to give a new proof of the local existence of solution h of (1.10) satisfying (1.11) for any constants $\lambda \in \mathbb{R}$, $c_0 > 0$, and $2 \leq n \in \mathbb{Z}^+$. For $\lambda \geq 0$, we will then use a continuation method to extend the local singular solutions of (1.10) and (1.11) to global solutions of (1.10) and (1.11). We will also prove the higher-order asymptotic behavior of the local solutions of (1.10) and (1.11), near the origin.

The main results we obtain in this paper are the following.

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Theorem 1.1 *Let $2 \le n \in \mathbb{Z}^+$, $\lambda \ge 0$, $\alpha = \sqrt{n} - 1$, $c_0 > 0$, $c_1 \in \mathbb{R}$, and*

$$c_2 := \frac{(n-1)(\alpha-1)}{2} = \frac{(n-1)(\sqrt{n}-2)}{2}. \tag{1.17}$$

There exists a unique solution $h \in C^2((0,\infty))$ of (1.10) in $(0,\infty)$ which satisfies (1.11) and (2.4) in $(0,\varepsilon)$ with w given by (1.12) for some constant $\varepsilon > 0$.

Theorem 1.2 *Let $4 < n \in \mathbb{Z}^+$, $\lambda \ge 0$, $\alpha = \sqrt{n} - 1$, $c_0 > 0$, $c_1 \in \mathbb{R}$, and c_2 be given by (1.17). Then there exists a constant $0 < \delta_0 < 1$ such that (1.10) has a unique solution $h \in C^2((0,\infty))$ in $(0,\infty)$ which satisfies (1.11) and*

$$\begin{aligned} h(r) = &\frac{1}{r^\alpha}\left\{c_0 - \frac{c_2}{\alpha}r^\alpha + \left(\frac{c_1}{\alpha+1} - \frac{\alpha\lambda}{2(\alpha+1)^2}\right)r^{\alpha+1} + \frac{\alpha\lambda}{2\alpha+2}r^{\alpha+1}\log r\right. \\ &\left. + \frac{c_2^2 + (n-1)c_2}{4c_0\alpha(\alpha-1)}r^{2\alpha} + o(1)r^{2\alpha}\right\} \quad \forall 0 < r \le \delta_0. \end{aligned} \tag{1.18}$$

Moreover,

$$\begin{aligned} h_r(r) = &\frac{1}{r^{\alpha+1}}\left\{-\alpha c_0 + \left(\frac{c_1}{\alpha+1} + \frac{\alpha^2\lambda}{2(\alpha+1)^2}\right)r^{\alpha+1} + \frac{\alpha\lambda}{2\alpha+2}r^{\alpha+1}\log r\right. \\ &\left. + \frac{c_2^2 + (n-1)c_2}{4c_0(\alpha-1)}r^{2\alpha} + o(1)r^{2\alpha}\right\} \quad \forall 0 < r \le \delta_0. \end{aligned} \tag{1.19}$$

Theorem 1.3 *Let $n \in \{2,3,4\}$, $\alpha = \sqrt{n} - 1$, $\lambda \ge 0$, $c_0 > 0$, $c_1 \in \mathbb{R}$, and c_2 be given by (1.17). Let $h \in C^2((0,\infty))$ be given by Theorem 1.1. Then there exists a constant $0 < \delta_0 < 1$ such that*

$$h(r) = \begin{cases} \dfrac{1}{r^\alpha}\left(c_0 - \dfrac{c_2}{\alpha}r^\alpha - \dfrac{c_2(c_2+n-1)}{4c_0\alpha(1-\alpha)}r^{2\alpha} + o(1)r^{2\alpha}\right), & \forall 0 < r \le \delta_0 \ \text{ if } n = 2,3, \\ \dfrac{1}{r}\left(c_0 + \dfrac{\lambda}{4}r^2\log r + o(1)r^2|\log r|\right), & \forall 0 < r \le \delta_0 \quad \text{if } n = 4. \end{cases} \tag{1.20}$$

Moreover,

$$h_r(r) = \begin{cases} \dfrac{1}{r^{\alpha+1}}\left(-\alpha c_0 - \dfrac{c_2(c_2+n-1)}{4c_0(1-\alpha)}r^{2\alpha} + o(1)r^{2\alpha}\right), & \forall 0 < r \le \delta_0 \ \text{ if } n = 2,3, \\ \dfrac{1}{r^2}\left(-c_0 + \dfrac{\lambda}{4}r^2\log r + o(1)r^2|\log r|\right), & \forall 0 < r \le \delta_0 \quad \text{if } n = 4. \end{cases} \tag{1.21}$$

Note that the singular solutions h of (1.10) in $(0,\infty)$ given by Theorems 1.1–1.3 satisfy (1.11) with $\alpha = \sqrt{n} - 1$. Moreover, by (1.2), the solitons constructed in Theorems 1.1 and 1.2 are complete at $t = +\infty$. A natural question to ask is that does there exist any other singular solution of (1.10) in $(0,\varepsilon)$ for some constant $\varepsilon > 0$ which blow-up at a different rate at the origin. We answer this question in the negative. More precisely, we prove the following result.

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Theorem 1.4 *Let $2 \le n \in \mathbb{Z}^+$, $\lambda \in \mathbb{R}$, $\varepsilon > 0$, and $c_0 > 0$. Suppose $h \in C^2((0,\varepsilon))$ is a solution of (1.10) in $(0,\varepsilon)$ which satisfies (1.11) for some constant $\alpha > 0$. Then $\alpha = \sqrt{n} - 1$.*

The plan of the paper is as follows. In Section 2, we will prove the local existence of infinitely many singular solutions of (1.10) and (1.11), in a neighborhood of the origin, and conditions for uniqueness of local singular solutions are given. We will also prove the higher-order asymptotic behavior of these local solutions near the origin. In Section 3, we will prove the global existence of infinitely many singular solutions of (1.10) and (1.11) and conditions for uniqueness of global singular solution are given. In Section 4, we will prove the asymptotic behavior of $a(t)$ near the origin.

2 Local existence, uniqueness, and asymptotic behavior of singular solutions near the origin

In this section, for any $2 \le n \in \mathbb{Z}^+$, $\lambda \in \mathbb{R}$, and $c_0 > 0$, we will prove the local existence of infinitely many singular solutions of (1.10) in $(0,\varepsilon)$ which satisfy (1.11) for some constant $\varepsilon > 0$. Under some mild conditions on the singular solutions of (1.10) in $(0,\varepsilon)$, we will also prove the uniqueness of local singular solutions of (1.10) in $(0,\varepsilon)$ satisfying (1.11). We first observe that if $h \in C^2((0,\varepsilon];\mathbb{R}^+)$ is a solution of (1.10) in $(0,\varepsilon]$ for some constant $\varepsilon > 0$ which satisfies (1.11) for some constant $c_0 > 0$ and w is given by (1.12) with $\alpha = \sqrt{n} - 1$, then by (1.11), (1.12), and (1.14), $w > 0$ satisfies

$$\begin{aligned} w_{rr}(r) =& \frac{\alpha}{r} w_r(r) + \frac{(n-1)(\alpha-1)}{2} r^{\alpha-2} + \frac{\alpha\lambda}{2} r^{\alpha-1} - \frac{(n-1) r^{\alpha-1} w_r(r)}{2w(r)} \\ & - \frac{\lambda r^\alpha w_r(r)}{2w(r)} + \frac{w_r(r)^2}{2w(r)} \end{aligned} \tag{2.1}$$

in $(0,\varepsilon]$ and

$$w(0) = c_0 \tag{2.2}$$

if $w \in C([0,\varepsilon];\mathbb{R}^+)$. Hence, the existence of solution $h \in C^2((0,\varepsilon];\mathbb{R}^+)$ of (1.10) in $(0,\varepsilon]$ which satisfies (1.11) is equivalent to the existence of solution $w \in C^2((0,\varepsilon];\mathbb{R}^+) \cap C([0,\varepsilon];\mathbb{R}^+)$ of (2.1) in $(0,\varepsilon]$ which satisfies (2.2). Note that (2.1) is equivalent to

