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Asymptotic Dimension of Proper CAT(0) Spaces that are Homeomorphic to the Plane

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Abstract. In this paper, we investigate a proper CAT(0) space (X, d) that is homeomorphic to $\mathbb{R}^2$ and we show that the asymptotic dimension $\operatorname{asdim}(X, d)$ is equal to 2.

1 Introduction and Preliminaries

In this paper, we study the asymptotic dimension of proper CAT(0) spaces that are homeomorphic to $\mathbb{R}^2$.

A metric space (X, d) is *proper* if all closed bounded sets in (X, d) are compact. We say that a metric space (X, d) is a *geodesic space* if for any $x, y \in X$, there exists an isometric embedding $\xi\colon [0, d(x, y)] \to X$ such that $\xi(0) = x$ and $\xi(d(x, y)) = y$ (such a ξ is called a *geodesic*).

Let (X, d) be a geodesic space and let T be a geodesic triangle in X. A *comparison triangle* for T is a geodesic triangle $\overline{T}$ in the Euclidean plane $\mathbb{R}^2$ with the same edge lengths as T. Choose two points x and y in T. Let $\bar{x}$ and $\bar{y}$ denote the corresponding points in $\overline{T}$. Then the inequality $d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$ is called the CAT(0)-*inequality*, where $d_{\mathbb{R}^2}$ is the usual metric on $\mathbb{R}^2$. A geodesic space X is called a CAT(0) *space* if the CAT(0)-inequality holds for all geodesic triangles T and for all choices of two points x and y in T. Details of CAT(0) spaces are found in [1].

In Section 2, we first investigate proper CAT(0) spaces that are homeomorphic to $\mathbb{R}^2$ and we show the following.

Proposition 1.1 *Let (X, d) be a proper* CAT(0) *space that is homeomorphic to $\mathbb{R}^2$. Then $S(x, r)$ is homeomorphic to $\mathbb{S}^1$ for all $x \in X$ and all $r > 0$. Hence the boundary ∂X is homeomorphic to a circle $\mathbb{S}^1$.*

Let (X, d) be a metric space and let $\mathcal{U}$ be a family of subsets of (X, d). The family $\mathcal{U}$ is said to be *uniformly bounded* if there exists a positive number K such that $\operatorname{diam} U \leq K$ for all $U \in \mathcal{U}$. The family $\mathcal{U}$ is said to be *r-disjoint* if $d(U, U') > r$ for any $U, U' \in \mathcal{U}$ with $U \neq U'$.

The *asymptotic dimension* of a metric space (X, d) does not exceed n and we write $\operatorname{asdim}(X, d) \leq n$, if for every $r > 0$ there exist uniformly bounded, r-disjoint families

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$\mathcal{U}^0, \mathcal{U}^1, \dots, \mathcal{U}^n$ of subsets of X such that $\bigcup_{k=0}^n \mathcal{U}^k$ covers X. The *asymptotic dimension* of a metric space (X, d) is equal to n, and we write $\operatorname{asdim}(X, d) = n$, if $\operatorname{asdim}(X, d) \leq n$ and $\operatorname{asdim}(X, d) \nleq n - 1$.

The asymptotic dimension of a group relates to the Novikov conjecture, and there is some interesting recent research on asymptotic dimensions [2, 5–7, 9, 15]. In [9], Gromov remarks that word hyperbolic groups have finite asymptotic dimension, and Roe gives details of the proof in [12]. The asymptotic dimension of CAT(0) groups and CAT(0) spaces is unknown in general.

The purpose of this paper is to prove the following theorem.

Theorem 1.2 *If (X, d) is a proper* CAT(0) *space that is homeomorphic to $\mathbb{R}^2$, then* $\operatorname{asdim}(X, d) = 2$.

We note that the proper CAT(0) space (X, d) in this theorem need not have an action of some group. We give an example in Section 4.

2 Proper CAT(0) Spaces that are Homeomorphic to $\mathbb{R}^2$

We first give notation used in this paper.

Notation 2.1 Let the set of all natural numbers, real numbers, and $[0, \infty)$ be denoted by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$, respectively. Set $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+$, $\mathbb{B}^n = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 \leq 1\}$, and $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$. Let Y be a subspace of a metric space (X, d). The interior and the closure of Y in a space X will be denoted by $\operatorname{Int}_X Y$ and $\operatorname{Cl}_X Y$, respectively. Also set $B(x, r) = \{y \in X : d(x, y) \leq r\}$ and $S(x, r) = \{y \in X : d(x, y) = r\}$. We denote the geodesic from x to y in a CAT(0) space (X, d) by $[x, y]$ (cf. [1, Proposition II 1.4]). Set $[x, y) = [x, y] \setminus \{y\}$, $(x, y] = [x, y] \setminus \{x\}$ and $(x, y) = [x, y] \setminus \{x, y\}$.

The following lemma is known.

Lemma 2.2 *Let (X, d) be a proper* CAT(0) *space, $r > 0$ and $x_0 \in X$. Then, the following are satisfied:*

(i) *$B(x_0, r)$ is a convex set;*
(ii) *$x_0 \notin [x, y] \subset B(x_0, r)$ and $(x, y) \subset B(x_0, r) \setminus S(x_0, r)$ for any $x, y \in S(x_0, r)$ with $d(x, y) < 2r$;*
(iii) (cf.[1, Lemma II 5.8 and Proposition II 5.12]) *If X is a manifold, for each $x \in X \setminus \{x_0\}$, there exists a geodesic line $\xi\colon \mathbb{R} \to X$ such that $\xi(0) = x_0$ and $\xi(d(x_0, x)) = x$.*

We investigate a proper CAT(0) space that is homeomorphic to $\mathbb{R}^2$.

