K. Shikishima-Tsuji and M. Katsura
Nagoya Math. J.
Vol. 125 (1992), 93–103

HYPERTRANSCENDENTAL ELEMENTS OF A FORMAL POWER-SERIES RING OF POSITIVE CHARACTERISTIC

KAYOKO SHIKISHIMA-TSUJI AND MASASHI KATSURA

§ 0. Introduction

Throughout this paper, we denote by **N**, **Q** and **R** the set of all natural numbers containing 0, the set of all rational numbers, and the set of all real numbers, respectively.

Let K be a fixed field of positive characteristic p and K_a an algebraic closure of K. We denote by $K[\![X]\!]$ the formal power-series ring and by $d = (d_\mu;\ \mu \in \mathbf{N})$ the formal derivation of $K[\![X]\!]$, i.e., for every $A = \sum_{i=0}^{\infty} a_i X^i \in K[\![X]\!]$, the μ-th derivative $d_\mu A$ of A is defined by

$$d_\mu A = \sum_{i=\mu}^{\infty} \binom{i}{\mu} a_i X^{i-\mu}.$$

For differential rings and differential fields of positive characteristic, see Okugawa [4].

This paper contains three theorems. Let A be an element $\sum_{i=0}^{\infty} a_i X^i$ of $K[\![X]\!]$. We say that A is *hypertranscendental* over K, if, for every $\mu \in \mathbf{N}$, $A, d_1 A, \cdots, d_\mu A$ are algebraically independent over $K(X)$. When the characteristic of the field is zero, the existence of hypertranscendental elements is well-known (see D. Hilbert [1], O. Hölder [2], F. Kuiper [3]). Theorem 1 shows the existence of hypertranscendental elements in case of positive characteristic.

Let L be a differential field and S a subset of a differential extension field of L. We say that S is *differentially independent* over L or all the elements of S are *differentially independent* over L, if for every $\mu \in \mathbf{N}$ and elements $s_1, \cdots, s_\mu$ of S, there are no nonzero differential polynomial $F(X_1, \cdots, X_\mu) \in L\{X_1, \cdots, X_\mu\}$ such that $F(s_1, \cdots, s_\mu) = 0$.

Theorem 2 states that there are infinitely many hypertranscendental

Received February 21, 1991.

elements in $K[\![X]\!]$ over K which are differentially independent over K.

If an element of $K[\![X]\!]$ is differentially quasi-algebraic over K (see K. Shikishima-Tsuji [5]), then

$$\lim_{s\to\infty} \frac{1}{s} \operatorname{tr} \deg \{d_\mu A;\ \mu < s\}/K(X) = 0\,.$$

If A is hypertranscendental over K, then

$$\lim_{s\to\infty} \frac{1}{s} \operatorname{tr} \deg \{d_\mu A;\ \mu < s\}/K(X) = 1\,.$$

Let A be hypertranscendental over K. It can be easily shown that, for every $0 < r < p$, the formal power series $B = d_{p-r}A$ satisfies the equation

$$\lim_{s\to\infty} \frac{1}{s} \operatorname{tr} \deg \{d_\mu B;\ \mu < s\}/K(X) = \frac{r}{p}\,.$$

For every $\alpha \in \mathbf{R}$ $(0 \leq \alpha \leq 1)$, there exists a formal power series B_α of $K[\![X]\!]$ such that

$$\lim_{s\to\infty} \frac{1}{s} \operatorname{tr} \deg \{d_\mu B_\alpha;\ \mu < s\}/K(X) = \alpha\,.$$

This is Theorem 3.

§ 1.

For $m, n \in \mathbf{N}$, the binomial coefficient $\binom{m}{n}$ equals $\dfrac{m!}{n!(m-n)!}$ in case $m \geq n$, otherwise zero.

LEMMA 1. *Let $m, n \in \mathbf{N}$. If $m = \sum_{i=0}^{e} m_i p^i$ and $n = \sum_{i=0}^{e} n_i p^i$ are the p-adic expressions of m and n respectively, then*

$$\binom{m}{n} \equiv \binom{m_0}{n_0} \cdots \binom{m_e}{n_e} \pmod{p}. \tag{1}$$

Proof. By expanding both sides of the identity over the prime field of characteristic p:

$$(1 + x)^m = (1 + x)^{m_0}(1 + x^p)^{m_1}(1 + x^{p^2})^{m_2} \cdots (1 + x^{p^e})^{m_e}\,,$$

and comparing the coefficients of x^n, we obtain the congruence (1). q.e.d.

