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SIMPLE QUOTIENTS OF EUCLIDEAN LIE ALGEBRAS

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Introduction. In [**2**], we considered a class of Lie algebras generalizing the classical simple Lie algebras. Using a field Φ of characteristic zero and a square matrix (A_{ij}) of integers with the properties (1) $A_{ii} = 2$, (2) $A_{ij} \leqq 0$ if $i \neq j$, (3) $A_{ij} = 0$ if and only if $A_{ji} = 0$, and (4) $(A_{ij})\text{diag}\{\epsilon_1, \ldots, \epsilon_l\}$ is symmetric for some appropriate non-zero rational ϵ_i, a Lie algebra $E = E((A_{ij}))$ over Φ can be constructed, together with the usual accoutrements: a root system, invariant bilinear form, and Weyl group.

For indecomposable (A_{ij}), E is simple except when (A_{ij}) is singular and removal of any row and corresponding column of (A_{ij}) leaves a Cartan matrix. The non-simple Es, Euclidean Lie algebras, were our object of study in [**3**] as well as in the present paper. They are infinite-dimensional, have ascending chain condition on ideals, and proper ideals are of finite codimension. Furthermore, there exists a special bijective linear mapping $'\colon E \to E$ with the property $[a'b] = [ab]'$ for all $a, b \in E$. This shift map is determined only up to a scalar multiple. For $\mu \in \Phi^\times = \Phi - \{0\}$, $I(\mu) = \{a' - \mu a \mid a \in E\}$ is an ideal, and the quotient $E(\mu)$ of E by this ideal is finite-dimensional central simple over Φ. In this paper we are concerned with the structure of these simple quotient algebras.

Every Euclidean Lie algebra has a tier number associated with it (see § 1), and this is one of 1, 2, or 3. In [**3**] we proved that when the tier number is 1, $E(\mu) \cong E(1)$ for all $\mu \in \Phi^\times$ and is a split simple Lie algebra of easily determined type. For 2-tiered Lie algebras (with the exception of $F_{4,2}$) we showed that $E(\mu)$ has type independent of μ but $E(\mu)$ and $E(\nu)$ may fail to be isomorphic for some μ and ν in $\Phi^\times$.

The main results of this paper are the following.

THEOREM 1. *If E is a 2-tiered Euclidean Lie algebra over a field Φ of characteristic zero, then the shift map can be chosen so that $E(\mu)$ splits over $\Phi(\sqrt{\mu})$ (relative to the Cartan subalgebra $(H + E_\xi)\pi_\mu$) for all $\mu \in \Phi^\times$.*

THEOREM 3. *Under the assumptions of Theorem 1, $E(\mu) \cong E(\nu)$ if and only if $\mu\nu^{-1}$ is a square.*

We also see along the way that $F_{4,2}(\mu)$ is of type E_6 for all μ.

In the last section we show that the adjoint group of $E(\mu)$, where μ is a non-square of $\Phi^\times$, is the Steinberg twisted group arising from the split algebra $\Phi(\sqrt{\mu}) \otimes_\Phi E(\mu)$ relative to the non-trivial automorphism of $\Phi(\sqrt{\mu})$ over Φ.

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1. Proof of Theorem 1. Let $\mathbf{L} = \{0, 1, \ldots, l\}$ where l is some positive integer and let Φ be a field of characteristic zero. We identify the integers $\mathbf{Z}$ with the prime subring of Φ. Let E be a 2-tiered Euclidean Lie algebra over Φ with generators $\{e_i, f_i, h_i | i \in \mathbf{L}\}$. We briefly recall the relevant facts about E and refer the reader to [**3**] for the definitions and further details.

Let Δ be the root system for E relative to the given generators (we consider 0 to be in Δ). For each $\alpha \in \Delta$ let E_α denote the corresponding root space. $H \equiv \Phi h_0 + \ldots + \Phi h_l$ is E_0 and forms an abelian subalgebra of E of dimension l (not $l + 1$).

Denote by Π the fundamental system of roots $\{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ and by $\Delta_{\mathbf{Z}}$ the group generated by them. $\Delta_{\mathbf{Z}}$ is a free abelian group of rank $l + 1$ and the real space $\mathbf{R} \otimes_{\mathbf{Z}} \Delta_{\mathbf{Z}}$ is equipped with a positive semi-definite form (,) of rank l. The reflections $r_0, r_1, \ldots, r_l$ defined by $\alpha_0, \alpha_1, \ldots, \alpha_l$ generate an affine reflection group W which stabilizes Δ. The radical of (,) intersects Δ in a cyclic group Z and we can take as a generator for Z the positive element $\xi = \sum_{i \in \mathbf{L}} \xi_i \alpha_i$ of least height in Z. Our α_0 is chosen (to within symmetries of the diagram of Π) by the following conditions: (1) $\Pi - \alpha_0$ is connected, (2) $\xi_0 = 1$, (3) (used only when (1) and (2) do not characterize a root to within diagram symmetries) $\alpha_0 + \xi \in \Delta$.