$$\begin{aligned} & (r^{-\alpha} w_r)_r(r) = c_2 r^{-2} + \frac{\alpha\lambda}{2} r^{-1} - \frac{(n-1) r^{-1} w_r(r)}{2w(r)} - \frac{\lambda w_r(r)}{2w(r)} + \frac{r^{-\alpha} w_r(r)^2}{2w(r)} \qquad (2.3) \\ \Leftrightarrow \quad & r^{-\alpha} w_r(r) = -c_2 r^{-1} + c_1 + \frac{\alpha\lambda}{2} \log r + \frac{(n-1)}{2} \int_r^\varepsilon \frac{\rho^{-1} w_r(\rho)}{w(\rho)}\, d\rho \\ & \qquad - \frac{\lambda}{2} \int_0^r \frac{w_r(\rho)}{w(\rho)}\, d\rho - \frac{1}{2} \int_r^\varepsilon \frac{\rho^{-\alpha} w_r(\rho)^2}{w(\rho)}\, d\rho \quad \forall 0 < r \le \varepsilon \\ \Leftrightarrow \quad & w_r(r) = -c_2 r^{\alpha-1} + c_1 r^\alpha + \frac{\alpha\lambda}{2} r^\alpha \log r + r^\alpha \left\{ \frac{(n-1)}{2} \int_r^\varepsilon \frac{\rho^{-1} w_r(\rho)}{w(\rho)}\, d\rho \right. \\ & \left. \qquad - \frac{\lambda}{2} \int_0^r \frac{w_r(\rho)}{w(\rho)}\, d\rho - \frac{1}{2} \int_r^\varepsilon \frac{\rho^{-\alpha} w_r(\rho)^2}{w(\rho)}\, d\rho \right\} \quad \forall 0 < r \le \varepsilon \qquad (2.4) \end{aligned}$$

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for some constant $c_1 \in \mathbb{R}$. This suggests one to use fixed-point argument to prove the existence of solution $w \in C^2((0,\varepsilon];\mathbb{R}^+) \cap C([0,\varepsilon];\mathbb{R}^+)$ of (2.1) in $(0,\varepsilon]$ which satisfies (2.2).

Proposition *Let $2 \le n \in \mathbb{Z}^+$, $\alpha = \sqrt{n}-1$, $\lambda, c_1 \in \mathbb{R}$, $c_0 > 0$, and let c_2 be given by (1.17). Then there exists a constant $0 < \varepsilon < 1$ such that (2.1) has a unique solution $w \in C^2((0,\varepsilon];\mathbb{R}^+) \cap C([0,\varepsilon];\mathbb{R}^+)$ in $(0,\varepsilon]$ which satisfies (2.2) and (2.4). Moreover,*

$$\lim_{r\to 0^+} r^{1-\alpha} w_r(r) = -c_2 \tag{2.5}$$

holds.

Proof For any $\varepsilon > 0$, we define the Banach space

$$\begin{aligned}\mathcal{X}_\varepsilon :=& \{(w,v) : w \in C([0,\varepsilon];\mathbb{R}^+), v \in C((0,\varepsilon];\mathbb{R}) \text{ such that } r^{1-\alpha}v(r) \text{ can be}\\ & \text{extended to a function in } C([0,\varepsilon];\mathbb{R})\}\end{aligned}$$

with a norm given by

$$\|(w,v)\|_{\mathcal{X}_\varepsilon} = \max\left(\|w\|_{L^\infty([0,\varepsilon])}, \|r^{1-\alpha}v(r)\|_{L^\infty([0,\varepsilon])}\right).$$

For any $(w,v) \in \mathcal{X}_\varepsilon$, we define

$$\Phi(w,v) := (\Phi_1(w,v), \Phi_2(w,v)),$$

where

$$\begin{cases} \Phi_1(w,v)(r) = c_0 + \displaystyle\int_0^r v(\rho)\,d\rho, \\ \Phi_2(w,v)(r) = -c_2 r^{\alpha-1} + c_1 r^\alpha + \dfrac{\alpha\lambda}{2} r^\alpha \log r + r^\alpha \left\{ \dfrac{(n-1)}{2} \displaystyle\int_r^\varepsilon \frac{\rho^{-1}v(\rho)}{w(\rho)}\,d\rho \right. \\ \qquad\qquad \left. -\dfrac{\lambda}{2}\displaystyle\int_0^r \frac{v(\rho)}{w(\rho)}\,d\rho - \frac{1}{2}\int_r^\varepsilon \frac{\rho^{-\alpha}v(\rho)^2}{w(\rho)}\,d\rho \right\} \end{cases} \tag{2.6}$$

for any $0 < r \le \varepsilon$. Let

$$\mathcal{D}_\varepsilon := \left\{ \|(w,v) - (c_0, -c_2 r^{\alpha-1})\|_{\mathcal{X}_\varepsilon} \le c_0/10 \right\}. \tag{2.7}$$

Since $(c_0, -c_2 r^{\alpha-1}) \in \mathcal{D}_\varepsilon$, $\mathcal{D}_\varepsilon \neq \phi$. We will show that there exists $\varepsilon \in (0,1/2)$ such that the map $(w,v) \mapsto \Phi(w,v)$ has a unique fixed point in the closed subspace $\mathcal{D}_\varepsilon$. Let

$$\varepsilon_1 = \min\left(\frac{1}{2}, \left(\frac{c_0\alpha}{10|c_2| + c_0}\right)^{1/\alpha}\right).$$

We first prove that $\Phi(\mathcal{D}_\varepsilon) \subset \mathcal{D}_\varepsilon$ for sufficiently small $\varepsilon \in (0,\varepsilon_1)$. For any $\varepsilon \in (0,\varepsilon_1)$, $(w,v) \in \mathcal{D}_\varepsilon$, $0 \le r < \varepsilon$, by (2.7), we have

$$\frac{9c_0}{10} \le w(r) \le \frac{11c_0}{10} \quad \forall 0 < r \le \varepsilon \tag{2.8}$$

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and

$$|v(r)| \le c_3 r^{\alpha-1} \quad \forall 0 < r \le \varepsilon, \tag{2.9}$$

where $c_3 = |c_2| + (c_0/10)$. Hence, by (2.9),

$$\begin{aligned} &|\Phi_1(w,v)(r) - c_0| \le \int_0^r c_3 \rho^{\alpha-1}\, d\rho = (c_3/\alpha) r^\alpha \le (c_3/\alpha)\varepsilon^\alpha \le c_0/10 \quad \forall 0 < r \le \varepsilon \\ &\Rightarrow \quad \|\Phi_1(w,v) - c_0\|_{L^\infty([0,\varepsilon])} \le c_0/10 \quad \text{and} \quad \|\Phi_1(w,v)\|_{L^\infty([0,\varepsilon])} \le 11c_0/10. \end{aligned} \tag{2.10}$$

