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THE JACOBSON RADICAL AND REGULAR MODULES

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Let A be an associative, but not necessarily commutative, ring with identity, and $J=J(A)$ its Jacobson radical. A (unital) module is *regular* iff every submodule is pure (see (1)). The *regular socle* $R(M)$ of a module M is the sum of all its submodules which are regular. These concepts have been introduced and studied in (2).

THEOREM 1. *For every regular module M we have $J \cdot M=0$.*

Proof. If P is a finitely generated submodule of JM then $JP=JM \cap P=P$ since P is pure in M and hence $P=0$ by Nakayama's Lemma.

COROLLARY. *$J= \cap AnnM$, with the intersection taken over all regular modules.*

THEOREM 2. *A ring A is semi-local (i.e. A/J is artinian) iff A/J is von Neumann regular and $R(M)=S(M)$, the usual socle of M, for every A-module (left or right).*

Proof. If A/J is artinian then it is semi-simple (in the sense of Bourbaki) and hence regular. To show that $R(M)=S(M)$ for all M it suffices to show that every regular module is semi-simple. But this holds since for regular modules M we have $J \cdot M=0$ and hence M is an A/J-module.

Conversely if A/J is a regular ring then $M=A/J$ is a regular A-module, hence semi-simple as an A-module, and therefore as an $A/AnnM$-module; i.e. $A/J= A/AnnM$ is a semi-simple ring.

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