The hypothesis (2-tiered) on E means that if an element α is in Δ, then so are all its translates by elements of $\mathbf{Z}2\xi$ (but there exist elements α in Δ for which $\alpha + \xi \notin \Delta$). If X is any subset of Δ, $\bar{X}$ will denote the set of equivalence classes represented by the elements of X in Δ taken modulo 2ξ. The shift map $'$ of E onto itself has the following properties: (1) $[ab]' = [a'b]$ for all $a, b \in E$ and (2) $E_\alpha' = E_{\alpha+2\xi}$ for all $\alpha \in \Delta$. These properties characterize it to within a scalar multiple. For each $\mu \in \Phi^\times$ let π_μ be the natural homomorphism of E onto $E(\mu)$. Then

$$E(\mu) = \sum \bigoplus_{\substack{\alpha \in \Delta \\ 0 \leqq \mathrm{ht}\,\alpha < \mathrm{ht}\,2\xi}} (E_\alpha)\pi_\mu,$$

and $(H + E_\xi)\pi_\mu$ is a Cartan subalgebra [**3**, Lemma 5].

In the following we will be considering $E(1)$ for the most part. We will denote $E(1)$ by $\bar{E}$ and $(E_\alpha)\pi_1$ by $\bar{E}_\alpha$ for all $\alpha \in \Delta$. Since $\bar{E}_\alpha = \bar{E}_\beta$ if and only if $\alpha \equiv \beta \pmod{2\xi}$, there is no ambiguity in speaking of $\bar{E}_\alpha$, where $\alpha \in \bar{\Delta}$. Since H and $\bar{H}$ are canonically isomorphic, we often consider elements of Δ as functions on $\bar{H}$. $\bar{H} + \bar{E}_\xi$ will be denoted by K. Finally we define some subsets of Δ: $\Delta_0 = \{\alpha \in \Delta - Z | \alpha + \xi \notin \Delta\}$, $\Delta_1 = \{\alpha \in \Delta - Z | \alpha + \xi \in \Delta\}$. For $\alpha = \sum_{i \in \mathbf{L}} z_i \alpha_i$ define $t(\alpha) = z_0$. Let $\Gamma = \{\alpha | t(\alpha) \text{ is odd}\}$.

For each $\beta \in \Delta - Z$ and for each $x \in E_\beta$, ad x is nilpotent, and hence $\exp(\operatorname{ad} x)$ is a well-defined automorphism of E. The group $\mathfrak{X}_\beta$, generated by these as x ranges over E_β is isomorphic to Φ^+. As usual, we call the group G generated by the $\mathfrak{X}_\beta$, as β ranges over $\Delta - Z$, the adjoint group of E, and see that it is generated by $\mathfrak{X}_{\pm\alpha_0}, \ldots, \mathfrak{X}_{\pm\alpha_l}$ [**6**]. For $x \in E_\beta$, $l \in E$, $l' \exp(\operatorname{ad} x) = (l \exp(\operatorname{ad} x))'$, whence G commutes with the shift map. Thus G stabilizes

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every ideal of E and induces a group of automorphisms on each quotient of E. The group induced on $E(\mu)$ will be denoted G_μ.

For each $w \in W$ there is an automorphism $\theta = \theta(w) \in G$ (not unique) such that $E_\beta\theta = E_{\beta w}$ for all $\beta \in \Delta$ [**1**, Theorem 2]. We will need the fact that if $w = r_{i_1} \ldots r_{i_k}$, then $\theta(w)$ can be chosen to be a product of the automorphisms exp ad e_j and exp ad f_j, $j = i_1, \ldots, i_k$. We will then say that θ is defined over $\alpha_{i_1}, \ldots, \alpha_{i_k}$.

LEMMA 1. *If* $\alpha \in \Delta_1$ *and* $a \in E_\alpha$, $a \neq 0$, *then* $[aE_\xi] \neq (0)$.

Proof. Using a suitable automorphism of W if necessary, we can suppose that $\alpha = \alpha_i \in \Pi$ and $a = e_i$. Let $K_i = \Phi e_i + \Phi h_i + \Phi f_i$ and let $b \in E_{\alpha_i+\xi}$. By [**3**, Lemma 2], the K_i-module M generated by b is irreducible. Since $(\alpha_i + \xi)(h_i) = 2$, $M \cap E_\xi \neq (0)$, and hence $[e_iE_\xi] \cap M \neq (0)$.