Notation 2.3 Let $(X, d), r, x_0, x$, and y be as in Lemma 2.2(ii). Suppose that X is homeomorphic to $\mathbb{R}^2$. By Lemma 2.2, there exist two geodesic rays $\xi_{x_0,x}, \xi_{x_0,y}\colon \mathbb{R}_+ \to X$ such that $\xi_{x_0,x}(0) = \xi_{x_0,y}(0) = x_0$, $\xi_{x_0,x}(r) = x$ and $\xi_{x_0,y}(r) = y$. By Lemma 2.2, $\xi_{x_0,x}([r, \infty)) \cup [x, y] \cup \xi_{x_0,y}([r, \infty))$ is homeomorphic to $\mathbb{R}$. Since X is homeomorphic to $\mathbb{R}^2$, by Schönflies Theorem there exists the component C of $X \setminus \xi_{x_0,x}([r, \infty)) \cup [x, y] \cup \xi_{x_0,y}([r, \infty))$ such that $x_0 \notin C$. Set $\ell(x, y) = S(x_0, r) \cap \operatorname{Cl}_X C$.

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We show some lemmas.

Lemma 2.4 *Let (X, d) be a proper* CAT(0) *space that is homeomorphic to* $\mathbb{R}^2$*. Then, $S(x, r)$ is a continuum for all $x \in X$ and all $r > 0$.*

Proof Let $x_0 \in X$ and $r > 0$. Since $B(x_0, r)$ is a convex set, by duality (cf. [13]),

$$\bar{H}_0(X \setminus B(x_0, r)) \cong \check{H}^1(B(x_0, r)) = 0,$$

thus, $X \setminus \mathrm{Int}_X B(x_0, r) = \mathrm{Cl}_X(X \setminus B(x_0, r))$ is connected. Since there exists a deformation retraction of $X \setminus \mathrm{Int}_X B(x_0, r)$ onto $S(x_0, r)$, $S(x_0, r)$ is connected. ∎

Lemma 2.5 *Let (X, d) be a proper* CAT(0) *space that is homeomorphic to* $\mathbb{R}^2$*, $r > 0$, $x_0 \in X$, $x, y \in S(x_0, r)$ with $0 < d(x, y) < 2r$ and $z \in \ell(x, y)$. Then*

(i) *$\ell(x, y)$ is a continuum,*
(ii) *$[x, y] \cap [x_0, z] \neq \varnothing$, and*
(iii) *$d(x, z) \leq d(x, y)$.*

Proof (i) By Notation 2.3, there exists the component D of $X \setminus R$ such that $C \subset D$, where $R = \xi_{x_0,x}(\mathbb{R}_+) \cup \xi_{x_0,y}(\mathbb{R}_+)$. Since X is homeomorphic to $\mathbb{R}^2$, by Schönflies Theorem, $\mathrm{Cl}_X D$ is homeomorphic to $\mathbb{R}^2_+$. Let D' be a copy of D. Define an equivalent relation: $\sim$ in $D \cup D'$ as follows: for $a \in D$ and $a' \in D'$, $a \sim a'$ if and only if $a = a'$, $a \in R$, and $a' \in R'$. Set $B = B(x_0, r) \cap \mathrm{Cl}_X D$, $\widetilde{D} = (D \cup D')/{\sim}$ and $\widetilde{B} = (B \cup B')/{\sim}$. Then there exists a deformation retraction $\mathrm{Cl}_X(D \setminus B)$ onto $\ell(x, y)$, $\widetilde{D}$ is homeomorphic to $\mathbb{R}^2$ and $\widetilde{B}$ is a contractible compact set. By the same method as in the proof of Lemma 2.4, we can show that $\ell(x, y) \cup (\ell(x, y))'/{\sim}$ is connected. Since there exists the natural surjective map from $\ell(x, y) \cup (\ell(x, y))'/{\sim}$ onto $\ell(x, y)$, $\ell(x, y)$ is connected.

(ii) We may assume that $z \notin \{x, y\}$. By Notation 2.3, there exists the component C of $X \setminus \xi_{x_0,x}([r, \infty)) \cup [x, y] \cup \xi_{x_0,y}([r, \infty))$ such that $x_0 \notin C$ and $z \in C$. Thus, $\xi_{x_0,x}([r, \infty)) \cup [x, y] \cup \xi_{x_0,y}([r, \infty))$ separates x_0 and z in X. Since $[x_0, z] \subset B(x_0, r)$ is an arc connecting x_0 and z in X, $[x, y] \cap [x_0, z] \neq \varnothing$.

(iii) On the contrary, suppose $d(x, z) > d(x, y)$. By (ii), there exists $z' \in [x, y] \cap [x_0, z]$. Since $z' \in [x, y]$,

$$d(x, z') + d(z', y) = d(x, y) < d(x, z) \leq d(x, z') + d(z', z),$$

thus, $d(z', y) < d(z', z)$. Then,

$$r = d(x_0, y) \leq d(x_0, z') + d(z', y) < d(x_0, z') + d(z', z) = d(x_0, z) = r,$$

a contradiction. ∎

Lemma 2.6 *Let (X, d) be a proper* CAT(0) *space that is homeomorphic to* $\mathbb{R}^2$*, $r, t > 0$, $x_0 \in X$ and $y_0 \in S(x_0, r)$. Then $S(x_0, r) \cap B(y_0, t)$ is connected.*

Proof Set $N = S(x_0, r) \cap B(y_0, t)$. If $t \geq 2r$, $S(x_0, r) \subset B(y_0, t)$. By Lemma 2.4, N is connected. We may assume that $t < 2r$. Take $x \in N$. Since $d(y_0, x) \leq t < 2r$, by Lemma 2.5, we have

$$\ell(y_0, x) \subset S(x_0, r) \cap B(y_0, d(y_0, x)) \subset S(x_0, r) \cap B(y_0, t) = N.$$

Therefore, by Lemma 2.5, $N = \bigcup\{\ell(y_0, x) : x \in N\}$ is connected, which proves the lemma. ■

We obtain the following proposition from the lemmas above.

