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A SELECTION THEOREM AND ITS APPLICATIONS

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In this paper, we first prove an improved version of the selection theorem of Yannelis-Prabhakar and next prove a fixed point theorem in a non-compact product space. As applications, an intersection theorem and two equilibrium existence theorems for a non-compact abstract economy are given.

1. INTRODUCTION

In convex analysis, the Fan-Browder fixed point theorem [**2**] is an essential tool in proving existence theorems of numerous nonlinear problems (for example see [**2, 7, 13, 15**]). Actually, the Fan-Browder fixed point theorem can be proved by constructing a continuous selection.

In [**15**], Yannelis-Prabhakar proved a continuous selection theorem and obtained a fixed point theorem in paracompact convex sets. Using this fixed point theorem, they obtained an equilibrium existence theorem for a compact abstract economy.

In this paper, we first give an improved version of the selection theorem of Yannelis-Prabhakar [**15**]. By applying this result, we prove a fixed point theorem in non-compact product spaces. As an application of our fixed point theorem, we first prove an intersection theorem which is closely related to a non-compact generalisation of Fan's intersection theorem [**6**] due to Shih-Tan [**12**]. Next, two equilibrium existence theorems are obtained which are either closely related to or generalisations of those results of Borglin-Keiding [**1**], Shafer-Sonnenschein [**11**], Tarafdar [**14**] and Yannelis-Prabhakar [**15**].

We shall need the following notations and definitions. Let A be a non-empty set. We shall denote by 2^A the family of all subsets of A. If A is a non-empty subset of a topological space X, we shall denote by $cl_X A$ the closure of A in X. If A is a subset of a vector space, $co\, A$ denotes the convex hull of A. Let X, Y be topological spaces and $\phi : X \to 2^Y$ be a correspondence.

(i) If $A \subset X$, we shall denote the restriction of ϕ to A by $\phi|_A$, that is, $\phi|_A : A \to 2^Y$ is the correspondence defined by $\phi|_A(x) = \phi(x)$ for all $x \in A$.

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(ii) ϕ is said to be *upper semicontinuous* if for each open subset V of Y, the set $\{x \in X : \phi(x) \subset V\}$ is open in X.

(iii) $f : X \to Y$ is a *continuous selection* of ϕ if f is continuous and $f(x) \in \phi(x)$ for all $x \in X$.

(iv) If Y is a vector space, the correspondence $co\,\phi : X \to 2^Y$ is defined by $(co\,\phi)(x) = co\,\phi(x)$ for all $x \in X$.

2. Selection and fixed point theorems

We shall first generalise a selection theorem of Yannelis-Prabhakar [**15**, Theorem 3.1] as follows :

Theorem 1. *Let X be a non-empty paracompact Hausdorff topological space and Y be a non-empty convex subset of a topological vector space. Suppose $S, T : X \to 2^Y$ are correspondences such that*

(1) *for each $x \in X$, $co\,S(x) \subset T(x)$ and $S(x) \neq \emptyset$,*

(2) *for each $y \in Y$, $S^{-1}(y)$ is open in X.*

Then T has a continuous selection.

Proof: By (1), $X = \bigcup_{y \in Y} S^{-1}(y)$. Since X is paracompact, by (2) and Lemma 1 of Michael [**10**], there exists an open locally finite refinement $\mathcal{F} = \{U_a : a \in A\}$ of the family $\{S^{-1}(y) : y \in Y\}$ where A is an index set and U_a is an open subset of X. By Proposition 2 of Michael [**10**], there exists a family of continuous functions $\{g_a : a \in A\}$ such that $g_a : X \to [0,1]$, $g_a(x) = 0$ for $x \notin U_a$ and $\sum_{a \in A} g_a(x) = 1$ for all $x \in X$. For each $a \in A$, choose $y_a \in Y$ such that $U_a \subset S^{-1}(y_a)$. This can be done since $\mathcal{F}$ is a refinement of $\{S^{-1}(y) : y \in Y\}$. Define $f : X \to Y$ by