Fix $\alpha \in \Delta_1$ and let $V^{(\alpha)} = \bar{E}_\alpha + \bar{E}_{\alpha+\xi}$. V is a two-dimensional K-module, and hence for some finite extension P of Φ, $V_P{}^{(\alpha)} = P \otimes_\Phi V^{(\alpha)}$ has a weight vector $u_1 \neq 0$, say with weight φ_1. We suppose that P is chosen to be minimal. Let $a \in \bar{E}_\alpha - \{0\}$, $a^* \in \bar{E}_{\alpha+\xi} - \{0\}$. It is a consequence of Lemma 1 that $P \otimes \bar{E}_{\alpha+\xi}$ is not a weight space for K. We can suppose then, that $u_1 = a + \lambda a^*$, with $\lambda \in P$. It follows that $\varphi_1|\bar{H} = \alpha$, and for all $k \in \bar{E}_\xi$,

$$(1) \qquad \varphi_1(k)a = \lambda[a^*k], \qquad \lambda\varphi_1(k)a^* = [ak].$$

By Lemma 1, choose $k_0 \in \bar{E}_\xi$ such that $[ak_0] \neq 0$. Using k_0 in (1), we have $\lambda \neq 0$ and $\varphi_1|\bar{E}_\xi \neq 0$. Again from (1), it is apparent that $u_2 = a - \lambda a^*$ is a weight vector relative to K with weight φ_2 satisfying

$$\varphi_2|\bar{H} = \alpha, \qquad \varphi_2|\bar{E}_\xi = -\varphi_1|\bar{E}_\xi.$$

$V_P{}^{(\alpha)} = Pu_1 \oplus Pu_2$. Now, the characteristic polynomial of $\mathrm{ad}_V k_0$ has as its splitting field, Σ, in P either Φ or $\Phi(\sqrt{\mu})$, for some $\mu \in \Phi$. As $a \pm \lambda a^*$ are characteristic vectors for ad k_0 considered as a transformation on $V_P{}^{(\alpha)}$, and $a, a^* \in V^{(\alpha)}$, we find $\lambda \in \Sigma$. With k_0 in (1), we see that $\varphi_1(k_0)/\lambda$ and $\lambda\varphi_1(k_0)$ are in Φ, whence $\lambda^2 \in \Phi$ and $P = \Phi(\lambda)$ is an extension of degree at most two.

All this depends on our choice of $\alpha \in \Delta_1$ which has played no part up to this point. In the cases $C_{l,2}$, $F_{4,2}$, and $A_{1,2}$, Δ_1 is a single orbit under W, and hence, using suitable automorphisms $\theta(w)$, we obtain immediately that $V_P{}^{(\beta)}$ decomposes into two weight spaces relative to K for all $\beta \in \Delta_1$. In the case $B_{l,2}$, the two orbits making up Δ_1 are interchanged by the natural automorphism which effects $e_i \leftrightarrow e_{l-i}$, $f_i \leftrightarrow f_{l-i}$, $h_i \leftrightarrow -h_{l-i}$, $i \in \mathbf{L}$. There remains $BC_{l,2}$. Again there are two orbits, and we can take α_{l-1} and α_l as representatives of them (indexing as in [**3**, Table 2]). Let us suppose that the $\alpha \in \Delta_1$ of the previous discussion was taken to be α_l. Then, since $\alpha_{l-1} + \alpha_l$ and $-\alpha_l$ are in the same orbit as α_l, $V_P{}^{(\alpha_{l-1}+\alpha_l)}$ and $V_P{}^{(-\alpha_l)}$ both break into two weight spaces over P. Let $a \pm a^*$ and $b \pm b^*$ be corresponding weight vectors. [**3**, Lemma 2] shows that neither $[ab]$ nor $[a^*b]$ can be zero. All

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four of the products $[a \pm a^*, b \pm b^*]$ are weight vectors for K in $V_P^{(\alpha_l - 1)}$ and it is easy to pick two independent ones from amongst them. This shows that $V_P^{(\alpha_l - 1)}$ decomposes into two weight spaces relative to K, and hence so does $V_P^{(\beta)}$ for all $\beta \in \Delta_1$.

We have proved that if *E is a 2-tiered Euclidean Lie algebra over a field Φ of characteristic zero, $\bar{E} = E(1)$ splits relative to $\bar{H} + \bar{E}_\xi$ in some extension* P *of Φ of degree at most two.*

We obtain Theorem 1 by judiciously scaling the shift map. Fix $\alpha \in \Delta_1$ and return to the discussion centering around equation (1). If we replace a and a^* by their respective pre-images under π_1 in E_α and $E_{\alpha+\xi}$, respectively, and rewrite (1) in E, we obtain

$$\varphi_1(k)a' = \lambda[a^*k], \qquad \lambda\varphi_1(k)a^* = [ak] \tag{2}$$

for all $k \in E_\xi$.