Proposition 2.7 *Let (X, d) be a proper* CAT(0) *space that is homeomorphic to $\mathbb{R}^2$. Then, $S(x, r)$ is homeomorphic to $\mathbb{S}^1$ for all $x \in X$ and all $r > 0$.*

Proof By Lemma 2.4 and [14, Theorem 11.21], it suffices to show the following:

(i) $S(x_0, r) \setminus \{y_0, y_1\}$ is nonconnected for any $y_0, y_1 \in S(x_0, r)$ with $y_0 \neq y_1$, and
(ii) $S(x_0, r) \setminus \{y_0\}$ is connected for each $y_0 \in S(x_0, r)$.

We take two points $y_0, y_1 \in S(x_0, r)$ with $y_0 \neq y_1$. By Lemma 2.2, there exist geodesic rays $\xi_{x_0,y_0}, \xi_{x_0,y_1} : \mathbb{R}_+ \to X$ such that $\xi_{x_0,y_0}(0) = \xi_{x_0,y_1}(0) = x_0$, $\xi_{x_0,y_0}(r) = y_0$ and $\xi_{x_0,y_1}(r) = y_1$. By Schönflies Theorem, there exist closed sets Z_0, Z_1 of X such that Z_i is homeomorphic to $\mathbb{R}^2_+$ for $i = 0, 1$, $X = Z_0 \cup Z_1$, and $Z_0 \cap Z_1 \subset \xi_{x_0,y_0}(\mathbb{R}_+) \cup \xi_{x_0,y_1}(\mathbb{R}_+)$ is homeomorphic to $\mathbb{R}$. Since $S(x_0, r) \cap \mathrm{Int}_X Z_i \neq \varnothing$ for $i = 0, 1$, $S(x_0, r) \setminus \{y_0, y_1\}$ is nonconnected, which proves (i).

Let $x, y \in S(x_0, r) \setminus \{y_0\}$ with $x \neq y$. By Lemma 2.2, there exist geodesic rays $\xi_{x_0,x}, \xi_{x_0,y} : \mathbb{R}_+ \to X$ such that $\xi_{x_0,x}(0) = \xi_{x_0,y}(0) = x_0$, $\xi_{x_0,x}(r) = x$ and $\xi_{x_0,y}(r) = y$. Set $R = \xi_{x_0,x}(\mathbb{R}_+) \cup \xi_{x_0,y}(\mathbb{R}_+)$. By [1, Proposition 1.4(1), p.160], there exists $z \in [x_0, x)$ such that $\xi_{x_0,x}(\mathbb{R}_+) \cap \xi_{x_0,y}(\mathbb{R}_+) = [x_0, z]$. By Schönflies Theorem, there exists the component C of $X \setminus R$ such that $y_0 \notin C$ and $E_{x,y} = \mathrm{Cl}_X C$ is homeomorphic to $\mathbb{R}^2_+$. Set $L_{x,y} = E_{x,y} \cap S(x_0, r)$. We see

$$B(x_0, d(x_0, z)) \subset E_{x,y} \text{ or } B(x_0, d(x_0, z)) \cap E_{x,y} = \{z\}.$$

Suppose that $B(x_0, d(x_0, z)) \subset E_{x,y}$. We note that $L_{x,y}$, $B(x_0, d(x_0, z))$ and $\{z\}$ are deformation retracts of $\mathrm{Cl}_X(E_{x,y} \setminus B(x_0, r))$, $E_{x,y} \cap B(x_0, r)$ and $B(x_0, d(x_0, z))$, respectively. Thus, $\{z\}$ is a deformation retract of $E_{x,y} \cap B(x_0, r)$. Using the same method as in the proof of Lemma 2.5(i), we can show that $L_{x,y} \cup (L_{x,y})'/\!\sim$ is a deformation retract of $\mathrm{Cl}_X(E_{x,y} \setminus B(x_0, r)) \cup (\mathrm{Cl}_X(E_{x,y} \setminus B(x_0, r)))'/\!\sim$ and $\{z\}$ is a deformation retract of $(E_{x,y} \cap B(x_0, r)) \cup (E_{x,y} \cap B(x_0, r))'/\!\sim$, thus, $L_{x,y}$ is connected. Suppose that $B(x_0, d(x_0, z)) \cap E_{x,y} = \{z\}$. Since $\{z\}$ is a deformation retract of $E_{x,y} \cap B(x_0, r)$, by the same method above, we can show that $L_{x,y}$ is connected.

Fix $y_0' \in S(x_0, r) \setminus \{y_0\}$. Since $S(x_0, r) \setminus \{y_0\} = \bigcup\{L_{x,y_0'} : x \in S(x_0, r) \setminus \{y_0, y_0'\}\}$, it is connected, which proves (ii). ■

Corollary 2.8 *If (X, d) is a proper* CAT(0) *space that is homeomorphic to $\mathbb{R}^2$, then the boundary ∂X of X is homeomorphic to $\mathbb{S}^1$.*

We show the following lemma that is used in the proof of the main theorem.