$$f(x) = \sum_{a \in A} g_a(x)\, y_a \qquad \text{for each } x \in X.$$

From the local finiteness of $\mathcal{F}$, it follows that for each $x \in X$, at least one, and at most finitely many, $g_a(x)$ is not zero, and f is a well-defined continuous function from X to Y. Let $x \in X$ and $a \in A$ be such that $g_a(x) \neq 0$, then $x \in U_a \subset S^{-1}(y_a)$ so that $y_a \in S(x)$. By (1) and the definition of f, we have $f(x) \in co\,S(x) \subset T(x)$ for each $x \in X$. This completes the proof. ∎

If $S = T$, Theorem 1 reduces to Theorem 3.1 of Yannelis-Prabhakar [**15**].

We shall need the following lemma.

Lemma 1. *Let D be a non-empty compact subset of a topological vector space E. Then $co\,D$ is σ-compact and hence is paracompact.*

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PROOF: The proof that $co\,D$ is σ-compact can be found in [9, p.49]. For completeness, we shall include the simple proof here. For each $n \in N$, let $S_n = \{(\lambda_1, \dots, \lambda_n) : \lambda_1, \dots, \lambda_n \geqslant 0$ with $\sum_{i=1}^{n} \lambda_i = 1\}$ and define $f_n : S_n \times \prod_{i=1}^{n} D \to E$ by

$$f_n(\lambda_1, \dots, \lambda_n, x_1, \dots, x_n) = \sum_{i=1}^{n} \lambda_i x_i.$$

Then f_n is continuous. Since $S_n \times \prod_{i=1}^{n} D$ is compact, $f_n(S_n \times \prod_{i=1}^{n} D)$ is compact. But then $co\,D = \bigcup_{n=1}^{\infty} f_n\left(S_n \times \prod_{i=1}^{n} D\right)$ is σ-compact. It follows that $co\,D$ is Lindelöf. Since $co\,D$ is regular, $co\,D$ is paracompact by Corollary 33.15 in [3, p.341]. This completes the proof. □

We remark here that the topological vector space E in the above lemma is not assumed to be Hausdorff.

We shall prove the following fixed point theorem.

THEOREM 2. *Let $\{X_i\}_{i \in I}$ be a family of non-empty convex sets, each in a locally convex Hausdorff topological vector space E_i, where I is an index set. For each $i \in I$, let D_i be a non-empty compact subset of X_i and $S_i, T_i : X = \prod_{i \in I} X_i \to 2^{D_i}$ be such that for each $i \in I$,*

(1) *for each $x \in X$, $co\,S_i(x) \subset T_i(x)$ and $S_i(x) \neq \emptyset$,*
(2) *for each $y_i \in D_i$, $S_i^{-1}(y_i)$ is open in X.*

Then there exists a point $\hat{x} \in D = \prod_{i \in I} D_i$ such that $\hat{x} \in T(\hat{x}) = \prod_{i \in I} T_i(\hat{x})$, that is, $\hat{x}_i \in T_i(\hat{x})$ for all $i \in I$, where $\hat{x}_i$ is the projection of $\hat{x}$ onto X_i for each $i \in I$.

PROOF: Since $D = \prod_{i \in I} D_i$ is compact in X, it follows from Lemma 1 that $co\,D$ is paracompact in X. For each $i \in I$, let S_i^*, T_i^* be the restrictions of S_i, T_i on $co\,D$, then we have

(a) for each $x \in co\,D$, $co\,S_i^*(x) \subset co\,T_i^*(x)$ and $co\,S_i^*(x) \neq \emptyset$,
(b) for each $y_i \in D_i$,

$$\begin{aligned}(S_i^*)^{-1}(y_i) &= \{x \in co\,D : y_i \in S_i^*(x)\} \\ &= \{x \in co\,D : y_i \in S_i(x)\} \\ &= co\,D \cap S_i^{-1}(y_i)\end{aligned}$$

is open in $co\,D$.