We have seen that $\varphi_1(k)/\lambda$, $\lambda\varphi_1(k)$, and λ^2 are all in Φ for all $k \in E_\xi$. Define $''$ to be the shift map $'$ scaled by λ^{-2}, and define ψ by $\psi(k) = \lambda\varphi_1(k)$. Equations (2) become

$$\psi(k)a'' = [a^*k], \qquad \psi(k)a^* = [ak], \qquad k \in E_\xi. \tag{2'}$$

Thus replacing $'$ by $''$, the new $\bar{E} = E(1)$ is already split in Φ relative to $K = \bar{H} + \bar{E}_\xi$. Theorem 1 follows.

2. The root system of $E(1)$. Let $\Delta' \subset K^*$ be the root system for $\bar{E}$ relative to K. To avoid confusion, we denote the root space for $\varphi \in \Delta'$ by L_φ. Each $\alpha \in \bar{\Delta}_0$ yields a non-zero root $\varphi \in \Delta'$ with $L_\varphi = \bar{E}_\alpha$, $\varphi|\bar{H} = \alpha$, $\varphi|\bar{E}_\xi = 0$. Each pair α, $\alpha + \xi \in \bar{\Delta}_1$ ($\alpha + \xi$ has an obvious interpretation in $\bar{\Delta}_1$) yields two roots φ and $\tilde{\varphi}$ with $\varphi|\bar{H} = \alpha = \tilde{\varphi}|\bar{H}$, $\varphi|\bar{E}_\xi = -\tilde{\varphi}|\bar{E}_\xi$, and $L_\varphi + L_{\tilde{\varphi}} = \bar{E}_\alpha + \bar{E}_{\alpha+\xi}$. All the roots of $\Delta' - \{0\}$ are obtained in these two ways. For $\psi \in \Delta'$, we define $\tilde{\psi}$ by $\tilde{\psi}|\bar{H} = \psi|\bar{H}$, $\psi|\bar{E}_\xi = -\tilde{\psi}|\bar{E}_\xi$. $\tilde{}$ maps Δ' onto itself, is of order two, and extends the use of $\tilde{}$ above.

Define a map ι of $\bar{E}$ into itself by $\iota|\bar{H} = 1$, $\iota|\bar{E}_\xi = -1$, $\iota|\bar{E}_\alpha = 1$ if $\alpha \in \Delta - (Z \cup \Gamma)$, $\iota|\bar{E}_\alpha = -1$ if $\alpha \in \Gamma$. Then ι is an automorphism of period two since its value on any space $\bar{E}_\alpha$ depends on whether the number $t(\alpha)$ is even or odd.

Suppose that $\Delta_1 \cap (\Pi - \{\alpha_0\}) = \{\alpha_1, \ldots, \alpha_k\}$.

LEMMA 2. *Let e_i^* (f_i^*) be non-zero elements of $E_{\alpha_i+\xi}$ ($E_{-\alpha_i+\xi}$), $i = 1, \ldots, k$. Then $e_1, \ldots, e_l, f_1, \ldots, f_l, e_1^*, \ldots, e_k^*, f_1^*, \ldots, f_k^*$ generate E.*

Proof. Let A be the subalgebra of E which they generate. In the cases $E = B_{l,2}$, $C_{l,2}$, $F_{4,2}$, $\alpha_0\langle r_1, \ldots, r_l\rangle$ is the set of all roots of Δ of the form $\alpha_0 + \sum_{i=1}^{l} \lambda_i\alpha_i$ [**3**, Lemma 9]. Fix $i \in \{1, \ldots, k\}$. $\alpha_i + \xi \in \Delta$ and is in

$$\alpha_0\langle r_1, \ldots, r_l\rangle - \text{say } (\alpha_i + \xi)w = \alpha_0, \qquad w \in \langle r_1, \ldots, r_l\rangle.$$

We can produce a $\theta = \theta(w)$ defined over $\alpha_1, \ldots, \alpha_l$. Then $e_i^*\theta \in A \cap E_{\alpha_0}$. Thus e_0, and similarly f_0, is in A; whence $A = E$.

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In the cases $BC_{l,2}$ and $A_{1,2}$, with the notation of [**3**, Table 2], $2\alpha_l + \xi$ and $2\alpha_1 + \xi$ are roots, respectively, and the corresponding root spaces are clearly in A. Using [**3**, Lemma 11] and the fact that $2\alpha_l + \xi$ and $2\alpha_1 + \xi$ are of weight 4, we see that they are in $\alpha_0\langle r_1, \ldots, r_l\rangle$. We now proceed as above.

Let $\varphi_1, \ldots, \varphi_s$ be the set of all roots $\psi \in \Delta'$ such that $\psi|\bar{H} \in \Pi - \{\alpha_0\}$ (considered as functions on $\bar{H}$).