Lemma 2.9 *Let (X, d) be a proper* $\mathrm{CAT}(0)$ *space that is homeomorphic to* $\mathbb{R}^2$, $x_0 \in X$, $r, t > 0$ *with* $2t < r$ *and* $x, x' \in S(x_0, r)$ *with* $3t \leq d(x, x') < 2r$. *Then there exist* $y_0, \dots, y_{3n-1} \in S(x_0, r)$, *and* $m \in \mathbb{N}$ *with* $0 < 3m < 3n - 1$ *such that* $y_0 = x$, $y_{3m} = x'$, $t \leq \operatorname{diam} \ell(y_i, y_{i+1}) \leq 2t$, $\{y_0, \dots, y_{3n-1}\} \cap \ell(y_i, y_{i+1}) = \{y_i, y_{i+1}\}$ *for each* $i = 0, \dots, 3n - 1$, $S(x_0, r) = \ell(y_0, y_1) \cup \cdots \cup \ell(y_{3n-1}, y_{3n})$, *and* $\ell(x, x') = \ell(y_0, y_1) \cup \cdots \cup \ell(y_{3m-1}, y_{3m})$, *where* $y_{3n} = y_0$.

Proof Set $z_0 = y_0 = x$. By Proposition 2.7, $S(x_0, r)$ is homeomorphic to $\mathbb{S}^1$. Since $S(x_0, r) \not\subset B(z_0, t)$, by Lemma 2.6, $S(x_0, r) \cap B(z_0, t)$, $\ell(x, x')$, and $\ell(x, x') \cap B(z_0, t)$ are arcs. Let z_1 be the end point of $\ell(x, x') \cap B(z_0, t)$ with $z_0 \neq z_1$. By Lemma 2.6, we have $\ell(z_0, z_1) = \ell(x, x') \cap B(z_0, t)$. By Lemma 2.6, $S(x_0, r) \cap B(z_1, t)$ is an arc. Since z_0 and z_1 are the end points of $\ell(z_0, z_1)$ with $d(z_0, z_1) = t$, there exists the end point z_2 of $S(x_0, r) \cap B(z_1, t)$ such that $\operatorname{diam} \ell(z_1, z_2) = t$, $\ell(z_0, z_1) \cap \ell(z_1, z_2) = \{z_1\}$ and $\ell(z_0, z_1) \cup \ell(z_1, z_2) = \ell(x, x') \cap B(z_1, t)$. Thus, by induction, we can take $z_2, \dots, z_{p+1} \in S(x_0, r)$ and an arc $\ell(z_{i-1}, z_i)$ in $\ell(x, x')$ with the end points $\{z_{i-1}, z_i\}$ such that $z_{p+1} = x'$, $\ell(z_{i-1}, z_i) \cap \ell(z_i, z_{i+1}) = \{z_i\}$ for each $i = 1, \dots, p$, $\ell(z_{i-1}, z_i) \cup \ell(z_i, z_{i+1}) = \ell(x, x') \cap B(z_i, t)$, for each $i = 1, \dots, p$, $\ell(x, x') = \bigcup_{i=1}^{p+1} \ell(z_{i-1}, z_i)$, $\operatorname{diam} \ell(z_{i-1}, z_i) = t$ for any $i = 1, \dots, p$ and $\operatorname{diam} \ell(z_p, z_{p+1}) \leq t$. Let $k \in \mathbb{N}$ and $\delta = 0, 1, 2$ such that $p = 3k + \delta$. Set $m = k$ and $y_{3m} = z_{p+1}$. If $\delta = 0$, set $y_i = z_i$ for each $i = 1, \dots, 3m-1$. If $\delta = 1$, set $y_i = z_i$ for each $i = 1, \dots, 3m - 2$ and $y_{3m-1} = z_{p-1}$. If $\delta = 2$, set $y_i = z_i$ for each $i = 1, \dots, 3(m - 1)$, $y_{3m-2} = z_{p-3}$ and $y_{3m-1} = z_{p-1}$. Similarly, we have $y_{3m+1}, \dots, y_{3n-1} \in \mathrm{Cl}_X(S(x_0, r) \setminus \ell(x, x'))$, which proves the lemma. ■

3 Asymptotic Dimension of Proper $\mathrm{CAT}(0)$ Spaces that are Homeomorphic to $\mathbb{R}^2$

First we show the following.

Lemma 3.1 *Let (X, d) be a proper* $\mathrm{CAT}(0)$ *space that is homeomorphic to* $\mathbb{R}^2$. *Then,* $\operatorname{asdim}(X, d) \geq 2$.

Proof On the contrary, suppose that $\operatorname{asdim}(X, d) \leq 1$. Let $r > 0$. There exist uniformly bounded, $3r$-disjoint families $\mathcal{U}^0, \mathcal{U}^1$ of subsets of X such that $\mathcal{U}^0 \cup \mathcal{U}^1$ covers X. Since X is homeomorphic to $\mathbb{R}^2$, there exist uniformly bounded, r-disjoint families $\mathcal{V}^0, \mathcal{V}^1$ of subsets of X satisfying the following:

(i) $\mathcal{V}^0 \cup \mathcal{V}^1$ covers X;
(ii) every $V \in \mathcal{V}^0 \cup \mathcal{V}^1$ is a compact topological 2-manifold with boundary.