We now choose $c_4 > 1$ such that

$$|\log r| \le c_4 r^{-1/2} \quad \forall 0 < r \le 1/2. \tag{2.11}$$

Then, by (2.8), (2.9), and (2.11), for any $0 < r \le \varepsilon$,

$$\begin{aligned} r\left|\int_r^\varepsilon \frac{\rho^{-1} v(\rho)}{w(\rho)}\, d\rho\right| &\le \frac{10c_3 r}{9c_0} \int_r^\varepsilon \rho^{\alpha-2}\, d\rho \le \begin{cases} \dfrac{10c_3 r(r^{\alpha-1} + \varepsilon^{\alpha-1})}{9c_0|\alpha-1|}, & \text{if } n \ne 4 \\ \dfrac{10c_3 r(|\log r| + |\log \varepsilon|)}{9c_0}, & \text{if } n = 4 \end{cases} \\ &\le \begin{cases} \dfrac{20c_3\varepsilon^\alpha}{9c_0|\alpha-1|}, & \text{if } n \ne 4, \\ \dfrac{20c_3 c_4 \varepsilon^{1/2}}{9c_0}, & \text{if } n = 4, \end{cases} \end{aligned} \tag{2.12}$$

$$r\left|\int_0^r \frac{v(\rho)}{w(\rho)}\, d\rho\right| \le \frac{10c_3 r}{9c_0} \int_0^r \rho^{\alpha-1}\, d\rho = \frac{10c_3 r^{\alpha+1}}{9c_0\alpha} \le \frac{10c_3\varepsilon^{\alpha+1}}{9c_0\alpha} \tag{2.13}$$

and

$$\begin{aligned} r\left|\int_r^\varepsilon \frac{\rho^{-\alpha} v(\rho)^2}{w(\rho)}\, d\rho\right| &\le \frac{10c_3^2 r}{9c_0} \int_r^\varepsilon \rho^{\alpha-2}\, d\rho \le \begin{cases} \dfrac{10c_3^2 r(r^{\alpha-1} + \varepsilon^{\alpha-1})}{9c_0|\alpha-1|}, & \text{if } n \ne 4, \\ \dfrac{20c_3^2 r(|\log r| + |\log \varepsilon|)}{9c_0}, & \text{if } n = 4, \end{cases} \\ &\le \begin{cases} \dfrac{20c_3^2\varepsilon^\alpha}{9c_0|\alpha-1|}, & \text{if } n \ne 4, \\ \dfrac{20c_3^2 c_4 \varepsilon^{1/2}}{9c_0}, & \text{if } n = 4. \end{cases} \end{aligned} \tag{2.14}$$

Let

$$c_5 = \begin{cases} \dfrac{4n(c_3 + c_3^2)}{c_0|\alpha-1|} + \dfrac{c_3|\lambda|}{c_0\alpha}, & \text{if } n \ne 4, \\ \dfrac{4nc_4(c_3 + c_3^2)}{c_0} + \dfrac{c_3|\lambda|}{c_0\alpha}, & \text{if } n = 4. \end{cases}$$

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By (2.6) and (2.11)–(2.14),

$$\begin{aligned}
&r^{1-\alpha}\left|\Phi_2(w,v)(r)+c_2r^{\alpha-1}\right|\\
\le&|c_1|r+\frac{\alpha|\lambda|}{2}r|\log r|+\frac{(n-1)r}{2}\left|\int_r^\varepsilon\frac{\rho^{-1}v(\rho)}{w(\rho)}\,d\rho\right|+\frac{|\lambda|r}{2}\left|\int_0^r\frac{v(\rho)}{w(\rho)}\,d\rho\right|+\frac{r}{2}\left|\int_r^\varepsilon\frac{\rho^{-\alpha}v(\rho)^2}{w(\rho)}\,d\rho\right| \qquad (2.15)\\
\le&|c_1|\varepsilon+\frac{\alpha|\lambda|c_4}{2}\varepsilon^{1/2}+c_5(\varepsilon^\alpha+\varepsilon^{1/2})\quad\forall 0<r\le\varepsilon.
\end{aligned}$$

Let

$$\varepsilon_2=\min\left(\varepsilon_1,\frac{c_0}{30(|c_1|+1)},\left(\frac{c_0}{30c_5}\right)^{\frac{1}{\alpha}},\frac{c_0^2}{900(\alpha|\lambda|c_4+c_5)^2}\right)$$

and $\varepsilon\in(0,\varepsilon_2)$. Then, by (2.15),

$$\begin{aligned}
&r^{1-\alpha}\left|\Phi_2(w,v)(r)+c_2r^{\alpha-1}\right|\le c_0/10\quad\forall 0<r\le\varepsilon\\
\Rightarrow\quad&\|r^{1-\alpha}(\Phi_2(w,v)(r)+c_2r^{\alpha-1})\|_{L^\infty([0,\varepsilon])}\le c_0/10. \qquad (2.16)
\end{aligned}$$

By (2.10) and (2.16),

$$\|\Phi(w,v)-(c_0,-c_2r^{\alpha-1})\|_{\mathcal{X}_\varepsilon}\le c_0/10. \qquad (2.17)$$

Hence, $\Phi(\mathcal{D}_\varepsilon)\subset\mathcal{D}_\varepsilon$. Let $(w_1,v_1),(w_2,v_2)\in\mathcal{D}_\varepsilon$, $0<\varepsilon<\varepsilon_2$, $\delta_1=\|(w_1,v_1)-(w_2,v_2)\|_{\mathcal{X}_\varepsilon}$. Then

$$\frac{9c_0}{10}\le w_i(r)\le\frac{11c_0}{10}\quad\forall 0<r\le\varepsilon,\ i=1,2 \qquad (2.18)$$

and

$$|v_i(r)|\le c_3r^{\alpha-1}\quad\forall 0<r\le\varepsilon,\ i=1,2. \qquad (2.19)$$

Now

$$\begin{aligned}
|\Phi_1(w_1,v_1)(r)-\Phi_1(w_2,v_2)(r)|\le&\int_0^r|v_1(\rho)-v_2(\rho)|\,d\rho\\
\le&\|r^{1-\alpha}(v_1(r)-v_2(r))\|_{L^\infty([0,\varepsilon])}\int_0^r\rho^{\alpha-1}\,d\rho\\
\le&(\delta_1/\alpha)\varepsilon^\alpha\quad\forall 0<r\le\varepsilon, \qquad (2.20)
\end{aligned}$$

and by (2.11), (2.18), and (2.19), for any $0<r\le\varepsilon$,

$$\begin{aligned}
&r\left|\int_r^\varepsilon\frac{\rho^{-1}v_1(\rho)}{w_1(\rho)}\,d\rho-\int_r^\varepsilon\frac{\rho^{-1}v_2(\rho)}{w_2(\rho)}\,d\rho\right|\\
\le&r\int_r^\varepsilon\frac{\rho^{-1}|v_1(\rho)-v_2(\rho)|}{w_1(\rho)}\,d\rho+r\int_r^\varepsilon\rho^{-1}|v_2(\rho)|\left|\frac{1}{w_1(\rho)}-\frac{1}{w_2(\rho)}\right|d\rho\\
\le&\frac{10\|\rho^{1-\alpha}|v_1-v_2|(\rho)\|_{L^\infty([0,\varepsilon])}r}{9c_0}\int_r^\varepsilon\rho^{\alpha-2}\,d\rho\\
&+\frac{100c_3\|w_1-w_2\|_{L^\infty([0,\varepsilon])}r}{81c_0^2}\int_r^\varepsilon\rho^{\alpha-2}\,d\rho
\end{aligned}$$