LEMMA 3. *$\{\varphi_1, \ldots, \varphi_s\}$ are linearly independent elements of K^*.*

Proof. Suppose that the φs are indexed so that $\varphi_i|\bar{H} = \alpha_i$, $i = 1, \ldots, k$. Let e_i^* (f_i^*) be a non-zero element of $E_{\alpha_i+\xi}$ $(E_{-\alpha_i+\xi})$, $i = 1, \ldots, k$. Then $[e_1^*f_1], \ldots, [e_k^*f_k]$ is a basis of E_ξ [**3**, Lemma 6 and the discussion in cases (1)–(6) following Proposition 8].

$\varphi_1|\bar{E}_\xi, \ldots, \varphi_k|\bar{E}_\xi$ are independent functions: for if not, there is a $g \in E_\xi - \{0\}$ such that $\bar{g} \in \bar{E}_\xi - \{0\}$ satisfies $\varphi_i(\bar{g}) = 0$, $i = 1, \ldots, k$. Then by (1), $[e_ig] = 0$ for $i = 1, \ldots, k$, and hence for $i = 1, \ldots, l$. Also from (1), $[e_i^*g] = 0$ for $i = 1, \ldots, k$. Now $-\varphi_i|\bar{H} = -\alpha_i|\bar{H}$ and hence $[f_ig] = 0$, $i \in \mathbf{L}$. For $i = 1, \ldots, k$, $f_i^* = [h_i^*f_i]$ for suitable $h_i^* \in E_\xi$, from which $[f_i^*g] = 0$. Lemma 2 implies that g is in the centre of E whence $g = 0$.

Put $S = \{\varphi_1, \varphi_2, \ldots, \varphi_s\} - \{\varphi_1, \tilde{\varphi}_1, \ldots, \varphi_k, \tilde{\varphi}_k\}$. Now, if

$$\sum_{i=1}^{k} \lambda_i\varphi_i + \sum_{i=1}^{k} \lambda_i'\tilde{\varphi}_i + \sum_{\varphi \in S} \lambda_\varphi \varphi = 0,$$

where all $\lambda \in \Phi$, then restricting to $\bar{H}$ and $\bar{E}_\xi$ in turn, we see that all the λs are zero. This proves the lemma.

LEMMA 4: *If $i \neq j$, $\varphi_i - \varphi_j \notin \Delta'$.*

Proof. Suppose that $\varphi_i - \varphi_j \in \Delta'$. $(\varphi_i - \varphi_j)|\bar{H}$ is the function $\alpha_m - \alpha_n$, where $\alpha_m = \varphi_i|\bar{H}$, $\alpha_n = \varphi_j|\bar{H}$. Since $\varphi_i - \varphi_j$ is induced by a non-zero root (or root pair) of $\Delta - Z$, we conclude that $\alpha_m - \alpha_n$ or $\alpha_m - \alpha_n + \xi$ is in Δ. The only possibility is the latter. [**3**, Lemmas 9 and 11] show that

$$\beta = \alpha_m - \alpha_n + \xi \in \alpha_0\langle r_1, \ldots, r_l\rangle,$$

and since the weight of β is greater than 1 and $l \geqq 2$ $(m, n \in \mathbf{L} - \{0\})$, E must be $BC_{l,2}$. Then weight β is 4. Now, the proof of [**3**, Lemma 11] actually shows that there exist $r_{i_1}, \ldots, r_{i_t} \in \{r_1, \ldots, r_l\}$ such that $\alpha_0 = \beta r_{i_1} \ldots r_{i_t}$ and $\mathrm{ht}\beta > \mathrm{ht}\beta r_{i_1} > \mathrm{ht}\beta r_{i_1}r_{i_2} \ldots > \mathrm{ht}\alpha_0$. We see that each r_{i_j} must be r_m or r_n. Then $l = 2$, and a short calculation leads to a contradiction.

THEOREM 2. *$\{\varphi_1, \ldots, \varphi_s\}$ is a simple system of roots for $\bar{E}$ relative to K. The automorphism ι is a diagram automorphism of $\bar{E}$ relative to this system.*

Proof. The assertion about ι is obvious if $\{\varphi_1, \ldots, \varphi_s\}$ is a simple system. $L_{\pm\varphi_1}, \ldots, L_{\pm\varphi_s}$ generate $\bar{E}$ by Lemma 2. Note that if $i \neq j$, then