LEMMA 2. *Let m, n, e, t be natural numbers. For $t < p^e$, we have the*

https://doi.org/10.1017/S0027763000003913 Published online by Cambridge University Press

following statements:

(1) *If* $m \equiv n \pmod{p^e}$, *then* $\binom{m}{t} \equiv \binom{n}{t} \pmod{p}$.

(2) *If* $m \equiv r \pmod{p^e}$ *and* $0 \leq r \leq t-1$, *then* $\binom{m}{t} \equiv 0 \pmod{p}$.

(3) *If* $m \equiv t \pmod{p^e}$, *then* $\binom{m}{t} \equiv 1 \pmod{p}$.

Proof. (1) Let $m = \sum_{i=0}^{\alpha} m_i p^i$, $n = \sum_{i=0}^{\alpha} n_i p^i$ and $t = \sum_{i=0}^{\alpha} t_i p^i$ be the p-adic expressions of m and n respectively. Since $m \equiv n \pmod{p^e}$, we have $m_0 = n_0, \cdots, m_{e-1} = n_{e-1}$. Lemma 1 implies that

$$\binom{m}{t} \equiv \binom{m_0}{t_0} \cdots \binom{m_{e-1}}{t_{e-1}}\binom{m_e}{0} \cdots \binom{m_\alpha}{0}$$
$$\equiv \binom{n_0}{t_0} \cdots \binom{n_{e-1}}{t_{e-1}}\binom{n_e}{0} \cdots \binom{n_\alpha}{0} \equiv \binom{n}{t} \pmod{p}.$$

(2) Since $r \leq t-1$, we have $\binom{r}{t} = 0$. By (1), we have

$$\binom{m}{t} \equiv \binom{r}{t} \pmod{p}.$$

(3) By (1), we have

$$\binom{m}{t} \equiv \binom{t}{t} \pmod{p}.$$ q.e.d.

Let B be a formal power series of $K[\![X]\!]$. We denote the leading degree of B by $v(B)$ (i.e., if $B = \sum_{i=r}^{\infty} b_i X^i$ and $b_r \neq 0$, then $v(B) = r$ and if $B = 0$, then $v(B) = \infty$).

THEOREM 1. *Let A be an element $\sum_{i=0}^{\infty} a_i X^{m_i}$ of $K[\![X]\!]$ with nonzero $a_i \in K$ $(i \in \mathbf{N})$ and $m_0 < m_1 < m_2 < \cdots$ be natural numbers. If A satisfies the following condition, then A is hypertranscendental over K.*

For any $e, s \in \mathbf{N}$, there exist natural numbers $i_0 < i_1 < i_2 < \cdots$ such that

(1) $$m_{i_j} \equiv s \pmod{p^e} \quad \textit{and} \quad \lim_{j\to\infty} \frac{m_{i_j}}{m_{i_{j-1}}} = \infty\,.$$

Proof. Suppose A is not hypertranscendental over K_a. Then, there is a positive integer μ such that $A, d_1A, \cdots, d_\mu A$ are algebraically dependent over $K_a(X)$, that is, there exists a non-zero polynomial $F(X, Y_0, \cdots, Y_\mu) \in K_a[X, Y_0, \cdots, Y_\mu]$ which satisfies the following two conditions:

https://doi.org/10.1017/S0027763000003913 Published online by Cambridge University Press

(2) $F(X, A, d_1A, \cdots, d_\mu A) = 0$.

(3) If $G(X, Y_0, \cdots, Y_\mu)$ is non-zero polynomial such that $G(X, A, d_1A, \cdots, d_\mu A) = 0$, then the total degree of G is not smaller than that of F.

We see that F is irreducible by the condition (3).

Let c_1 and c_2 be the degrees of F on X and on $Y_0, \cdots, Y_\mu$, respectively. We take a natural number e such that $\mu < p^e$. By the assumption (1), there exist $k_0, k_1, \cdots, k_\mu \in \mathbf{N}$ such that the following conditions hold for every s $(0 \leq s \leq \mu)$;

(4) $m_{k_s-1} \geq c_1 + \mu$,

(5) $m_{k_s} > (c_2 + 1)m_{k_s-1}$,

(6) $m_{k_s} \equiv s \pmod{p^e}$, and,

(7) $m_{k_s} > v\left(\frac{\partial F}{\partial Y_t}(X, A, d_1A, \cdots, d_\mu A)\right) + 2\mu$, for every t $(0 \leq t \leq \mu)$ such that $\frac{\partial F}{\partial Y_t}(X, A, d_1A, \cdots, d_\mu A) \neq 0$.