Let $\varepsilon > 0$ with $\varepsilon < r/2$, let $V \in \mathcal{V}^i$ and let M and M' be two components of V with $d(M, M') = d(M, V \setminus M) < \varepsilon$. Then there exists a disk A in X such that $M \cup A \cup M'$ is connected, $V \cup A$ is a compact topological 2-manifold with boundary and $d(V \cup A, V') > r - \varepsilon$ whenever $V' \in \mathcal{V}^i$ with $V \neq V'$. Thus, we may assume that

(iii) $d(M, M') \geq \varepsilon$ for each $V \in \mathcal{V}^0 \cup \mathcal{V}^1$ and each two components M, M' of V.

Since $\mathcal{V}^0 \cup \mathcal{V}^1$ is uniformly bounded, there exists $r \leq s = \sup\{\operatorname{diam} C : C$ is a component of $V \in \mathcal{V}^0 \cup \mathcal{V}^1\} < \infty$. Thus, we may assume that there exists a component C_0 of $V_0 \in \mathcal{V}^0$ such that $s - \varepsilon < \operatorname{diam} C_0 \leq s$. We have $c_0, c_1 \in C_0$ such

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that $d(c_0, c_1) = \operatorname{diam} C_0$. By Lemma 2.2, there exists a geodesic line $\xi \colon \mathbb{R} \to X$ such that $\xi(0) = c_0$ and $\xi(\operatorname{diam} C_0) = c_1$. Since $C_0 \cap \xi(\mathbb{R}) \subset \xi([0, \operatorname{diam} C_0])$, there exists the component N of ∂C_0 containing c_0, c_1 that is contained in the closure of the unbounded component of $X \setminus C_0$.

We note that $N \subset \bigcup\{V_1 : V_1 \in \mathcal{V}^1\}$ is homeomorphic to $\mathbb{S}^1$. Since $\mathcal{V}^1$ is r-disjoint, there exists a component C_1 of $V_1 \in \mathcal{V}^1$ such that $N \subset C_1$. Then, there exist $t_0, t_1 \in \mathbb{R}$ with $t_0 < 0 < \operatorname{diam} C_0 < t_1$ such that $\xi(t_0), \xi(t_1) \in C_1 \cap \xi(\mathbb{R}) \subset \xi([t_0, t_1])$. Using a similar argument as above, we can show there exist a component N' of ∂C_1 containing $\xi(t_0), \xi(t_1)$ and $V_2 \in \mathcal{V}^0$ containing N'. If $V_0 = V_2$, by (iii), $d(c_0, \xi(t_0)) \geq \varepsilon$ and $d(c_1, \xi(t_1)) \geq \varepsilon$, *i.e.*, $d(\xi(t_0), \xi(t_1)) > \operatorname{diam} C_0 + 2\varepsilon > s$, which contradicts the definition of s. If $V_0 \neq V_2$, $d(V_0, V_2) > r$. Thus, $d(c_0, \xi(t_0)) > r$ and $d(c_1, \xi(t_1)) > r$, *i.e.*, $d(\xi(t_0), \xi(t_1)) > \operatorname{diam} C_0 + 2r > s$, which contradicts the definition of s. ■

We prove the main theorem.

Theorem 3.2 *Let (X, d) be a proper* CAT(0) *space that is homeomorphic to $\mathbb{R}^2$. Then,* $\operatorname{asdim}(X, d) = 2$.

Proof By Lemma 3.1 it suffices to show that $\operatorname{asdim}(X, d) \leq 2$.

Let $r > 0$. Fix $x_0 \in X$ and $k \in \mathbb{N}$ with $k \geq 6$. By Lemma 2.9, there exist $y_{0,0}, \dots, y_{0,3n(0)-1} \in S(x_0, kr)$ such that

$$2r \leq \operatorname{diam} \ell(y_{0,i}, y_{0,i+1}) \leq 16r, \{y_{0,0}, \dots, y_{0,3n(0)-1}\} \cap \ell(y_{0,i}, y_{0,i+1}) = \{y_{0,i}, y_{0,i+1}\}$$

for each $i = 0, \dots, 3n(0) - 1$ and

$$S(x_0, kr) = \ell(y_{0,0}, y_{0,1}) \cup \cdots \cup \ell(y_{0,3n(0)-1}, y_{0,3n(0)}),$$

where $y_{0,3n(0)} = y_{0,0}$. See Figure 3.2.1. Set

$$\mathcal{V}_{0,\delta} = \{\ell(y_{0,3i+\delta}, y_{0,3i+1+\delta}) : i = 0, \dots, n(0) - 1\}$$

for each $\delta = 0, 1, 2$.

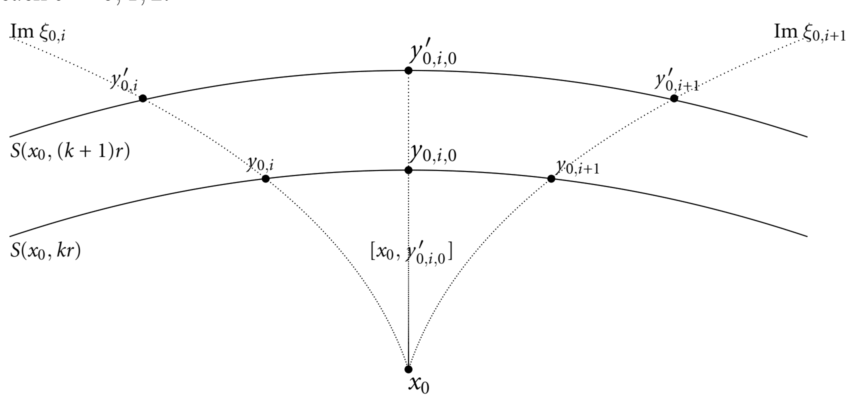

Figure 3.2.1

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For every $i = 0, \dots, 3n(0) - 1$, there exists a geodesic ray $\xi_{0,i}\colon \mathbb{R}_+ \to X$ such that $\xi_{0,i}(0) = x_0$ and $\xi_{0,i}(kr) = y_{0,i}$. Set $y'_{0,i} = \xi_{0,i}((k+1)r)$ for each $i = 0, \dots, 3n(0) - 1$. We note that $2r \le d(y_{0,i}, y_{0,i+1}) < d(y'_{0,i}, y'_{0,i+1}) \le 18r$ for each $i = 0, \dots, 3n(0) - 1$.