By Theorem 1, for each $i \in I$, T_i^* has a continuous selection $f_i : co\,D \to D_i$ such that $f_i(x) \in T_i^*(x) = T_i(x)$ for each $x \in co\,D$.

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Define $f : co\,D \to D$ and $T : co\,D \to 2^D$ by

$$f(x) = \prod_{i\in I} f_i(x) \quad \text{and} \quad T(x) = \prod_{i\in I} T_i(x) \quad \text{for each } x \in co\,D.$$

Then f is clearly continuous. By Theorem 4.5.1 of Smart [13], there exists $\widehat{x} \in D$ such that $\widehat{x} = f(\widehat{x}) \in T(\widehat{x})$. This completes the proof. ∎

Theorem 2 generalises Theorem 3.2 of Yannelis-Prabhakar [15] in several ways :

(i) I need not be a singleton set,
(ii) X_i need not be paracompact, and
(iii) S_i and T_i need not be identical.

3. Applications

Let $X_1, \dots, X_n$ ($n \geqslant 2$) be topological spaces and $X = \prod_{i=1}^{n} X_i$. Let $i \in \{1, \dots, n\}$ be arbitrarily fixed. Let $\widehat{X}_i = \prod_{\substack{j=1\\ j\neq i}}^{n} X_j$ and $\pi_i : X \to X_i$ and $\widehat{\pi}_i : X \to \widehat{X}_i$ be the projections. If $x \in X$, we can write $\pi_i(x) = x_i$ and $\widehat{\pi}_i(x) = \widehat{x}_i$. Let A be a subset of X, $x_i \in X_i$ and $\widehat{x}_i \in \widehat{X}_i$. Then $[x_i, \widehat{x}_i]$ denotes the point $x \in X$ such that $\pi_i(x) = x_i$ and $\widehat{\pi}_i(x) = \widehat{x}_i$ and we define $A(x_i) = \{\widehat{y}_i \in \widehat{X}_i : [x_i, \widehat{y}_i] \in A\}$ and $A(\widehat{x}_i) = \{y_i \in X_i : [y_i, \widehat{x}_i] \in A\}$. If $A_i \subset X_i$ and $\widehat{A}_i \subset \widehat{X}_i$, $A_i \otimes \widehat{A}_i$ denotes the set $\{[y_i, \widehat{y}_i] \in X : y_i \in A_i \text{ and } \widehat{y}_i \in \widehat{A}_i\}$.

We shall give an application of a fixed point theorem to an intersection theorem as follows:

Theorem 3. *Let $\{X_i\}_{i\in I}$ be a family of non-empty convex sets, each in a locally convex Hausdorff topological vector space E_i. For each $i \in I$, let D_i be a non-empty compact subset of X_i. Suppose that $\{A_i\}_{i\in I}, \{B_i\}_{i\in I}$ are two families of subsets of $X = \prod_{i\in I} X_i$ having the following properties:*

(1) *for each $i \in I$ and $x_i \in D_i$, the set $B_i(x_i)$ is open in $\widehat{X}_i$,*
(2) *for each $i \in I$, and $\widehat{y}_i \in \widehat{X}_i$, the set $B_i(\widehat{y}_i) \cap D_i$ $(= \{x_i \in D_i : [x_i, \widehat{y}_i] \in B_i\})$ $\neq \emptyset$ and $co(B_i(\widehat{y}_i) \cap D_i) \subset A_i(\widehat{y}_i) \cap D_i$.*

Then we have $\bigcap_{i\in I} A_i \neq \emptyset$.

Proof: Define S_i, $T_i : X \to 2^{D_i}$ as follows :

$$S_i(y) = B_i(\widehat{y}_i) \cap D_i,$$
$$T_i(y) = A_i(\widehat{y}_i) \cap D_i, \quad \text{for each } y \in X.$$

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Then by (2), for each $i \in I$ and $y \in X, co\, S_i(y) \subset T_i(y)$ and $S_i(y) \neq \emptyset$. By (1), for each $i \in I$ and $x_i \in D_i$,

$$\begin{aligned} S_i^{-1}(x_i) &= \{y \in X : x_i \in S_i(y)\} \\ &= \{y \in X : x_i \in B_i(\widehat{y}_i) \cap D_i\} \; (= \{y \in X : x_i \in B_i(\widehat{y}_i)\}) \\ &= \{y \in X : [x_i, \widehat{y}_i] \in B_i\} \\ &= X_i \otimes B_i(x_i) \end{aligned}$$

is open in X.