$$\varphi_i - \varphi_j \notin \Delta' \quad \text{(Lemma 4)}.$$

Let $\langle\ ,\ \rangle$ denote the bilinear form on K^* induced by the Killing form on $\bar{E}$. For all $i \neq j$, $\langle \varphi_i, \varphi_j \rangle \leqq 0$ since $\varphi_i - \varphi_j \notin \Delta'$. Thus the integers $B_{ij} = 2\langle \varphi_i, \varphi_j \rangle / \langle \varphi_j, \varphi_j \rangle$ form a generalized Cartan matrix. Let $N = E((B_{ij}))$ be the Lie algebra defined by (B_{ij}). Since $\varphi_1, \ldots, \varphi_s$ are linearly independent, we can use the subset of $\mathbf{Z}\varphi_1 + \ldots + \mathbf{Z}\varphi_s$ obtained from $\varphi_1, \ldots, \varphi_s$ under the maps R_j: $\varphi_i \mapsto \varphi_i - B_{ij}\varphi_j$, $j = 1, \ldots, s$, as the root system of N. Since $\varphi_i - B_{ij}\varphi_j \in \Delta'$, the root system of N is finite, and hence (B_{ij}) is a Cartan matrix and N is semi-simple. Choosing $\hat{e}_i \in L_{\varphi_i}, \hat{f}_i \in L_{-\varphi_i}, \hat{h}_i \in K$ such that $[\hat{e}_i\hat{f}_i] = \hat{h}_i$, $[\hat{e}_i\hat{h}_i] = 2\hat{e}_i$, $[\hat{f}_i\hat{h}_i] = -2\hat{f}_i$, we obtain a natural homomorphism of N onto $\bar{E}$. Clearly $N \cong \bar{E}$ and $N_\beta \to L_\beta$ for all $\beta \in \Delta'$, whence each $\beta \in \Delta'$ is either a non-positive or non-negative integral combination of $\varphi_1, \ldots, \varphi_s$.

3. Proof of Theorem 3. If $\mathfrak{L}$ is a semi-simple Lie algebra over a field Φ, split relative to a Cartan subalgebra $\mathfrak{H}$, and if (B_{ij}) is a Cartan matrix and $\beta_1, \ldots, \beta_l$ is a simple root system for $\mathfrak{L}$, we say that a set of generators $a_i \in \mathfrak{L}_{\beta_i}$, $b_i \in \mathfrak{L}_{-\beta_i}$, $i = 1, \ldots, l$, is a standard set of generators for $\mathfrak{L}$, if, putting $c_i = [a_ib_i]$ we have $[a_ic_j] = B_{ij}a_i$ and $[b_ic_j] = -B_{ij}b_i$.

We use the notation of the previous section. Then $\varphi_1, \ldots, \varphi_s$ is a simple system of roots for Δ' and we can choose $e_i^* \in E_{\alpha_i+\xi}, f_i^* \in E_{-\alpha_i+\xi}, i = 1, \ldots, k$, $\lambda_1, \ldots, \lambda_k \in \Phi^\times$ such that

$$\bar{e}_1 \pm \bar{e}_1^*, \ldots, \bar{e}_k \pm \bar{e}_k^*, \bar{e}_{k+1}, \ldots, \bar{e}_l, \lambda_1(\bar{f}_1 \pm \bar{f}_1^*), \ldots, \lambda_k(\bar{f}_k \pm \bar{f}_k^*), \bar{f}_{k+1}, \ldots, \bar{f}_l$$

is a standard set of generators for $\bar{E} = E(1)$. ($\bar{a}$ means $(a)\pi_1$.)

Let $\tilde{\Phi}$ be an algebraic closure of Φ and, for each $\mu \in \Phi^\times$, let $\sqrt{\mu}$ be a square root of μ in $\tilde{\Phi}$. Let $\mu \in \Phi^\times$ and let $\mathrm{P} = \Phi(\sqrt{\mu})$. Define a linear map κ: $E_\mathrm{P} \to E_\mathrm{P}$ by $e_\alpha \mapsto (\sqrt{\mu})^{-t(\alpha)}e_\alpha$ for all $\alpha \in \Delta$, and $e_\alpha \in E_\alpha$. κ is an automorphism of E and maps the ideal $I(1)$ onto $I(\mu)$. Thus there is an induced isomorphism κ' of $E_\mathrm{P}(1)$ onto $E_\mathrm{P}(\mu)$. If $\sqrt{\mu} \in \Phi$, this shows that $E(1) \cong E(\mu)$.

Suppose now that $\sqrt{\mu} \notin \Phi$. Let τ_0 denote the non-trivial automorphism of P over Φ and τ_0' some extension of τ_0 to an automorphism of $\tilde{\Phi}$ over Φ. The image under κ' of the standard basis of $E(1)$ given above is

$$e_1\pi_\mu \pm (\sqrt{\mu})^{-1}e_1^*\pi_\mu, \ldots, e_k\pi_\mu \pm (\sqrt{\mu})^{-1}e_k^*\pi_\mu, e_{k+1}\pi_\mu, \ldots, e_l\pi_\mu,$$