Let

$$G_s = \textstyle\sum_{i=0}^{k_s-1} a_i X^{m_i} \quad \text{and} \quad B_s = \sum_{i=k_s}^{\infty} a_i X^{m_i} \quad (0 \leq s \leq \mu).$$

By Taylor's expansion, we have

$$\begin{aligned} 0 &= F(X, A, d_1A, \cdots, d_\mu A) \\ &= F(X, G_s, d_1G_s, \cdots, d_\mu G_s) + \textstyle\sum_{t=0}^{\mu} d_t B_s \frac{\partial F}{\partial Y_t}(X, A, d_1A, \cdots, d_\mu A) - E_s \end{aligned}$$

where E_s is a sum of terms of degree ≥ 2 in $\{B_s, d_1B_r, \cdots, d_\mu B_s\}$. We have

$$\deg F(X, G_s, d_1G_s, \cdots, d_\mu G_s) \leq c_1 + c_2 m_{k_s-1},$$
$$v\left(d_t B_s \frac{\partial F}{\partial Y_t}(X, A, d_1A, \cdots, d_\mu A)\right) \geq v(d_t B_s) \geq m_{k_s} - t,$$

and

(8) $$v(E_s) \geq \min_{0 \leq t_1, t_2 \leq \mu} \{v(d_{t_1}B_s d_{t_2}B_s)\} \geq 2(m_{k_s} - \mu).$$

Hence, by (4) and (5), we have

$$\begin{aligned} &v\left(\textstyle\sum_{t=0}^{\mu} d_t B_s \frac{\partial F}{\partial Y_t}(X, A, d_1A, \cdots, d_\mu A) - E_s\right) \\ &\quad \geq m_{k_s} - \mu > (c_2 + 1)m_{k_s-1} - \mu \\ &\quad \geq c_2 m_{k_s-1} + c_1 \geq \deg F(X, G_s, d_1G_s, \cdots, d_\mu G_s). \end{aligned}$$

https://doi.org/10.1017/S0027763000003913 Published online by Cambridge University Press

Therefore, $F(X, G_s, d_1G_s, \cdots, d_\mu G_s) = 0$ and

$$\sum_{t=0}^{\mu} d_t B_s \frac{\partial F}{\partial Y_t}(X, A, d_1A, \cdots, d_\mu A) = E_s \quad (s = 0, 1, \cdots, \mu). \tag{9}$$

Let

$$W = \det\begin{pmatrix} B_0 & d_1B_0 & \cdots & d_\mu B_0 \\ & \cdots\cdots & & \\ B_\mu & d_1B_\mu & \cdots & d_\mu B_\mu \end{pmatrix},$$

and

$$V_t = \det\begin{pmatrix} B_0 & d_1B_0 & \cdots & d_{t-1}B_0 & E_0 & d_{t+1}B_0 & \cdots & d_\mu B_0 \\ & & & \cdots\cdots\cdots & & & & \\ B_\mu & d_1B_\mu & \cdots & d_{t-1}B_\mu & E_\mu & d_{t+1}B_\mu & \cdots & d_\mu B_\mu \end{pmatrix}.$$

On the other hand, $d_tB_s = \sum_{i=k_s}^{\infty}\binom{m_i}{t}a_iX^{m_i-t}$, and by (6) and Lemma 2,

$$\binom{m_{k_s}}{s} = 1, \quad \binom{m_{k_s}}{s+1} = \cdots = \binom{m_{k_s}}{\mu} = 0.$$

Hence, the coefficient of the leading form of the power series W is $a_{k_0}\cdots a_{k_\mu}$ and $v(W) = m_{k_0} + \cdots + m_{k_\mu} - \frac{\mu(\mu+1)}{2}$. Therefore, $W \neq 0$. By Cramer's rule, (9) implies

$$W\frac{\partial F}{\partial Y_t}(X, A, d_1A, \cdots, d_\mu A) = V_t. \tag{10}$$

We have

$$\begin{aligned} v(V_t) &\geq \min_{0\leq s\leq \nu}\left\{\left(m_{k_0} + \cdots + m_{k_\mu} - \frac{\mu(\mu+1)}{2}\right) - (m_{k_s} - t) + v(E_s)\right\} \\ &\geq v(W) + \min_{0\leq s\leq \mu}\{v(E_s) - m_{k_s}\}. \end{aligned}$$