By Theorem 2, there exists $x \in D = \prod_{i \in I} D_i$ such that $x \in T(x) = \prod_{i \in I} T_i(x)$, that is, $x_i \in A_i(\widehat{x}_i)$ for all $i \in I$ and hence $x = [x_i, \widehat{x}_i] \in \bigcap_{i \in I} A_i$. Therefore $\bigcap_{i \in I} A_i \neq \emptyset$. This completes the proof. ∎

We remark that Theorem 3 is closely related to but not comparable to Theorem 2 of Shih-Tan [**12**] which was a non-compact generalisation of Fan's intersection theorem [**6**] (in our case, the space E_i is required to be locally convex).

Next we shall give two equilibrium existence theorems for a non-compact abstract economy with an infinite number of commodities and an infinite number of agents. We first give some definitions in equilibrium theory. Let the set I of agents be any (possibly uncountable) set. An *abstract economy* $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ is defined as a family of ordered quaduples (X_i, A_i, B_i, P_i) where $A_i, B_i : \prod_{j \in I} X_j \to 2^{X_i}$ are constraint correspondences and $P_i : \prod_{j \in I} X_j \to 2^{X_i}$ is a preference correspondence. An *equilibrium* for Γ is a point $\widehat{x} \in X = \prod_{i \in I} X_i$ such that for each $i \in I$, $\widehat{x}_i \in cl_{X_i} B_i(\widehat{x})$ and $A_i(\widehat{x}) \cap P_i(\widehat{x}) = \emptyset$. When $A_i = B_i$ for each $i \in I$, our definitions of an abstract economy and an equilibrium coincide with the standard definitions, for example in Borglin-Keiding [**1**, p.315] or in Yannelis-Prabhakar [**15**, p.242].

We shall first show that by applying Himmelberg's fixed point theorem [**8**, Theorem 2] instead of Ky Fan's fixed point theorem [**5**], the proof of Theorem 6.1 of Yannelis-Prabhakar [**15**] can be used to prove its non-compact case.

THEOREM 4. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy such that for each $i \in I$,*

(1) *X_i is a non-empty convex subset of a locally convex Hausdorff topological vector space E_i and D_i is a non-empty compact subset of X_i,*
(2) *for each $x \in X = \prod_{i \in I} X_i, A_i(x)$ is non-empty, $A_i(x) \subset B_i(x) \subset D_i$ and $B_i(x)$ is convex,*
(3) *the correspondence $cl\, B_i : X \to 2^{X_i}$ defined by $(cl\, B_i)(x) = cl_{X_i} B_i(x)$ for each $x \in X$, is upper semicontinuous,*

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(4) *for each* $y \in D_i$, $A_i^{-1}(y)$ *is open in* X,
(5) *for each* $y \in X_i$, $P_i^{-1}(y)$ *is open in* X,
(6) *for each* $x \in X$, $x_i \notin co\,P_i(x)$,
(7) *the set* $\{x \in X : co\,A_i(x) \cap co\,P_i(x) \neq \emptyset\}$ *is paracompact.*

Then Γ *has an equilibrium* $\widehat{x} \in X$, *that is, for each* $i \in I$,

$$\widehat{x}_i \in cl_{X_i} B_i(\widehat{x}) \quad \text{and} \quad A_i(\widehat{x}) \cap P_i(\widehat{x}) = \emptyset.$$