Let $i \in \{0, \dots, 3n(0) - 1\}$.

Suppose that $d(y'_{0,i}, y'_{0,i+1}) < 12r$. We can take $y_{0,i,0} \in \ell(y_{0,i}, y_{0,i+1})$ and $y'_{0,i,0} \in \ell(y'_{0,i}, y'_{0,i+1})$ such that $r \le d(y_{0,i}, y_{0,i,0}) = d(y_{0,i+1}, y_{0,i,0}) = d(y_{0,i}, y_{0,i+1})/2 < 6r$ and $\{y_{0,i,0}\} = [x_0, y'_{0,i,0}] \cap S(x_0, kr)$. We note that $r < d(y'_{0,i}, y'_{0,i,0}), d(y'_{0,i+1}, y'_{0,i,0}) < 8r$.

Suppose that $12r \le d(y'_{0,i}, y'_{0,i+1})$. We note that $10r \le d(y_{0,i}, y_{0,i+1})$. There exist $z_{0,i,0}, z_{0,i,1} \in \ell(y_{0,i}, y_{0,i+1})$ and $z'_{0,i,0}, z'_{0,i,1} \in \ell(y'_{0,i}, y'_{0,i+1})$ such that $d(y_{0,i}, z_{0,i,0}) = d(y_{0,i+1}, z_{0,i,1}) = r$ and $\{z_{0,i,j}\} = [x_0, z'_{0,i,j}] \cap S(x_0, kr)$ for $j = 0, 1$. We note that $d(y'_{0,i}, z'_{0,i,0}), d(y'_{0,i+1}, z'_{0,i,1}) \le 3r$ and $6r \le d(z'_{0,i,0}, z'_{0,i,1})$. By Lemma 2.9, there exist $y'_{0,i,1}, \dots, y'_{0,i,3k_{0,i}-1} \in \ell(y'_{0,i,0}, y'_{0,i,3k_{0,i}})$ such that $2r \le d(y'_{0,i,j}, y'_{0,i,j+1}) \le 4r$ and $\ell(y'_{0,i,j}, y'_{0,i,j+1}) \cap \{y'_{0,i,0}, \dots, y'_{0,i,k_{0,i}}\} = \{y'_{0,i,j}, y'_{0,i,j+1}\}$ for each $j = 0, \dots, 3k_{0,i} - 1$, where $y'_{0,i,0} = z'_{0,i,0}$ and $y'_{0,i,3k_{0,i}} = z'_{0,i,1}$. See Figure 3.2.2.

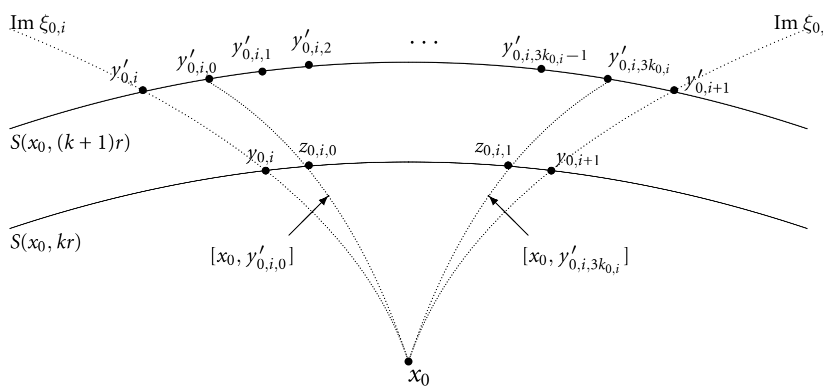

Figure 3.2.2

Set $Y_1 = \{y'_{0,i,j} : 0 \le i \le 3n(0) - 1 \text{ and } j = 0, \dots, 3k_{0,i}\}$ and $n(1) \in \mathbb{N}$ with $3n(1) - 1 = |Y_1|$. We can renumber $Y_1 = \{y_{1,i} : i = 0, \dots, 3n(1) - 1\}$ such that

$$\{y' \in S(x_0, (k+1)r) : y \in \bigcup \mathcal{V}_{0,1} \cap \bigcup \mathcal{V}_{0,2}\} \subset \bigcup \mathcal{V}_{1,0},$$
$$\{y' \in S(x_0, (k+1)r) : y \in \bigcup \mathcal{V}_{0,0} \cap \bigcup \mathcal{V}_{0,2}\} \subset \bigcup \mathcal{V}_{1,1},$$
$$\{y' \in S(x_0, (k+1)r) : y \in \bigcup \mathcal{V}_{0,0} \cap \bigcup \mathcal{V}_{0,1}\} \subset \bigcup \mathcal{V}_{1,2},$$

and $\ell(y_{1,i}, y_{1,i+1}) \cap Y_1 = \{y_{1,i}, y_{1,i+1}\}$ for each $i = 0, \dots, 3n(1) - 1$, where we let $y_{1,3n(1)} = y_{1,0}$ and $\mathcal{V}_{1,\delta} = \{\ell(y_{1,3i+\delta}, y_{1,3i+1+\delta}) : i = 0, \dots, n(1) - 1\}$ for each $\delta = 0, 1, 2$. We note that $2r \le \operatorname{diam} V \le 16r$ for all $\delta = 0, 1, 2$ and all $V \in \mathcal{V}_{1,\delta}$.