If $\frac{\partial F}{\partial Y_t}(X, A, d_1A, \cdots, d_\mu A) \neq 0$, then by (7), (8) and (10), we have

$$\begin{aligned} v\left(\frac{\partial F}{\partial Y_t}(X, A, d_1A, \cdots, d_\mu A)\right) &= v(V_t) - v(W) \\ &\geq \min_{0\leq s\leq \mu}\{v(E_s) - m_{k_s}\} \\ &\geq \min_{0\leq s\leq \mu}\{m_{k_s} - 2\mu\} > v\left(\frac{\partial F}{\partial Y_t}(X, A, d_1A, \cdots, d_\mu A)\right), \end{aligned}$$

which is a contradiction. Therefore, we have

https://doi.org/10.1017/S0027763000003913 Published online by Cambridge University Press

$$\frac{\partial F}{\partial Y_t}(X, A, d_1A, \cdots, d_\mu A) = 0 \quad (0 \leq t \leq \mu).$$

By the assumption (3), we have

$$\frac{\partial F}{\partial Y_t}(X, Y_0, \cdots, Y_\mu) = 0 \quad (0 \leq t \leq \mu).$$

Since F is irreducible, it follows that $F(X, Y_0, \cdots, Y_\mu) \in K_a[X, Y_0^p, \cdots, Y_\mu^p]$ and there exist $F_0, \cdots, F_{p-1} \in K_a[X^p, Y_0^p, \cdots, Y_\mu^p]$ such that

$$F(X, Y_0, \cdots, Y_\mu) = F_0(X, Y_0, \cdots, Y_\mu) + XF_1(X, Y_0, \cdots, Y_\mu) + \cdots + X^{p-1}F_{p-1}(X, Y_0, \cdots, Y_\mu).$$

Since $F(X, d_1A, \cdots, d_\mu A) = 0$ and $F_i(X, d_1A, \cdots, d_\mu A) \in K_a[\![X^p]\!]$ $(i = 0, \cdots, p-1)$, we have

$$F_i(X, d_1A, \cdots, d_\mu A) = 0 \quad (i = 0, \cdots, p-1).$$

Since K_a is perfect, there exist $G_0, \cdots, G_{p-1} \in K_a[X, Y_0, \cdots, Y_\mu]$ such that

$$F_i(X, Y_0, \cdots, Y_\mu) = (G_i(X, Y_0, \cdots, Y_\mu))^p \quad (i = 0, \cdots, p-1).$$

Since $G_i(X, d_1A, \cdots, d_\mu A) = 0$ $(i = 0, \cdots, p-1)$, (3) implies that

$$G_i(X, Y_0, \cdots, Y_\mu) = 0 \quad (i = 0, \cdots, p-1).$$

It follows that $F(X, Y_0, \cdots, Y_\mu) = 0$. This is a contradiction. q.e.d.

By this theorem, the power series

$$\textstyle\sum_{i=0}^{\infty} X^{p^{i^2}+i}, \quad \sum_{i=0}^{\infty} X^{i^i p+i} \quad \text{and} \quad \sum_{i=0}^{\infty} X^{i!+i}$$

are hypertranscendental.

§ 2.

Let $A = \sum_{i=0}^{\infty} a_i X^i$ be a formal power series of $K[\![X]\!]$. For $e \in \mathbf{N}$ and $k \in \{0, 1, \cdots, p^e - 1\}$ we denote the power series $\sum_{i=0}^{\infty} a_{k+ip^e} X^{ip^e}$ by $A_k^{(e)}$. Then, $A_0^{(e)}, \cdots, A_{p^e-1}^{(e)}$ are elements of $K[\![X^{p^e}]\!]$ and we have

$$A = A_0^{(e)} + XA_1^{(e)} + \cdots + X^{p^e-1}A_{p^e-1}^{(e)}.$$

THEOREM 2. *Let* $A = \sum_{i=1}^{\infty} a_i X^i$ *be hypertranscendental. For each* t $(t = 1, \cdots, p-1)$ *and* $s \in \mathbf{N} - \{0\}$, *let*

$$B_{s,t} = (A_{tp^s-1}^{(s)})^{p^{-s}} = \textstyle\sum_{i=0}^{\infty} (a_{tp^s-1+ip^s})^{p^{-s}} X^i.$$

https://doi.org/10.1017/S0027763000003913 Published online by Cambridge University Press

Then, $\{B_{s,t};\ s \in \mathbf{N} - \{0\},\ t = 1, \cdots, p-1\}$ *are differentially independent over* $K_a(X)$.

