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A note on limbless trees

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It is shown by enumeration that for any two non-isomorphic limbs with the same number of points the number of trees with p points that contain one limb equals the number of trees with p points that contain the other.

A *limb* [3] R at a point r in a tree T is a maximal subtree of T that contains r and one or more points adjacent to r. The point r is called the root of R. Schwenk [3] showed that the number of trees with a given number of points and containing a specified limb R depends only on the number of points in R and is independent of the structure of R. This was done by examining two cases defined by comparison of the number of points in the trees with the number of points in R. Schwenk then enumerated trees *not* containing any given limb R with n points by treating the special case in which R has $n - 1$ endpoints all adjacent to a common point. The purpose of this note is to present an alternative derivation of these results by enumerating directly the trees not containing any given limb R with n points, independently of the structure of R. Terminology follows [1].

Let $P(x)$ be the counting series for the trees which do not contain a copy of R and which are rooted at a point. Let $V(x)$ be the counting series for the trees that contain one or more copies of R and are rooted at a point which is common to all copies of R but is not a root of any copy of R. Let

$$L(x) = P(x) + V(x) . \tag{1}$$

Let $F(T)$ denote the collection of rooted trees obtained from a rooted

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tree T by deleting the root u of T and rooting the resulting trees at the points adjacent to u. Then T is a tree enumerated by $L(x)$ if and only if all members of $F(T)$ are enumerated by $L(x)$ and $F(R)$ is not a subcollection of $F(T)$. Now, as for ordinary rooted trees [1, p. 188], $x \exp \sum_{i=1}^{\infty} L(x^i)/i$ enumerates the rooted trees T for which all members of $F(T)$ are counted by $L(x)$. Similarly, $x^{n-1}x \exp \sum_{i=1}^{\infty} L(x^i)/i$ enumerates the rooted trees T for which all members of $F(T)$ are counted by $L(x)$ and $F(R)$ occurs at least once as a subcollection of $F(T)$. The factor x^{n-1} counts the points in one copy of $F(R)$. Hence

$$L(x) = (x-x^n) \exp \sum_{i=1}^{\infty} L(x^i)/i . \tag{2}$$

We now require Otter's Theorem [1, p. 189]: for any tree, the number of dissimilar points minus the number of dissimilar non-symmetric lines equals 1. Summing this equation over all trees not containing R we obtain

$$P(x) - Q(x) = l(x) , \tag{3}$$

where $l(x)$ is the counting series for trees not containing R and $Q(x)$ is the counting series for the trees that do not contain R and are rooted at a non-symmetric line. Let $E(x)$ denote the counting series for non-symmetrically line-rooted trees which contain one or more copies of R with the line-root in all copies of R. For a line-rooted tree T let $H(T)$ denote the pair of rooted-trees obtained from T by deleting its line-root (u, v) and rooting the resulting trees at u and v. Then T is a tree enumerated by $Q(x) + E(x)$ if and only if the members of $H(T)$ are enumerated by $L(x)$ and are not isomorphic. But ordered pairs of trees enumerated by $L(x)$ have counting series $L^2(x)$ and pairs of isomorphic trees enumerated by $L(x)$ have counting series $L(x^2)$. Hence

$$Q(x) + E(x) = \tfrac{1}{2}\left(L^2(x)-L(x^2)\right) . \tag{4}$$

Equations (1), (3), and (4) give us

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$$(5) \qquad l(x) = L(x) - V(x) + E(x) - \tfrac{1}{2}\left(L^2(x) - L\left(x^2\right)\right) .$$

We now show that $V(x) = E(x)$. Let T be a tree containing one or more copies of R no two of which are line disjoint. If r_1 and r_2 are distinct points in T and the roots of copies R_1 and R_2 of R respectively then r_1 $\left(r_2\right)$ is in r_2 $\left(R_1\right)$. Also, r_1 and r_2 are equidistant from the centre of T, otherwise the heights of R_1 and R_2 are different. (If T is bicentral we take the distance to the closer central point.) Thus the centre of T is in I, the subtree induced by the points common to all copies of R in T, and the set of points r_i which are the roots of copies R_i of R is a union of similarity classes of T. (Carrying the distance argument further we see that this set is actually a similarity class of T, which is essentially Lemma 4 in [3]; but we do not require this information here.) Hence, if A_i denotes the maximal limb at r_i that is line-disjoint from I, then the set of points (lines) in the collection of all A_i is a union of similarity classes of points (lines) of T. Moreover there is a one-to-one correspondence between the similarity classes of points (excluding r_i) and lines in each A_i determined by the distance of its points from r_i. Hence, by Otter's Theorem, the number of dissimilar (in T) points (excluding the r_i) in I equals the number of non-symmetric dissimilar (in T) lines in I. Therefore $V(x) = E(x)$ and equation (5) reduces to

$$(6) \qquad l(x) = L(x) - \tfrac{1}{2}\left(L^2(x) - L\left(x^2\right)\right) .$$

Equation (6) together with equation (2) produces the counting series for the trees without the limb R. Moreover, these equations were derived independently of the structure of R. Thus for any two non-isomorphic limbs with n points the number of trees with p points that contain one limb equals the number of trees with p points that contain the other.

Equations (2) and (6) were derived independently in [2] for the particular case in which R is B the 3-point tree rooted at its centre. This result was used to provide the counting series for stable trees (trees containing B). The author wishes to thank R.P. Hale (Gordon Institute of

Technology) for constructing a computer program that generates a significant number of coefficients of $L(x)$ and $l(x)$ for any n. We have used the program to count the number of stable trees with p points for $1 \le p \le 199$.

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