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$$\leq \begin{cases} \left(\dfrac{10}{9c_0|\alpha-1|}+\dfrac{100c_3}{81c_0^2|\alpha-1|}\right)\delta_1 r(r^{\alpha-1}+\varepsilon^{\alpha-1}), & \text{if } n\neq 4 \\ \left(\dfrac{10}{9c_0}+\dfrac{100c_3}{81c_0^2}\right)\delta_1 r|\log r|, & \text{if } n=4 \end{cases}$$

$$\leq \begin{cases} \dfrac{c_6\delta_1\varepsilon^\alpha}{|\alpha-1|}, & \text{if } n\neq 4, \\ c_4c_6\delta_1\varepsilon^{1/2}, & \text{if } n=4, \end{cases} \tag{2.21}$$

where

$$c_6=\frac{20}{9c_0}+\frac{200c_3}{81c_0^2},$$

$$\begin{aligned} &r\left|\int_0^r\frac{\nu_1(\rho)}{w_1(\rho)}\,d\rho-\int_0^r\frac{\nu_2(\rho)}{w_2(\rho)}\,d\rho\right| \\ \leq& r\int_0^r\frac{|\nu_1(\rho)-\nu_2(\rho)|}{w_1(\rho)}\,d\rho+r\int_0^r|\nu_2(\rho)|\left|\frac{1}{w_1(\rho)}-\frac{1}{w_2(\rho)}\right|d\rho \\ \leq& \frac{10\|\rho^{1-\alpha}|\nu_1-\nu_2|(\rho)\|_{L^\infty([0,\varepsilon])}r}{9c_0}\int_0^r\rho^{\alpha-1}\,d\rho \\ &+\frac{100c_3\|w_1-w_2\|_{L^\infty([0,\varepsilon])}r}{81c_0^2}\int_0^r\rho^{\alpha-1}\,d\rho \\ =&\left(\frac{10}{9c_0\alpha}+\frac{100c_3}{81c_0^2\alpha}\right)\delta_1 r^{\alpha+1}\leq\frac{c_6\delta_1\varepsilon^{\alpha+1}}{\alpha} \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} &r\left|\int_r^\varepsilon\frac{\rho^{-\alpha}\nu_1(\rho)^2}{w_1(\rho)}\,d\rho-\int_r^\varepsilon\frac{\rho^{-\alpha}\nu_2(\rho)^2}{w_2(\rho)}\,d\rho\right| \\ \leq& r\int_r^\varepsilon\frac{\rho^{-\alpha}|\nu_1(\rho)-\nu_2(\rho)|(|\nu_1(\rho)|+|\nu_2(\rho)|)}{w_1(\rho)}\,d\rho \\ &+r\int_r^\varepsilon\rho^{-\alpha}|\nu_2(\rho)|^2\left|\frac{1}{w_1(\rho)}-\frac{1}{w_2(\rho)}\right|d\rho \\ \leq& \frac{20c_3\|\rho^{1-\alpha}|\nu_1-\nu_2|(\rho)\|_{L^\infty([0,\varepsilon])}r}{9c_0}\int_r^\varepsilon\rho^{\alpha-2}\,d\rho \\ &+\frac{100c_3^2\|w_1-w_2\|_{L^\infty([0,\varepsilon])}r}{81c_0^2}\int_r^\varepsilon\rho^{\alpha-2}\,d\rho \\ \leq& \begin{cases} \left(\dfrac{20c_3}{9c_0|\alpha-1|}+\dfrac{100c_3^2}{81c_0^2|\alpha-1|}\right)\delta_1 r(r^{\alpha-1}+\varepsilon^{\alpha-1}), & \text{if } n\neq 4 \\ \left(\dfrac{20c_3}{9c_0}+\dfrac{100c_3^2}{81c_0^2}\right)\delta_1 r|\log r|, & \text{if } n=4 \end{cases} \end{aligned}$$

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$$\leq \begin{cases} \dfrac{2c_3c_6\delta_1\varepsilon^{\alpha}}{|\alpha-1|}, & \text{if } n \neq 4, \\ 2c_3c_4c_6\delta_1\varepsilon^{1/2}, & \text{if } n = 4. \end{cases} \tag{2.23}$$

By (2.6) and (2.21)–(2.23),

$$r^{1-\alpha}|\Phi_2(w_1,v_1)(r) - \Phi_2(w_2,v_2)(r)| \leq c_7\delta_1(\varepsilon^{\alpha} + \varepsilon^{1/2}) \quad \forall 0 < r \leq \varepsilon, \tag{2.24}$$

where

$$c_7 = \begin{cases} c_6\left(\dfrac{n(1+c_3)}{|\alpha-1|} + \dfrac{|\lambda|}{\alpha}\right), & \text{if } n \neq 4, \\ c_6\left(nc_4(1+c_3) + \dfrac{|\lambda|}{\alpha}\right), & \text{if } n = 4. \end{cases}$$

Let

$$\varepsilon_3 = \min\left(\varepsilon_2, (\alpha/6)^{1/\alpha}, (6c_7)^{-1/\alpha}, (6c_7)^{-2}\right)$$

and $0 < \varepsilon < \varepsilon_3$. By (2.20) and (2.24),

$$\|\Phi(w_1,v_1) - \Phi(w_2,v_2)\|_{\mathfrak{X}_\varepsilon} \leq \delta_1/2. \tag{2.25}$$

Hence, Φ is a contraction map on $\mathcal{D}_\varepsilon$. Therefore, by the contraction map theorem, there exists a unique fixed point $(w,v) = \Phi(w,v)$ in $\mathcal{D}_\varepsilon$. Thus,

$$\begin{cases} w(r) = c_0 + \displaystyle\int_0^r v(\rho)\,d\rho, \\ v(r) = -c_2r^{\alpha-1} + c_1r^{\alpha} + \dfrac{\alpha\lambda}{2}r^{\alpha}\log r + r^{\alpha}\left\{\dfrac{(n-1)}{2}\displaystyle\int_r^{\varepsilon}\frac{\rho^{-1}v(\rho)}{w(\rho)}\,d\rho\right. \\ \qquad \left. -\dfrac{\lambda}{2}\displaystyle\int_0^r\frac{v(\rho)}{w(\rho)}\,d\rho - \frac{1}{2}\int_r^{\varepsilon}\frac{\rho^{-\alpha}v(\rho)^2}{w(\rho)}\,d\rho\right\}. \end{cases} \tag{2.26}$$

By (2.26), $v(r) = w_r(r)$ for any $0 < r \leq \varepsilon$ and $w \in C^2((0,\varepsilon];\mathbb{R}^+) \cap C([0,\varepsilon],\mathbb{R}^+)$ satisfies (2.2) and (2.4). Hence, w satisfies (2.1). By (2.4) and (2.12)–(2.14), we get (2.5) and the proposition follows. ∎

By an argument similar to the proof of Proposition 2.1, we have the following result.

Proposition *Let $n \in \mathbb{Z}^+$, $n > 4$, $\alpha = \sqrt{n} - 1$, $\lambda, c_1 \in \mathbb{R}$, $c_0 > 0$, and let c_2 be given by (1.17). Then there exists a constant $0 < \varepsilon < 1$ such that (2.1) has a unique solution $w \in C^2((0,\varepsilon];\mathbb{R}^+) \cap C([0,\varepsilon];\mathbb{R}^+)$ in $(0,\varepsilon]$ which satisfies (2.2) and*

$$\begin{aligned} w_r(r) = {} & -c_2r^{\alpha-1} + c_1r^{\alpha} + \frac{\alpha\lambda}{2}r^{\alpha}\log r + r^{\alpha}\left\{-\frac{(n-1)}{2}\int_0^r\frac{\rho^{-1}w_r(\rho)}{w(\rho)}\,d\rho\right. \\ & \left. -\frac{\lambda}{2}\int_0^r\frac{w_r(\rho)}{w(\rho)}\,d\rho + \frac{1}{2}\int_0^r\frac{\rho^{-\alpha}w_r(\rho)^2}{w(\rho)}\,d\rho\right\} \qquad \forall 0 < r \leq \varepsilon. \end{aligned} \tag{2.27}$$

Moreover, (2.5) holds.