Remark. Let $m_0 < m_1 < m_2 < \cdots$ be a sequence of natural numbers satisfying the condition (1) of Theorem 1. The power series $A = \sum_{i=0}^{\infty} a_i X^i$ where $a_i = 1$ if i equals some m_j $(j \in \mathbf{N})$, otherwise 0, is hypertranscendental over K by Theorem 1. Therefore, by Theorem 2, $B_{s,t} = \sum_{i=0}^{\infty} a_{tp^{s-1}+ip^s} X^i$ $(s \in \mathbf{N} - \{0\},\ t = 1, \cdots, p-1)$ are differentially independent over $K(X)$.

Proof of Theorem 2. By $A_0^{(s-1)} = \sum_{t=0}^{p-1} X^{tp^{s-1}} A_{tp^s-1}^{(s)}$, we have

$$A = A_0^{(e)} + \sum_{s=1}^{e} \sum_{t=1}^{p-1} X^{tp^{s-1}} A_{tp^s-1}^{(s)}.$$

Hence, for $1 \leq \mu \leq p^e - 1$,

$$d_\mu A = d_\mu A_0^{(e)} + \sum_{s=1}^{e} \sum_{t=1}^{p-1} \sum_{v_1+v_2=\mu} d_{v_1} X^{tp^{s-1}} d_{v_2} A_{tp^s-1}^{(s)}.$$

For every u, $d_v A_u^{(r)} \neq 0$ implies $p^r \mid v$. Then, $d_\mu A \in K(X, d_{vp^s} A_{tp^s-1}^{(s)};\ s = 1, 2, \cdots, e,\ t = 1, 2, \cdots, p-1,\ v = 0, 1, \cdots, p^{e-s} - 1)$. Hence,

$$\begin{aligned} &K(X, d_\mu A;\ \mu = 1, 2, \cdots, p^e - 1) \\ &\quad \subseteq K(X, d_{vp^s} A_{tp^s-1}^{(s)};\ s = 1, 2, \cdots, e,\ t = 1, 2, \cdots, p-1,\ v = 0, 1, \cdots, p^{e-s} - 1). \end{aligned}$$

Since A is hypertranscendental,

$$\begin{aligned} \operatorname{tr\,deg} \{&d_{vp^s} A_{tp^s-1}^{(s)};\ s = 1, 2, \cdots, e,\ t = 1, 2, \cdots, p-1,\ v = 1, 2, \cdots, \\ &p^{e-s} - 1\}/K_a(X) \\ &\qquad \geq \operatorname{tr\,deg} \{d_\mu A;\ \mu = 1, 2, \cdots, p^e - 1\}/K_a(X) = p^e - 1. \end{aligned}$$

However, the cardinality of the set $\{(s, t, v);\ s = 1, 2, \cdots, e,\ t = 1, 2, \cdots, p-1,\ v = 0, 1, \cdots, p^{e-s} - 1\}$ is $(p-1)(p^{e-1} + p^{e-2} + \cdots + p + 1) = p^e - 1$. Hence, $\{d_{vp^s} A_{tp^s-1}^{(s)};\ s = 1, 2, \cdots, e,\ t = 1, 2, \cdots, p-1,\ v = 0, 1, \cdots, p^{e-s} - 1\}$ are algebraically independent over $K_a(X)$. Since, $d_{vp^s} A_{tp^s-1}^{(s)} = d_{vp^s}(B_{s,t})^{p^s} = (d_v B_{s,t})^{p^s}$, we see that

$$\{d_v B_{s,t};\ s = 1, 2, \cdots, e,\ t = 1, 2, \cdots, p-1,\ v = 0, 1, \cdots, p^{e-s} - 1\}$$

are algebraically independent over $K_a(X)$. Thus, we have the conclusion. q.e.d.

§ 3.

For $k \in \mathbf{N}$, we associate the real number $\langle\!\langle k \rangle\!\rangle$ as follows: If

$$k = k_0 + k_1 p + \cdots + k_{e-1} p^{e-1} \quad (0 \leq k_i \leq p - 1)$$

is the p-adic expression, then

$$\langle\!\langle k\rangle\!\rangle = \frac{k_0}{p} + \frac{k_1}{p^2} + \cdots + \frac{k_{e-1}}{p^e}.$$

For a set S, the cardinal number of S is denoted by $\sharp S$.