By induction, for every $m \in \mathbb{N}$ with $m \ge 2$, there exists

$$Y_m = \{y_{m,i} : i = 0, \dots, 3n(m) - 1\} \subset S(x_0, (k+m)r)$$

such that

$$\{y' \in S(x_0,(k+m)r) : y \in \bigcup \mathcal{V}_{m-1,1} \cap \bigcup \mathcal{V}_{m-1,2}\} \subset \bigcup \mathcal{V}_{m,0},$$
$$\{y' \in S(x_0,(k+m)r) : y \in \bigcup \mathcal{V}_{m-1,0} \cap \bigcup \mathcal{V}_{m-1,2}\} \subset \bigcup \mathcal{V}_{m,1},$$
$$\{y' \in S(x_0,(k+m)r) : y \in \bigcup \mathcal{V}_{m-1,0} \cap \bigcup \mathcal{V}_{m-1,1}\} \subset \bigcup \mathcal{V}_{m,2},$$

$\ell(y_{m,i}, y_{m,i+1}) \cap Y_m = \{y_{m,i}, y_{m,i+1}\}$ for each $i = 0,\dots,3n(m)-1$ and $2r \leq \operatorname{diam} V \leq 16r$ for all $\delta = 0,1,2$ and all $V \in \mathcal{V}_{m,\delta}$, where we let

$$\mathcal{V}_{m,\delta} = \{\ell(y_{m,3i+\delta}, y_{m,3i+1+\delta}) : i = 0,\dots,n(m)-1\}$$

for each $\delta = 0,1,2$.

For each $V \in \mathcal{V}_{m,\delta}$ and each $\delta = 0,1,2$, set

$$\overline{V} = \{x \in B(x_0,(k+m+1)r) \setminus \operatorname{Int}_X B(x_0,(k+m)r) : [x_0,x] \cap V \neq \varnothing\},$$

$\overline{\mathcal{V}_{m,\delta}} = \{\overline{V} : V \in \mathcal{V}_{m,\delta}\}$, and $\mathcal{W}_\delta = \{W : W \text{ is a component of } \bigcup_{m=0}^{\infty} \overline{\mathcal{V}_{m,\delta}}\}$.

By construction, we have the following:

(i) for $V \in \mathcal{V}_{m,\delta}$, $\operatorname{diam} \overline{V} \cap S(x_0,(k+m+1)r) < 12r$ if and only if $\overline{V} \cap \bigcup \overline{\mathcal{V}_{m+1,\delta}} = \varnothing$;
(ii) let $\mathcal{V}_{m+1}(V) = \{U \in \mathcal{V}_{m+1,\delta} : \overline{V} \cap U \neq \varnothing\}$ for each $V \in \mathcal{V}_{m,\delta}$, then $U \subset \overline{V}$ for $U \in \mathcal{V}_{m+1}(V)$;
(iii) we have $\mathcal{V}_{m+2}(U) = \varnothing$ for each $V \in \mathcal{V}_{m,\delta}$ and each $U \in \mathcal{V}_{m+1}(V)$, because $\operatorname{diam} U < 12r$ by construction.

For every $\delta = 0,1,2$ and every $W \in \mathcal{W}_\delta$, we have

$$\begin{aligned}\operatorname{diam} W &\leq \sup\{\operatorname{diam} V : V \in \mathcal{V}_{m,\delta} \text{ for } m \geq 0 \text{ and } \delta = 0,1,2\} + 4r\\ &\leq 16r + 4r = 20r.\end{aligned}$$

Let $V_i, V_j \in \mathcal{V}_{m,\delta}$ with $V_i \neq V_j$. We show that $d(V_i,V_j) \geq r$. On the contrary, suppose that $d(x,y) < r$ for some $x \in V_i$ and some $y \in V_j$. By Lemma 2.6, let $\ell(x,y)$ denote the arc in $S(x_0,(k+m)r) \cap B(x,d(x,y))$ with the end points $\{x,y\}$. By construction, we have $i = 0,\dots,n(m)-1$ such that $\ell(y_{m,i},y_{m,i+1}) \subsetneq \ell(x,y)$. However, $r \leq \operatorname{diam} \ell(y_{m,i},y_{m,i+1}) \leq \operatorname{diam} \ell(x,y) = d(x,y) < r$, a contradiction.

Let $\overline{V_i}, \overline{V_j} \in \overline{\mathcal{V}_{m,\delta}}$ with $\overline{V_i} \neq \overline{V_j}$. We show that $d(\overline{V_i},\overline{V_j}) \geq r$. Let $x' \in \overline{V_i}$ and $y' \in \overline{V_j}$. Set $\{x\} = [x_0,x'] \cap V_i$ and $\{y\} = [x_0,y'] \cap V_j$. By the above, $r \leq d(V_i,V_j) \leq d(x,y)$. Let T be the geodesic triangle consisting of three points x_0, x', y', let $\overline{T}$ be a comparison triangle for T in $\mathbb{R}^2$, and let $\overline{x_0}, \overline{x}, \overline{y}, \overline{x'}$, and $\overline{y'}$ denote the corresponding points in $\overline{T}$. Since X is a CAT(0) space, we have

$$r \leq d(x,y) \leq d_{\mathbb{R}^2}(\overline{x},\overline{y}) \leq d_{\mathbb{R}^2}(\overline{x'},\overline{y'}) = d(x',y'),$$

thus, $d(\overline{V_i},\overline{V_j}) \geq r$.