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Corollary *Let $2 \le n \in \mathbb{Z}^+$, $\alpha = \sqrt{n} - 1$, $\lambda, c_1 \in \mathbb{R}$, $c_0 > 0$, and let c_2 be given by (1.17). Then there exists a constant $0 < \varepsilon < 1$ such that (1.10) has infinitely many solutions $h \in C^2((0, \varepsilon])$ in $(0, \varepsilon]$ which satisfies (1.11). Moreover, (1.10) has a unique solution $h \in C^2((0, \varepsilon])$ in $(0, \varepsilon]$ which satisfies (1.11) and (2.4) with w being given by (1.12). Moreover, (2.5) holds.*

Corollary *Let $n \in \mathbb{Z}^+$, $n > 4$, $\alpha = \sqrt{n} - 1$, $\lambda, c_1 \in \mathbb{R}$, $c_0 > 0$, and let c_2 be given by (1.17). Then there exists a constant $0 < \varepsilon < 1$ such that (1.10) has a unique solution $h \in C^2((0, \varepsilon])$ in $(0, \varepsilon]$ which satisfies (1.11) and (2.27) with w being given by (1.12). Moreover, (2.5) holds.*

Proposition *Let $n \in \mathbb{Z}^+$, $n > 4$, $\alpha = \sqrt{n} - 1$, $\lambda, c_1 \in \mathbb{R}$, $c_0 > 0$, and let c_2 be given by (1.17). Then there exists a constant $0 < \varepsilon < 1$ such that (1.10) has a unique solution $h \in C^2((0, \varepsilon])$ in $(0, \varepsilon]$ which satisfies (1.11) and (1.18) for some constant $0 < \delta_0 < \varepsilon$. Moreover, (1.19) and (2.5) hold with w being given by (1.12).*

Proof Since $n > 4$, $\alpha > 1$ and $c_2 > 0$. Let w be given by (1.12). By Corollary 2.4, there exists a constant $0 < \varepsilon < 1$ such that (1.10) has a unique solution $h \in C^2((0, \varepsilon])$ in $(0, \varepsilon]$ which satisfies (1.11), (2.5), and (2.27). Let

$$0 < \delta_1 < \min\left(\varepsilon, \frac{|c_2|}{2c_0}, \frac{c_2^2}{2c_0}\right). \tag{2.28}$$

By (1.11) and (2.5), there exist constants $\delta_0 \in (0, \varepsilon)$ and $c_8 > 0$ such that

$$-\frac{c_2}{c_0} - \delta_1 \le \frac{r^{1-\alpha} w_r(r)}{w(r)} \le -\frac{c_2}{c_0} + \delta_1 \quad \forall 0 < r \le \delta_0 \tag{2.29}$$

and

$$\frac{c_2^2}{c_0} - \delta_1 \le \frac{(r^{1-\alpha} w_r(r))^2}{w(r)} \le \frac{c_2^2}{c_0} + \delta_1 \quad \forall 0 < r \le \delta_0 \tag{2.30}$$

holds. Then, by (2.29) and (2.30),

$$\begin{aligned} &-\frac{(n-1)}{2}\int_0^r \frac{\rho^{-1} w_r(\rho)}{w(\rho)}\,d\rho - \frac{\lambda}{2}\int_0^r \frac{w_r(\rho)}{w(\rho)}\,d\rho + \frac{1}{2}\int_0^r \frac{\rho^{-\alpha} w_r(\rho)^2}{w(\rho)}\,d\rho \\ &\le \frac{(n-1)}{2}\left(\frac{c_2}{c_0} + \delta_1\right)\int_0^r \rho^{\alpha-2}\,d\rho + \frac{\lambda}{2}\left(\frac{c_2}{c_0} + \operatorname{sign}(\lambda)\delta_1\right)\int_0^r \rho^{\alpha-1}\,d\rho \\ &\quad + \frac{1}{2}\left(\frac{c_2^2}{c_0} + \delta_1\right)\int_0^r \rho^{\alpha-2}\,d\rho \\ &\le \frac{c_2(n-1+c_2)}{2c_0(\alpha-1)} r^{\alpha-1} + \frac{\lambda c_2}{2c_0\alpha} r^{\alpha} + \frac{n\delta_1}{2(\alpha-1)} r^{\alpha-1} + \frac{|\lambda|\delta_1}{2\alpha} r^{\alpha} \quad \forall 0 < r \le \delta_0 \end{aligned} \tag{2.31}$$

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and

$$\begin{aligned}
&-\frac{(n-1)}{2}\int_0^r \frac{\rho^{-1}w_r(\rho)}{w(\rho)}\,d\rho-\frac{\lambda}{2}\int_0^r \frac{w_r(\rho)}{w(\rho)}\,d\rho+\frac{1}{2}\int_0^r \frac{\rho^{-\alpha}w_r(\rho)^2}{w(\rho)}\,d\rho\\
&\geq\frac{(n-1)}{2}\left(\frac{c_2}{c_0}-\delta_1\right)\int_0^r \rho^{\alpha-2}\,d\rho+\frac{\lambda}{2}\left(\frac{c_2}{c_0}-\operatorname{sign}(\lambda)\delta_1\right)\int_0^r \rho^{\alpha-1}\,d\rho\\
&\quad+\frac{1}{2}\left(\frac{c_2^2}{c_0}-\delta_1\right)\int_0^r \rho^{\alpha-2}\,d\rho\\
&\geq\frac{c_2(n-1+c_2)}{2c_0(\alpha-1)}r^{\alpha-1}+\frac{\lambda c_2}{2c_0\alpha}r^{\alpha}-\frac{n\delta_1}{2(\alpha-1)}r^{\alpha-1}-\frac{|\lambda|\delta_1}{2\alpha}r^{\alpha}\quad\forall 0<r\leq\delta_0. \qquad (2.32)
\end{aligned}$$

Hence, by (1.11), (2.27), (2.31), and (2.32),

$$\begin{aligned}
&w_r(r)=-c_2r^{\alpha-1}+c_1r^{\alpha}+\frac{\alpha\lambda}{2}r^{\alpha}\log r+\frac{c_2(n-1+c_2)}{2c_0(\alpha-1)}r^{2\alpha-1}+o(1)(r^{2\alpha-1}) \qquad (2.33)\\
\Rightarrow\quad &r^{\alpha}h(r)=c_0-\frac{c_2}{\alpha}r^{\alpha}+\left(\frac{c_1}{\alpha+1}-\frac{\alpha\lambda}{2(\alpha+1)^2}\right)r^{\alpha+1}+\frac{\alpha\lambda}{2\alpha+2}r^{\alpha+1}\log r\\
&\quad+\frac{c_2(n-1+c_2)}{4c_0\alpha(\alpha-1)}r^{2\alpha}+o(1)r^{2\alpha}\quad\forall 0<r\leq\delta_0
\end{aligned}$$

and (1.18) follows. Since

$$w_r(r)=\alpha r^{\alpha-1}h(r)+r^{\alpha}h_r(r)\quad\forall r>0, \qquad (2.34)$$

by (1.18) and (2.33), we get (1.19).

Suppose $h_1\in C^2((0,\varepsilon))$ is another solution of (1.10) which satisfies (1.11) and

$$\begin{aligned}
h_1(r)=&\frac{1}{r^{\alpha}}\left\{c_0-\frac{c_2}{\alpha}r^{\alpha}+\left(\frac{c_1}{\alpha+1}-\frac{\alpha\lambda}{2(\alpha+1)^2}\right)r^{\alpha+1}+\frac{\alpha\lambda}{2\alpha+2}r^{\alpha+1}\log r\right.\\
&\left.+\frac{c_2(n-1+c_2)}{4c_0\alpha(\alpha-1)}r^{2\alpha}+o(1)r^{2\alpha}\right\}\quad\forall 0<r\leq\delta_0. \qquad (2.35)
\end{aligned}$$