LEMMA 3. *Let* $\alpha \in \mathbf{R}$ $(0 \le \alpha \le 1)$. *Then,*

$$\lim_{s\to\infty} \frac{1}{s} \sharp\{\lambda \in \mathbf{N};\ \lambda \le s-1,\ \langle\!\langle \lambda\rangle\!\rangle \le \alpha\} = \alpha \quad (s \in \mathbf{N}).$$

Proof. Let $\alpha = \dfrac{\alpha_0}{p} + \dfrac{\alpha_1}{p^2} + \cdots$ be the p-adic expression of α, where there is no n such that $\alpha_n = \alpha_{n+1} = \cdots = p-1$. We fix a natural number s and associate $e = e(s) \in \mathbf{N}$ with s by $p^{e-1} \le s < p^e$. The set

$$\left\{\lambda \in \mathbf{N};\ \lambda \le s-1,\ \langle\!\langle \lambda\rangle\!\rangle < \frac{\alpha_0}{p} + \cdots + \frac{\alpha_{e-1}}{p^e}\right\}$$

is the disjoint union of the following sets:

$$T_{ij} = \{\lambda = \lambda_0 + \lambda_1 p + \cdots + \lambda_{e-1}p^{e-1};\ \lambda_0 = \alpha_0, \lambda_1 = \alpha_1, \cdots, \lambda_{i-1} = \alpha_{i-1},$$
$$\lambda_i = j, \lambda \le s-1\} \quad (i = 0, 1, \cdots, e-1, j = 0, 1, \cdots, \alpha_i - 1).$$

Let $s = s_0 + s_1 p + \cdots + s_{e-1}p^{e-1}$ be the p-adic expressions of s. If $\alpha_0 + \alpha_1 p + \cdots + \alpha_{i-1}p^{i-1} + jp^i < s_0 + s_1 p + \cdots s_i p^i$, then

$$\sharp T_{ij} = s_{i+1} + s_{i+2}p + \cdots + s_{e-1}p^{e-i-2}.$$

If $\alpha_0 + \alpha_1 p + \cdots + \alpha_{i-1}p^{i-1} + jp^i \ge s_0 + s_1 p + \cdots + s_i p^i$, then

$$\sharp T_{ij} = s_{i+1} + s_{i+2}p + \cdots + s_{e-1}p^{e-i-2} - 1.$$

In any case, we have

$$\frac{s}{p^{i+1}} - 1 \le \sharp T_{ij} \le \frac{s}{p^{i+1}}.$$

It follows that

$$\begin{aligned}
s&\left(\alpha - \frac{1}{p^{e(s)}}\right) - (p-1)e(s)\\
&\le s\left(\frac{\alpha_0}{p} + \frac{\alpha_1}{p^2} + \cdots + \frac{\alpha_{e-1}}{p^e}\right) - (\alpha_0 + \alpha_1 + \cdots + \alpha_{e-1})\\
&= \alpha_0\left(\frac{s}{p} - 1\right) + \alpha_1\left(\frac{s}{p^2} - 1\right) + \alpha_{e-1}\left(\frac{s}{p^e} - 1\right)\\
&\le \textstyle\sum_{i=0}^{e-1} \sum_{j=0}^{\alpha_i - 1} \sharp T_{ij}
\end{aligned}$$

https://doi.org/10.1017/S0027763000003913 Published online by Cambridge University Press

$$\leq \#\left\{\lambda \in \mathbf{N} \mid \lambda \leq s-1,\ \langle\!\langle \lambda \rangle\!\rangle < \frac{\alpha_0}{p} + \frac{\alpha_1}{p^2} + \cdots + \frac{\alpha_{e-1}}{p^e}\right\}$$
$$\leq \#\{\lambda \in \mathbf{N} \mid \lambda \leq s-1,\ \langle\!\langle \lambda \rangle\!\rangle \leq \alpha\}$$
$$\leq \#\left\{\lambda \in \mathbf{N} \mid \lambda \leq s-1,\ \langle\!\langle \lambda \rangle\!\rangle < \frac{\alpha_0}{p} + \frac{\alpha_1}{p^2} + \cdots + \frac{\alpha_{e-1}}{p^e}\right\} + 1$$
$$\leq \textstyle\sum_{i=0}^{e-1} \sum_{j=0}^{\alpha_i - 1} \# T_{ij} + 1$$
$$\leq \alpha_0 \frac{s}{p} + \alpha_1 \frac{s}{p^2} + \alpha_{e-1} \frac{s}{p^e} + 1$$
$$= s\left(\frac{\alpha_0}{p} + \frac{\alpha_1}{p^2} + \cdots + \frac{\alpha_{e-1}}{p^e}\right) + 1$$
$$\leq s\alpha + 1 .$$

Since $\lim_{s\to\infty}\left(\alpha - \dfrac{1}{p^{e(s)}} - \dfrac{(p-1)e(s)}{s}\right) = \lim_{s\to\infty}\left(\alpha + \dfrac{1}{s}\right) = \alpha$, we have the conclusion. q.e.d.