Let $\overline{V_i} \in \overline{\mathcal{V}_{m,\delta}}$ and $\overline{V_j} \in \overline{\mathcal{V}_{m+1,\delta}}$ with $\overline{V_i} \cap \overline{V_j} = \varnothing$. Set $W_j = \{[x_0,x] \cap S(x_0,(k+m)) : x \in \overline{V_j}\}$. By the definition of $y'_{m,i,j}$'s, similarly, we can show $d(V_i,W_j) \geq r$. Since X is a CAT(0) space, we can obtain that $d(\overline{V_i},\overline{V_j}) \geq r$ by the same method. By (i), (ii), and (iii), we have $d(W,W') \geq r$ for any $W,W' \in \mathcal{W}_\delta$ with $W \neq W'$.

Let $\mathcal{U}_0 = \{U : U \text{ is a component of } B(x_0,kr) \cup \bigcup \mathcal{W}_0\}$ and $\mathcal{U}_\delta = \mathcal{W}_\delta$ for $\delta = 1,2$. By the above, $\mathcal{U}_0 \cup \mathcal{U}_1 \cup \mathcal{U}_2$ is a uniformly bounded cover of (X,d) and $d(U,U') \geq r$ for any $U,U' \in \mathcal{U}_\delta$ with $U \neq U'$, which proves the theorem. ■

4 Application

As an application of Theorem 3.2, we obtain the following corollary.

Corollary 4.1 *Let (W,S) be a Coxeter system. If the boundary $\partial\Sigma(W,S)$ of $\Sigma(W,S)$ is homeomorphic to $\mathbb{S}^1$, then* $\operatorname{asdim} W = 2$.

Proof Let (W,S) be a Coxeter system whose boundary $\partial\Sigma(W,S)$ is homeomorphic to $\mathbb{S}^1$. Then the Coxeter group W is a virtual Poincaré duality group, and $W = W_{\tilde{S}} \times W_{S\setminus\tilde{S}}$, for some $\tilde{S} \subset S$, where the nerve $N(W_{\tilde{S}}, \tilde{S})$ is homeomorphic to $\mathbb{S}^1$ and $W_{S\setminus\tilde{S}}$ is finite ([4], cf. [10]). Then the Davis complex $\Sigma(W,S)$ splits as

$$\Sigma(W,S) = \Sigma(W_{\tilde{S}}, \tilde{S}) \times \Sigma(W_{S\setminus\tilde{S}}, S\setminus\tilde{S}).$$

Here $\Sigma(W_{\tilde{S}}, \tilde{S})$ is homeomorphic to $\mathbb{R}^2$ and $\Sigma(W_{S\setminus\tilde{S}}, S\setminus\tilde{S})$ is bounded. By Theorem 3.2, we obtain that $\operatorname{asdim}\Sigma(W,S) = 2$. Hence $\operatorname{asdim} W = 2$. ■

In general, it is known that every Coxeter group has finite asymptotic dimension ([6], cf. [8]).

Example 4.2 Let $m \in \mathbb{N}$ and let $D_m \subset \mathbb{R}^2$ be a regular m-polygon with a metric $d_m = d_{\mathbb{R}^2}|_{D_m}$ and edges $e_1, \dots, e_m$ such that $\operatorname{diam} e_i = 1$ for each $i = 1, \dots, n$. We consider a noncompact cell 2-complex (Σ, d) with a triangulation $\mathcal{T}$ as follows:

(i) for every $\sigma \in \mathcal{T} \setminus \mathcal{T}^{(1)}$ there exist $m(\sigma) \in \mathbb{N}$ and a simplicial isometry f_σ from $(D_{m(\sigma)}, d_{m(\sigma)})$ onto $(|\sigma|, d|_{|\sigma|})$;

(ii) $|\{\sigma \in \mathcal{T} \setminus \mathcal{T}^{(1)} : \tau < \sigma\}| = 2$ for each $\tau \in \mathcal{T}^{(1)} \setminus \mathcal{T}^{(0)}$;

(iii) $r(v) = \sum\{\pi - 2\pi/m(\sigma) : v < \sigma \in \mathcal{T} \setminus \mathcal{T}^{(1)}\} \geq 2\pi$ for each $v \in \mathcal{T}^{(0)}$;

(iv) for any $x, y \in \Sigma$

$$d(x,y) = \min\{\sum_{j=1}^{k} d_{m(\sigma)}(f_{\sigma_j}^{-1}(x_{j-1}), f_{\sigma_j}^{-1}(x_j)) : \\ x = x_0 \in |\sigma_1|, x_j \in |\sigma_j| \cap |\sigma_{j+1}| (1 \leq j < k), y = x_k \in |\sigma_k|\}.$$

By [3] or [11], every (Σ, d) above is a CAT(0) space that is homeomorphic to $\mathbb{R}^2$, hence we obtain that $\operatorname{asdim}(\Sigma, d) = 2$ from Theorem 3.2. Here we note that (Σ, d) need not have an action of some group, and (Σ, d) is neither a Euclidean nor a hyperbolic plane in general.

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