Let $w_1(r)=r^{\alpha}h_1(r)$. Then w_1 satisfies (2.3). Integrating equation (2.3) for w_1 over $(0,r)$, we get

$$\begin{aligned}
w_{1,r}(r)=&-c_2r^{\alpha-1}+c_1'r^{\alpha}+\frac{\alpha\lambda}{2}r^{\alpha}\log r+r^{\alpha}\left\{-\frac{(n-1)}{2}\int_0^r \frac{\rho^{-1}w_{1,r}(\rho)}{w_1(\rho)}\,d\rho\right.\\
&\left.-\frac{\lambda}{2}\int_0^r \frac{w_{1,r}(\rho)}{w_1(\rho)}\,d\rho+\frac{1}{2}\int_0^r \frac{\rho^{-\alpha}w_{1,r}(\rho)^2}{w_1(\rho)}\,d\rho\right\}\qquad\forall 0<r\leq\varepsilon \qquad (2.36)
\end{aligned}$$

for some constant $c_1'\in\mathbb{R}$. By (2.36) and a similar argument as before, we get

$$\begin{aligned}
h_1(r)=&\frac{1}{r^{\alpha}}\left\{c_0-\frac{c_2}{\alpha}r^{\alpha}+\left(\frac{c_1'}{\alpha+1}-\frac{\alpha\lambda}{2(\alpha+1)^2}\right)r^{\alpha+1}+\frac{\alpha\lambda}{2\alpha+2}r^{\alpha+1}\log r\right.\\
&\left.+\frac{c_2(n-1+c_2)}{4c_0\alpha(\alpha-1)}r^{2\alpha}+o(1)r^{2\alpha}\right\}\quad\forall 0<r\leq\delta_0. \qquad (2.37)
\end{aligned}$$

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By (2.35) and (2.37),

$$c_1 - c_1' = o(1)(r^{\alpha-1}) \quad \forall 0 < r \le \delta_0 \quad \Rightarrow \quad c_1 = c_1' \quad \text{as } r \to 0^+.$$

Hence, both w and w_1 satisfy (2.27). Then, by Proposition 2.2, $w \equiv w_1$ on $[0, \varepsilon]$. Thus, $h = h_1$ on $[0, \varepsilon]$ and the solution h is unique. ∎

Proposition *Let $n \in \{2, 3, 4\}$, $\alpha = \sqrt{n} - 1$, $\lambda, c_1 \in \mathbb{R}$, $c_0 > 0$, and let c_2 be given by (1.17). Let $0 < \varepsilon < 1$, and let $h \in C^2((0, \varepsilon])$ be the unique solution of (1.10) in $(0, \varepsilon]$ given by Corollary 2.3, which satisfies (1.11), (2.4), and (2.5) with w being given by (1.12). Then there exists a constant $0 < \delta_0 < \varepsilon$ such that (1.20) and (1.21) hold.*

Proof Note that $\alpha < 1$, $1/(1-\alpha) \le 4$, and $c_2 < 0$ when $n = 2$ or 3, and $\alpha = 1$ and $c_2 = 0$, when $n = 4$. Let δ_1 satisfy (2.28) when $n = 2, 3$ and $0 < \delta_1 < \varepsilon$ when $n = 4$. By (1.11) and (2.5), there exists a constant $0 < \delta_0 < \varepsilon$ such that (2.29) and (2.30) hold. Hence, by (2.29) and (2.30), for any $0 < r < \delta_0$,

$$\begin{aligned}
&\frac{(n-1)}{2}\int_r^\varepsilon \frac{\rho^{-1}w_r(\rho)}{w(\rho)}\,d\rho - \frac{\lambda}{2}\int_0^r \frac{w_r(\rho)}{w(\rho)}\,d\rho - \frac{1}{2}\int_r^\varepsilon \frac{\rho^{-\alpha}w_r(\rho)^2}{w(\rho)}\,d\rho \\
\le &\frac{(n-1)}{2}\left(-\frac{c_2}{c_0} + \delta_1\right)\int_r^{\delta_0} \rho^{\alpha-2}\,d\rho + \frac{\lambda}{2}\left(\frac{c_2}{c_0} + \operatorname{sign}(\lambda)\delta_1\right)\int_0^r \rho^{\alpha-1}\,d\rho \\
&- \frac{1}{2}\left(\frac{c_2^2}{c_0} - \delta_1\right)\int_r^{\delta_0} \rho^{\alpha-2}\,d\rho + c_8
\end{aligned}$$

(2.38)

$$\le \begin{cases} -\dfrac{c_2(c_2+n-1)}{2c_0(1-\alpha)}(r^{\alpha-1} - \delta_0^{\alpha-1}) + \dfrac{\lambda c_2}{2c_0\alpha}r^\alpha + \dfrac{n\delta_1}{2(1-\alpha)}(r^{\alpha-1} - \delta_0^{\alpha-1}) \\ \qquad + \dfrac{|\lambda|\delta_1}{2\alpha}r^\alpha + c_8 \end{cases}, \quad \text{if } n = 2, 3$$

$$\frac{n\delta_1}{2}(\log\delta_0 - \log r) + \frac{|\lambda|\delta_1}{2}r + c_8, \qquad \text{if } n = 4$$

and

$$\begin{aligned}
&\frac{(n-1)}{2}\int_r^\varepsilon \frac{\rho^{-1}w_r(\rho)}{w(\rho)}\,d\rho - \frac{\lambda}{2}\int_0^r \frac{w_r(\rho)}{w(\rho)}\,d\rho - \frac{1}{2}\int_r^\varepsilon \frac{\rho^{-\alpha}w_r(\rho)^2}{w(\rho)}\,d\rho \\
\ge &\frac{(n-1)}{2}\left(-\frac{c_2}{c_0} - \delta_1\right)\int_r^{\delta_0} \rho^{\alpha-2}\,d\rho + \frac{\lambda}{2}\left(\frac{c_2}{c_0} - \operatorname{sign}(\lambda)\delta_1\right)\int_0^r \rho^{\alpha-1}\,d\rho \\
&- \frac{1}{2}\left(\frac{c_2^2}{c_0} + \delta_1\right)\int_r^{\delta_0} \rho^{\alpha-2}\,d\rho + c_8
\end{aligned}$$

(2.39)

$$\ge \begin{cases} -\dfrac{c_2(c_2+n-1)}{2c_0(1-\alpha)}(r^{\alpha-1} - \delta_0^{\alpha-1}) + \dfrac{\lambda c_2}{2c_0\alpha}r^\alpha - \dfrac{n\delta_1}{2(1-\alpha)}(r^{\alpha-1} - \delta_0^{\alpha-1}) \\ \qquad - \dfrac{|\lambda|\delta_1}{2\alpha}r^\alpha + c_8, \end{cases} \quad \text{if } n = 2, 3,$$

$$-\frac{n\delta_1}{2}(\log\delta_0 - \log r) - \frac{|\lambda|\delta_1}{2}r + c_8, \qquad \text{if } n = 4,$$

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where

$$c_8 = \frac{(n-1)}{2}\int_{\delta_0}^{\varepsilon}\frac{\rho^{-1}w_r(\rho)}{w(\rho)}\,d\rho - \frac{1}{2}\int_{\delta_0}^{\varepsilon}\frac{\rho^{-\alpha}w_r(\rho)^2}{w(\rho)}\,d\rho.$$

Thus, by (1.11), (2.4), (2.38), and (2.39),

(2.40)

$$w_r(r) = \begin{cases} -c_2 r^{\alpha-1} - \dfrac{c_2(c_2+n-1)}{2c_0(1-\alpha)}r^{2\alpha-1} + o(1)(r^{2\alpha-1}) & \forall 0 < r \le \delta_0, \quad \text{if } n = 2,3, \\ \dfrac{\lambda}{2} r\log r + o(1)(r|\log r|) & \forall 0 < r \le \delta_0, \quad \text{if } n = 4, \end{cases}$$

(2.41)

$$\Rightarrow\ w(r) = r^{\alpha}h(r) = \begin{cases} c_0 - \dfrac{c_2}{\alpha}r^{\alpha} - \dfrac{c_2(c_2+n-1)}{4c_0\alpha(1-\alpha)}r^{2\alpha} + o(1)r^{2\alpha} & \forall 0 < r \le \delta_0, \quad \text{if } n = 2,3, \\ c_0 + \dfrac{\lambda}{4}r^2\log r + o(1)r^2|\log r| & \forall 0 < r \le \delta_0, \quad \text{if } n = 4, \end{cases}$$

and (1.20) follows. By (1.20), (2.34), and (2.40), we get (1.21) and the proposition follows. ∎

3 Global existence and uniqueness of singular solutions

In this section, we will use a modification of the technique of Hsu [9] to prove the global existence of infinitely many singular solutions of (1.10) and (1.11) in $(0,\infty)$. We will also prove the uniqueness of the global singular solution of such equation in terms of its asymptotic behavior near the origin.