etc. These generate, over Φ, an algebra X isomorphic to the split algebra $E(1)$. The semi-linear automorphism $\tau = \tau_0 \otimes 1_{E(\mu)}$ of $E_\mathrm{P}(\mu) = \mathrm{P} \otimes_\Phi E(\mu)$ fixes $E(\mu)$ while performing a diagram automorphism on X. This is sufficient to prove that $X \not\cong E(\mu)$. In fact, let $\tau_X = \tau|X$, and let $\tilde{\tau}_X$ be the extension of τ_X to an automorphism of $\tilde{X} = X_{\tilde{\Phi}} = E(\mu)_{\tilde{\Phi}}$. Suppose, by way of contradiction, that ω: $E(\mu) \to X$ is an isomorphism. Let $\tilde{\omega}$ be the extension of ω to an automorphism of $\tilde{X}$ and τ^* the extension of τ to a τ_0'-semi-linear map of $\tilde{X}$ onto itself. From

$$\tau_X^{-1}\tau_X = \omega^{-1}1_{E(\mu)}\omega \tag{3}$$

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we obtain

$$(4) \qquad \tilde{\tau}_X{}^{-1}\tau^* = \tilde{\omega}^{-1}\tau^*\tilde{\omega}$$

since each side of (4) is the unique τ_0'-semi-linear extension to $\tilde{X}$ of the corresponding side of (3).

Thus we have $\tilde{\tau}_X = \tau^*\tilde{\omega}^{-1}(\tau^*)^{-1}\tilde{\omega}$. But $\tilde{\tau}_X$ is a diagram automorphism of $\tilde{X}$, and consequently $\tau^*\tilde{\omega}^{-1}(\tau^*)^{-1}\tilde{\omega}$ is an outer automorphism. This means that $\tilde{\omega}$ must be an outer automorphism. However, it is easy to see that the choice of ω can be made so that $\tilde{\omega}$ is inner. This shows that $E(\mu) \not\cong X \cong E(1)$.

The general case $\mu\nu^{-1} \notin \Phi^{\times 2}$ now follows immediately. We can suppose that neither μ nor ν is in $\Phi^{\times 2}$. $\nu \notin \mathrm{P}^{\times 2}$, where $\mathrm{P} = \Phi(\sqrt{\mu})$, and hence

$$E(\mu)_{\mathrm{P}} = E_{\mathrm{P}}(\mu) \cong E_{\mathrm{P}}(1) \not\cong E_{\mathrm{P}}(\nu) = E(\nu)_{\mathrm{P}}$$

from which $E(\mu) \not\cong E(\nu)$.

COROLLARY. *$F_{4,2}(\mu)$ is of type E_6 for all $\mu \in \Phi^{\times}$.*

Proof. From [**3**, p. 1453], $F_{4,2}(\mu)$ is of type B_6, C_6, or E_6. $F_{4,2}(1)$ is split and has a diagram automorphism, and hence is of type E_6. For other μ, $\Phi(\sqrt{\mu}) \otimes_\Phi F_{4,2}(\mu) \cong \Phi(\sqrt{\mu}) \otimes_\Phi F_{4,2}(1)$.

4. Connections with Chevalley groups. Our aim in this section is to prove the following result.

THEOREM 4. *If $\mu \notin \Phi^{\times 2}$ and τ_0 is the automorphism of $\mathrm{P} = \Phi(\sqrt{\mu})$ over Φ of period two, then the Steinberg group G_0 of the split simple algebra $E_{\mathrm{P}}(\mu)$ relative to τ_0 is precisely the group of automorphisms G_μ of $E(\mu)$ induced by G. (Every automorphism of $E(\mu)$ can obviously be identified with one of $E_{\mathrm{P}}(\mu)$.)*

G_0 is obtained as follows. Let X be a split simple Φ-subalgebra of $E_{\mathrm{P}}(\mu)$ such that $X_{\mathrm{P}} = E_{\mathrm{P}}(\mu)$. Let δ be the diagram automorphism of X and extend it to a semi-automorphism $\bar{\delta}$ of X_{P} with automorphism τ_0 on P. The elements of the adjoint group of $E_{\mathrm{P}}(\mu)$ invariant by conjugation by $\bar{\delta}$ form G_0.

For convenience we will denote $(\nu)\tau_0$ by $\bar{\nu}$, for $\nu \in \mathrm{P}$.

We select X as in § 3 so that τ is the semi-automorphism $\bar{\delta}$. Using the κ and κ' of § 3, we have the diagram

$$\begin{array}{ccccc}
\mathrm{P} \otimes E & \xrightarrow{1 \otimes \pi_\mu} & E_{\mathrm{P}}(\mu) & \longleftarrow & X \\
\uparrow \kappa & & \uparrow \kappa' & & \uparrow \kappa' \\
\mathrm{P} \otimes E & \xrightarrow{1 \otimes \pi_1} & E_{\mathrm{P}}(1) & \longleftarrow & E(1)
\end{array}$$

For each root space L_φ of $E(1)$, let L_φ' denote its image in X under κ'. Since we will be working entirely in $E(\mu)$, it is convenient to use e_i, e_i^*, etc., instead of $e_i\pi_\mu$, $e_i^*\pi_\mu$, etc.