LEMMA 4. *A power series A is hypertranscendental over K if and only if $\{A_0^{(e)}, \cdots, A_{p^e-1}^{(e)}\}$ is algebraically independent over $K(X)$ for every $e \in \mathbf{N}$.*

Proof. It is easy to see that if $\mu \leq p^e - 1$, then

$$d_\mu A_k^{(e)} = d_\mu\left(\textstyle\sum_{i=0}^{\infty} a_{k+ip^e} X^{ip^e}\right) = 0$$

for $k \in \{0, 1, \cdots, \mu\}$. Since $A = A_0^{(e)} + XA_1^{(e)} + \cdots + X^{p^e-1}A_{p^e-1}^{(e)}$, the vector space spanned by $A, Xd_1A, \cdots, X^{p^e-1}d_{p^e-1}A$ over K coincides with the vector space spanned by $A_0^{(e)}, XA_1^{(e)}, \cdots, X^{p^e-1}A_{p^e-1}^{(e)}$ over K. q.e.d.

THEOREM 3. *For any $\alpha \in \mathbf{R}$ $(0 \leq \alpha \leq 1)$, there exists a formal power series B of $K[\![X]\!]$ such that*

$$\lim_{s\to\infty} \frac{1}{s} \operatorname{tr\,deg} \{B, d_1B, \cdots, d_{s-1}B\}/K(X) = \alpha \quad (s \in \mathbf{N}).$$

Proof. Let $A = \sum_{i=0}^{\infty} a_i X^i$ be hypertranscendental. We consider the formal power series $B = \sum_{i=0}^{\infty} \varepsilon_i a_i X^i$ with $\varepsilon_i = 0$ if $\langle\!\langle i \rangle\!\rangle > \alpha$ and $\varepsilon_i = 1$ if $\langle\!\langle i \rangle\!\rangle \leq \alpha$. Let $\alpha = \dfrac{\alpha_0}{p} + \dfrac{\alpha_1}{p^2} + \cdots$ be the p-adic expression of α, where there is no n such that $\alpha_n = \alpha_{n+1} = \cdots = p-1$. We fix a natural number s and associate

$$e = e(s) \in \mathbf{N} \quad \text{by } p^{e-1} \leq s < p^e,$$
$$t = \alpha_0 + \alpha_1 p + \cdots + \alpha_{e-1} p^{e-1}$$

and

$$\beta = \langle\langle t \rangle\rangle = \frac{\alpha_0}{p} + \frac{\alpha_1}{p^2} + \cdots + \frac{\alpha_{e-1}}{p^e}.$$

For every $k \in \mathbf{N}$ $(k < p^e)$ such that $\langle\langle k \rangle\rangle > \alpha$ and every $i \in \mathbf{N}$, we have $\langle\langle ip^e + k \rangle\rangle \geq \langle\langle k \rangle\rangle > \alpha$. By the definition of B, we have

$$B_k^{(e)} = \sum_{i=0}^{\infty} \varepsilon_{k+ip^e} a_{k+ip^e} X^{ip^e} = 0.$$

Therefore,

(1) if $\langle\langle k \rangle\rangle > \alpha$ then $B_k^{(e)} = 0$.

For each $j \in \mathbf{N}$ $(j < p^e)$, either $\langle\langle k \rangle\rangle > \langle\langle j \rangle\rangle$ or else $\langle\langle j \rangle\rangle > \alpha$. In the former case, we have $\binom{j}{k} = 0$ by Lemma 1. In the latter case, we have $B_j^{(e)} = 0$ by (1). Hence we have

$$d_k B = \sum_{j=k}^{p^e-1} \binom{j}{k} B_j^{(e)} X^{j-k} = 0.$$

Therefore,

(2) if $\langle\langle k \rangle\rangle > \alpha$ then $d_k B = 0$.