Lemma 3.1 *Let $2 \le n \in \mathbb{Z}^+$, $\lambda \in \mathbb{R}$, and $L > 0$. Suppose $h \in C^2((0,L))$ satisfies (1.10) in $(0,L)$. Then*

(3.1)

$$h_r(r_1) = \frac{n-1}{r_1} + \lambda + \sqrt{\frac{h(r_1)}{h(r_2)}}\left(h_r(r_2) - \frac{n-1}{r_2} - \lambda\right) + \frac{(n-1)\sqrt{h(r_1)}}{2}\int_{r_2}^{r_1}\frac{h(\rho)+1}{\rho^2\sqrt{h(\rho)}}\,d\rho$$

holds for any $0 < r_2 < r_1 < L$.

Proof By (1.10),

$$(h^{-1/2}h_r)_r = \frac{(n-1)(h-1)}{2r^2h^{1/2}} - \frac{(n-1+\lambda r)h_r}{2rh^{3/2}} \quad \forall r > 0$$

$$\Rightarrow\ h_r(r_1) = \sqrt{h(r_1)}\left\{\frac{h_r(r_2)}{\sqrt{h(r_2)}} + \frac{(n-1)}{2}\int_{r_2}^{r_1}\frac{h(\rho)-1}{\rho^2\sqrt{h(\rho)}}\,d\rho - \int_{r_2}^{r_1}\frac{(n-1+\lambda\rho)h_r(\rho)}{2\rho h(\rho)^{3/2}}\,d\rho\right\}$$

$$= \sqrt{h(r_1)}\left\{\frac{h_r(r_2)}{\sqrt{h(r_2)}} + \frac{(n-1)}{2}\int_{r_2}^{r_1}\frac{h(\rho)-1}{\rho^2\sqrt{h(\rho)}}\,d\rho + \left(\frac{n-1}{r_1}+\lambda\right)\frac{1}{\sqrt{h(r_1)}}\right.$$

$$\left. - \left(\frac{n-1}{r_2}+\lambda\right)\frac{1}{\sqrt{h(r_2)}} + (n-1)\int_{r_2}^{r_1}\frac{d\rho}{\rho^2\sqrt{h(\rho)}}\right\} \quad \forall 0 < r_2 < r_1 < L$$

and (3.1) follows. ∎

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We now observe that by an argument similar to the proof of Lemmas 2.3–2.6 of [9] but with (1.10) and (3.1) replacing (1.6) and (2.25) of [9] in the proof there, we have the following results.

Lemma 3.2 (cf. Lemmas 2.3 and 2.4 of [9]) *Let $2 \le n \in \mathbb{Z}^+$ and $\lambda \in \mathbb{R}$. Suppose $h \in C^2((0, L))$ satisfies (1.10) in $(0, L)$ for some constant $L \in (0, \infty)$ such that $L < -(n-1)/\lambda$ if $\lambda < 0$. Then there exist constants $C_2 > C_1 > 0$ such that*

$$C_1 \le h(r) \le C_2 \quad \forall L/2 \le r \le L. \tag{3.2}$$

Lemma 3.3 (cf. Lemmas 2.5 and 2.6 of [9]) *Let $2 \le n \in \mathbb{Z}^+$ and $\lambda \in \mathbb{R}$. Suppose $h \in C^2((0, L))$ satisfies (1.10) in $(0, L)$ for some constant $L \in (0, \infty)$ such that $L < -(n-1)/\lambda$ if $\lambda < 0$. Then there exist constants $C_4 > C_3$ such that*

$$C_3 \le h_r(r) \le C_4 \quad \forall L/2 \le r \le L. \tag{3.3}$$

We next observe that by standard ODE theory, we have the following result.

Lemma 3.4 (cf. Lemma 2.7 of [9]) *Let $2 \le n \in \mathbb{Z}^+$, $\lambda \in \mathbb{R}$, $L > 0$, $b_0 \in (C_1, C_2)$, $b_1 \in (C_4, C_3)$ for some constants $C_2 > C_1 > 0$, and $C_3 > C_4$. Then there exists a constant $0 < \delta_1 < L/4$ depending only on C_1, C_2, C_3, C_4 such that for any $r_0 \in (L/2, L)$, (1.10) has a unique solution $\widetilde{h} \in C^2((r_0 - \delta_1, r_0 + \delta_1))$ in $(r_0 - \delta_1, r_0 + \delta_1)$ which satisfies*

$$\widetilde{h}(r_0) = b_0 \quad \text{and} \quad \widetilde{h}_r(r_0) = b_1. \tag{3.4}$$

We are now ready for the proof of Theorem 1.1.

Proof of Theorem 1.1 We will use a modification of the proof of Theorem 1.1 of [9] to prove the theorem. We first observe that by Corollary 2.3, there exists a constant $0 < \varepsilon < 1$ such that (1.10) has a unique solution $h \in C^2((0, \varepsilon])$ in $(0, \varepsilon]$ which satisfies (1.11) and (2.4) with w being given by (1.12). Moreover, (2.5) holds. Let $(0, L)$ be the maximal interval of existence of solution $h \in C^2((0, L))$ of (1.10) in $(0, L)$ which satisfies (1.11). Suppose $L < \infty$. Then, by Lemmas 3.2 and 3.3, there exist constants $C_2 > C_1 > 0$ and $C_3 > C_4$ such that (3.2) and (3.3) hold.

Then, by Lemma 3.4, there exists a constant $0 < \delta_1 < L/4$ depending only on C_1, C_2, C_3, C_4 such that for any $r_0 \in (L/2, L)$ (1.10) has a unique solution $\widetilde{h} \in C^2((r_0 - \delta_1, r_0 + \delta_1))$ in $(r_0 - \delta_1, r_0 + \delta_1)$ which satisfies (3.4) with $b_0 = h(r_0)$ and $b_1 = h_r(r_0)$. We now set $r_0 = L - (\delta_1/2)$ and extend h to a function on $[0, L + (\delta_1/2))$ by setting $h(r) = \widetilde{h}(r)$ for any $r \in [L, L + (\delta_1/2))$. Then $h \in C^2((0, L + (\delta_1/2)))$ is a solution of (1.10) in $(0, L + \delta_1)$ which satisfies (1.11) and (2.4). This contradicts the choice of L. Hence, $L = \infty$ and there exists a solution $h \in C^2((0, \infty))$ of (1.10) which satisfies (1.11) and (2.4).