G_μ is generated by the elements $\exp \lambda \operatorname{ad} e_i$, $\exp \lambda \operatorname{ad} f_i$, where $\lambda \in \Phi$ and

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$i = 0, 1, \ldots, l$. These extend uniquely to automorphisms in G_0, providing us with an injective homomorphism of G_μ into G_0. Conversely, each element of G_0 induces an automorphism of $E(\mu)$, and thus we have an injective homomorphism of G_0 into Aut $E(\mu)$.

X has $e_i \pm (\sqrt{\mu})^{-1}e_i^*$, $i = 1, \ldots, k$, $e_{k+1}, \ldots, e_l$, $\lambda_i(f_i \pm (\sqrt{\mu})^{-1}f_i)$, $i = 1, \ldots, k$, $\lambda_{k+1}f_{k+1}, \ldots, \lambda_l f_l$ as a standard set of generators. By [**5**, Lemmas 4.6 and 7.6], G_0 is generated by the following sets of elements:

(1) $\exp \operatorname{ad} \nu e_i$, $i = k+1, \ldots, l$, $\nu \in \Phi$;

(2) $\exp \operatorname{ad} \nu(e_i + (\sqrt{\mu})^{-1}e_i^*) \exp \operatorname{ad} \bar{\nu}(e_i - (\sqrt{\mu})^{-1}e_i^*)$, $i = 1, \ldots, k$, when $\varphi_i + \tilde{\varphi}_i \notin \Delta'$ ($\nu \in \mathrm{P}$);

(3) $\exp \operatorname{ad} \nu(e_i + (\sqrt{\mu})^{-1}e_i^*) \exp \operatorname{ad} \bar{\nu}(e_i - (\sqrt{\mu})^{-1}e_i^*) \exp \operatorname{ad} y$ for $i = 1, \ldots, k$, when $\varphi_i + \tilde{\varphi}_i \in \Delta'$ ($\nu \in \mathrm{P}$). Here y is in $\mathrm{P}(L'_{\varphi_i+\tilde{\varphi}_i})$ and its precise form is of no importance to us;

(4) the expressions resulting from (1), (2), and (3) when the es are replaced by fs.

Generators of type (1) are clearly in G_μ. In the case (2),

$$[e_i + (\sqrt{\mu})^{-1}e_i^*, e_i - (\sqrt{\mu})^{-1}e_i^*] = 0,$$

and so we obtain $(\exp \operatorname{ad}(\nu + \bar{\nu})e_i)(\exp (\sqrt{\mu})^{-1}(\nu - \bar{\nu})e_i^*)$, which is in G_μ. In case (3), put $x^+ = e_i + (\sqrt{\mu})^{-1}e_i^*$, $x^- = e_i - (\sqrt{\mu})^{-1}e_i^*$. Using the facts that $2\varphi_i + \tilde{\varphi}_i$ and $\varphi_i + 2\tilde{\varphi}_i$ are not in Δ', $y \in \mathrm{P}(L'_{\varphi_i+\tilde{\varphi}_i})$, and the Campbell-Hausdorff formula [**1**] we obtain

$$\begin{aligned}\exp \operatorname{ad} \nu x^+ \exp \operatorname{ad} \bar{\nu}x^- \exp \operatorname{ad} y &= \exp \operatorname{ad}(\nu x^+ + \bar{\nu}x^- + \tfrac{1}{2}\nu\bar{\nu}[x^+x^-]) \exp \operatorname{ad} y \\ &= \exp \operatorname{ad}(\nu x^+ + \bar{\nu}x^-) \exp \operatorname{ad}(\tfrac{1}{2}\nu\bar{\nu}[x^+x^-] + y).\end{aligned}$$

Since τ commutes with our generator, $g = \frac{1}{2}\nu\bar{\nu}[x^+x^-] + y$ is in $E(\mu)$. Thus $\exp \operatorname{ad} g \in G_\mu$. Writing $\nu x^+ + \bar{\nu}x^- = a + b$, where $a \in (E_{\alpha_i})\pi_\mu$, $b \in (E_{\alpha_i+\xi})\pi_\mu$, $\exp \operatorname{ad} a \exp \operatorname{ad} b = \exp \operatorname{ad}(a + b + \frac{1}{2}[ab]) = \exp \operatorname{ad}(a + b) \exp \operatorname{ad} \frac{1}{2}[ab]$, since $3\alpha_i$ and $3\alpha_i + \xi$ are not in Δ. Thus $\exp \operatorname{ad}(a + b) \in G_\mu$. Now the homomorphism $G_0 \to \operatorname{Aut} E(\mu)$ maps G_0 into G_μ, whence $G_0 = G_\mu$ on identification.

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