It follows that

$$K(X, B, d_1B, d_2B, \cdots, d_{s-1}B) = K(X)(d_kB;\ k \leq s-1,\ \langle\langle k \rangle\rangle \leq \alpha).$$

Hence

(3) $\operatorname{tr\,deg}\{B, d_1B, d_2B, \cdots, d_{s-1}B\}/K(X) \leq \#\{k \in \mathbf{N};\ k \leq s-1,\ \langle\langle k \rangle\rangle \leq \alpha\}$.

For every $k \in \mathbf{N}$ $(k < p^e)$ such that $\langle\langle k \rangle\rangle < \beta$ and every $i \in \mathbf{N}$, we have $\langle\langle ip^e + k \rangle\rangle < \langle\langle k \rangle\rangle + \dfrac{1}{p^e} \leq \alpha$. By the definition of B, we have

$$B_k^{(e)} = \sum_{i=0}^{\infty} \varepsilon_{k+ip^e} a_{k+ip^e} X^{ip^e} = A_k^{(e)}.$$

Therefore,

(4) if $\langle\langle k \rangle\rangle < \beta$ then $B_k^{(e)} = A_k^{(e)}$.

For any $k \in \mathbf{N}$ with $k < p^e$ and $\langle\langle k \rangle\rangle < \beta$ it follows from (1) and (4) that

$$d_k B = A_k^{(e)} + \binom{t}{k} B_t^{(e)} X^{t-k} + \sum \binom{i}{k} A_i^{(e)} X^{i-k}$$

where the summation ranges over all i with $k < i < p^e$, $\langle\langle i \rangle\rangle < \beta$. Therefore,

$$K(X, B_t^{(e)})(d_kB; k \leq s-1, k \neq t)(A_i^{(e)}; s \leq i < p^e, \langle\!\langle i \rangle\!\rangle < \beta) \\ = K(X, B_t^{(e)})(A_k^{(e)}; k < p^e, \langle\!\langle k \rangle\!\rangle < \beta).$$

Since $\{A_k^{(e)}; 0 \leq k < p^e, \langle\!\langle k \rangle\!\rangle < \beta\}$ is algebraically independent over $K(X)$ by Lemma 4, we have

$$\begin{aligned} &\operatorname{tr deg} \{B, d_1B, d_2B, \cdots, d_{s-1}B\}/K(X) \\ &\quad \geq \operatorname{tr deg} \{d_kB \mid k \leq s-1, k \neq t\}/K(X, B_t^{(e)}) \\ &\quad \geq \#\{k \in \mathbf{N}; k \leq s-1, \langle\!\langle k \rangle\!\rangle < \beta\} - 1. \end{aligned}$$

Since $\{k \in \mathbf{N}; k \leq s-1, \langle\!\langle k \rangle\!\rangle < \beta\} = \{k \in \mathbf{N}; k \leq s-1, \langle\!\langle k \rangle\!\rangle \leq \alpha\} - \{t\}$, we have

$$\begin{aligned} &\operatorname{tr deg} \{B, d_1B, d_2B, \cdots, d_{s-1}B\}/K(X) \\ &\quad \geq \#\{k \in \mathbf{N}; k \leq s-1, \langle\!\langle k \rangle\!\rangle \leq \alpha\} - 2. \end{aligned} \tag{5}$$

Now the conclusion follows from (3), (5) and Lemma 3. q.e.d.

The problem treated in this paper has been derived from a subject suggested by Professor Hideyuki Matsumura, for whom the authors are grateful.

References

[1] D. Hilbert, Mathematische Probleme, C.R. du 2. congrès international des math., Paris 1902.
[2] O. Hölder, Ueber die Eigenschaft der Gammafunction keiner algebraischen Differentialgleichung zu genügen, Math. Ann., **28** (1886), 1–13.
[3] F. Kuiper, On algebraic independence in differential rings, Proc. Kon. Ned. Akad. Wetensch., Ser. A**67** (1964), 90–103.
[4] K. Okugawa, "Differential Algebra of Nonzero Characteristic", Lectures in Mathematics, Department of Math. Kyoto Univ., **16**, 1987.
[5] K. Shikishima-Tsuji, Differentially quasi-algebraic extension of positive characteristic p, submitted.

Kayoko Shikishima-Tsuji
Institute of Mathematics
Yoshida College
Kyoto University
Kyoto 606, Japan

Masashi Katsura
Department of Mathematics
Faculty of Science
Kyoto Sangyo University
Kyoto 603, Japan