Suppose $h_1 \in C^2((0, \infty))$ is another solution of (1.10) which satisfies (1.11) and (2.4) with w being replaced by $w_1 = r^\alpha h_1(r)$. Then both w and w_1 satisfy (2.3). Hence, both w and w_1 satisfy (2.1) and (2.2) in $(0, \varepsilon]$. Therefore, by Proposition 2.1, $w(r) \equiv w_1(r)$ in $(0, \varepsilon]$. Hence, $h(r) = h_1(r)$ in $(0, \varepsilon]$. Then, by standard ODE theory, $h(r) = h_1(r)$ in $[\varepsilon, \infty)$. Thus, $h(r) = h_1(r)$ in $(0, \infty)$ and the solution h is unique. ∎

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Proof of Theorem 1.2 By Corollary 2.4 and an argument similar to the proof of Theorem 1.1, there exists a unique solution $h \in C^2((0,\infty))$ of (1.10) in $(0,\infty)$ which satisfies (1.11) and (2.27) in $(0,\varepsilon)$ with w given by (1.12) for some $0 < \varepsilon < 1$. By (2.27) and the same argument as the proof of Proposition 2.5, we get (1.18) and (1.19). Suppose $h_1 \in C^2((0,\infty))$ is another solution of (1.10) in $(0,\infty)$ which satisfies (1.11) and (1.18). Then, by an argument similar to the proof of Proposition 2.5, $h_1(r) \equiv h(r)$ in $(0,\varepsilon]$. Hence, by standard ODE uniqueness theory, $h_1(r) \equiv h(r)$ in $[\varepsilon,\infty)$ and the theorem follows. ∎

Finally, by Theorem 1.1 and an argument similar to the proof of Proposition 2.6, we get Theorem 1.3.

Proof of Theorem 1.4 Without loss of generality, we may assume that $\varepsilon = 2$. Let w be given by (1.12) and

$$q(r) = \frac{rh_r(r)}{h(r)}. \tag{3.5}$$

We first claim that there exists a decreasing sequence $\{r_i\}_{i=1}^{\infty} \subset (0,\varepsilon)$ such that

$$\lim_{i\to\infty} q(r_i) = -\alpha. \tag{3.6}$$

To prove the claim, we note that by (1.12),

$$w_r(r) = \alpha r^{\alpha-1}h(r) + r^{\alpha}h_r(r) = \frac{w(r)}{r}(\alpha + q(r)) \quad \forall 0 < r < \varepsilon. \tag{3.7}$$

For any $i \in \mathbb{Z}^+$ by the mean value theorem, there exists $r_i \in (1/(2i), 1/i)$ such that

$$w_r(r_i) = 2i(w(1/i) - w(1/(2i))). \tag{3.8}$$

By (1.11), (3.6), and (3.8),

$$|\alpha + q(r_i)| \le \frac{2ir_i|w(1/i) - w(1/(2i))|}{w(r_i)} \le \frac{2|w(1/i) - w(1/(2i))|}{w(r_i)} \quad \forall i \in \mathbb{Z}^+$$
$$\Rightarrow \quad \lim_{i\to\infty} |\alpha + q(r_i)| = 0$$

and the claim follows. By (1.10) and a direct computation, q satisfies

$$q_r(r) + \left(-\frac{1}{r} + \frac{\lambda}{2h(r)} + \frac{n-1}{2rh(r)}\right)q(r) = -\frac{1}{2r}\left(q(r)^2 - \frac{(n-1)(h(r)-1)}{h(r)}\right) \quad \forall 0 < r < \varepsilon. \tag{3.9}$$

Let

$$F(r) = \exp\left(\frac{\lambda}{2}\int_0^r \frac{d\rho}{h(\rho)} + \frac{n-1}{2}\int_0^r \frac{d\rho}{\rho h(\rho)}\right) \quad \forall 0 < r < \varepsilon. \tag{3.10}$$

Then, by (3.9),

$$(r^{-1}F(r)q(r))_r = -\frac{F(r)}{2r^2}\left(q(r)^2 - \frac{(n-1)(h(r)-1)}{h(r)}\right) \quad \forall 0 < r < \varepsilon$$
$$\Rightarrow \quad q(r) = \frac{1}{r^{-1}F(r)}(F(1)q(1) + I_1(r)) \quad \forall 0 < r < \varepsilon, \tag{3.11}$$

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where

$$I_1(r) = \int_r^1 \frac{F(\rho)}{2\rho^2}\left(q(\rho)^2 - \frac{(n-1)(h(\rho)-1)}{h(\rho)}\right)d\rho \quad \forall 0 < r < \varepsilon. \tag{3.12}$$

We now divide the proof into two cases.

Case 1: $\limsup_{i\to\infty}|I_1(r_i)| < \infty$.

By (3.6) and (3.11),

$$-\alpha = \lim_{i\to\infty} q(r_i) = 0,$$

which contradicts the assumption that $\alpha > 0$. Hence, Case 1 does not hold.

Case 2: $\limsup_{i\to\infty}|I_1(r_i)| = \infty$.

Then, we may assume, without loss of generality, that $\lim_{i\to\infty}|I_1(r_i)| = \infty$. Since $\alpha > 0$, by (1.11), (3.6), (3.11), and the l'Hospital rule,

$$-\alpha = \lim_{i\to\infty} q(r_i) = \lim_{i\to\infty} \frac{-\frac{F(r_i)}{2r_i^2}\left(q(r_i)^2 - \frac{(n-1)(h(r_i)-1)}{h(r_i)}\right)}{-r_i^{-2}F(r_i) + r_i^{-1}F(r_i)\left(\frac{\lambda}{2h(r_i)} + \frac{n-1}{2r_i h(r_i)}\right)} = \frac{\alpha^2 - (n-1)}{2}$$

$$\Rightarrow \quad \alpha^2 + 2\alpha - (n-1) = 0$$

$$\Rightarrow \quad \alpha = \sqrt{n} - 1$$

and the theorem follows. ∎

4 Asymptotic behavior of the function $a(t)$ near the origin

In this section, we will prove the asymptotic behavior of $a(t)$ near the origin.

Proposition *Let $2 \le n \in \mathbb{Z}^+$, $\alpha = \sqrt{n} - 1$, $\lambda \ge 0, c_1 \in \mathbb{R}$, $c_0 > 0$, and let c_2 be given by (1.17). For $n > 4$, let $h \in C^2((0,\infty))$ be the unique solution of (1.10) in $(0,\infty)$, which satisfies (1.11) and (1.18) for some constant $0 < \delta_0 < 1$ given by Theorem 1.2. For $n \in \{2,3,4\}$, let $h \in C^2((0,\infty))$ be given by Theorem 1.1, which satisfies (1.20) for some constant $0 < \delta_0 < 1$. Then*

$$a(t) \approx (\sqrt{nc_0}\,t)^{1/\sqrt{n}} \quad \text{as } t \to 0^+. \tag{4.1}$$

Proof By (1.18) and (1.20),

$$\begin{aligned}(h(\rho^2))^{-1/2} &= \begin{cases} c_0^{-1/2}\rho^\alpha\left(1 + O(\rho^{2\alpha})\right)^{-1/2}, & \text{if } n \ne 4 \\ c_0^{-1/2}\rho\left(1 + O(\rho^4|\log\rho|^2)\right)^{-1/2}, & \text{if } n = 4 \end{cases} \\ &= \begin{cases} c_0^{-1/2}\left(\rho^\alpha + O(\rho^{3\alpha})\right), & \text{if } n \ne 4, \\ c_0^{-1/2}\left(\rho + O(\rho^5|\log\rho|^2)\right), & \text{if } n = 4. \end{cases}\end{aligned} \tag{4.2}$$

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By (1.16) and (4.2),

$$t \approx \frac{a(t)^{\sqrt{n}}}{\sqrt{nc_0}} \quad \text{as } t \to 0^+$$

and (4.1) follows. ∎

By a similar argument, we have the following proposition.

Proposition *Let $2 \le n \in \mathbb{Z}^+$, $\alpha = \sqrt{n} - 1$, $\lambda, c_1 \in \mathbb{R}$, $c_0 > 0$, and let c_2 be given by (1.17). For $n > 4$, let $h \in C^2((0, \varepsilon])$ be the unique solution of (1.10) in $(0, \varepsilon]$, which satisfies (1.11) and (1.18) for some constants $0 < \delta_0 < \varepsilon < 1$, given by Proposition 2.5. For $n \in \{2, 3, 4\}$, let $h \in C^2((0, \varepsilon])$ be given by Corollary 2.3, which satisfies (1.20) for some constants $0 < \delta_0 < \varepsilon < 1$. Then (4.1) holds.*

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