PROOF: We first fix $i \in I$. Define $\phi_i : X \to 2^{X_i}$ by

$$\phi_i(x) = co\,A_i(x) \cap co\,P_i(x) \qquad \text{for each } x \in X.$$

By (4), (5) and Lemma 5.1 of Yannelis-Prabhakar [15], it is easy to see that for each $y \in X_i$, $\phi_i^{-1}(y)$ is open in X. Let $U_i = \{x \in X : \phi_i(x) \neq \emptyset\}$. Since $U_i = \bigcup_{y \in X_i} \phi_i^{-1}(y)$, U_i is open in X. By (7), U_i is paracompact. Note that $\phi_i|_{U_i} : U_i \to 2^{X_i}$ has the following properties :

(i) for each $x \in U_i$, $\phi_i|_{U_i}(x)$ is non-empty and convex,
(ii) for each $y \in X_i$, $\left(\phi_i|_{U_i}\right)^{-1}(y) = \phi_i^{-1}(y) \cap U_i$ is open in U_i.

By Theorem 3.1 of Yannelis-Prabhakar [15] (which is the case $S = T$ in our Theorem 1), there exists a continuous selection $f_i : U_i \to 2^{X_i}$ such that $f_i(x) \in \phi_i|_{U_i}(x)$ for all $x \in U_i$. Define $F_i : X \to 2^{X_i}$ by

$$F_i(x) = \begin{cases} \{f_i(x)\}, & \text{if } x \in U_i, \\ cl_{X_i} B_i(x), & \text{if } x \notin U_i. \end{cases}$$

By (3) and Lemma 6.1 of Yannelis-Prabhakar [15], $F_i : X \to 2^{X_i}$ is upper semicontinuous on X. Clearly for each $x \in X, F_i(x)$ is a non-empty closed convex subset of D_i by (2). Finally we define $F : X \to 2^X$ by

$$F(x) = \prod_{i \in I} F_i(x) \quad \text{for each } x \in X.$$

It follows from Lemma 3 of Fan [5] that F is upper semicontinuous on X. Obviously for each $x \in X, F(x)$ is a closed convex subset of $D = \prod_{i \in I} D_i$. By Tychonoff's product theorem (for example see Dugundji [4, p.224]), D is a compact subset of X. Hence by Theorem 2 of Himmelberg [8], there exists a point $\widehat{x} \in D$ such that $\widehat{x} \in F(\widehat{x})$. If $\widehat{x} \in U_i$ for some $i \in I$, then $\widehat{x}_i = f_i(\widehat{x}) \in co\,A_i(\widehat{x}) \cap co\,P_i(\widehat{x}) \subset co\,P_i(\widehat{x})$ which contradicts (6). Thus for each $i \in I$, we must have $\widehat{x} \notin U_i$ so that $\widehat{x}_i \in cl_{X_i} B_i(\widehat{x})$ and

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$co\,A_i(\hat{x}) \cap co\,P_i(\hat{x}) = \emptyset$. Consequently, $\hat{x}$ is an equilibrium for Γ. This completes the proof. □

As we have seen in the proof, we can obtain a stronger separation result, that is, for each $i \in I$, $co\,A_i(\hat{x}) \cap co\,P_i(\hat{x}) = \emptyset$.

Theorem 4 generalises Theorem 6.1 of Yannelis-Prabhakar [**15**] in the following ways :

(i) for each $i \in I$, the space E_i need not be metrisable,
(ii) for each $i \in I$, the set X_i need not be compact, and
(iii) the set I of agents need not be countable.

Theorem 5. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy such that for each $i \in I$, the following conditions hold:*

(1) *X_i is a non-empty convex subset of a locally convex Hausdorff topological vector space E_i and D_i be a non-empty compact subset of X_i,*
(2) *for each $x \in X$, $A_i(x)$ is non-empty and $co\,A_i(x) \subset B_i(x) \subset D_i$,*
(3) *for each $y_i \in D_i$, the set $[(co\,P_i)^{-1}(y_i) \cup F_i] \cap A_i^{-1}(y_i)$ is open in X, where $F_i = \{x \in X : A_i(x) \cap P_i(x) = \emptyset\}$,*
(4) *for each $x \in X$, $x_i \notin co\,P_i(x)$.*

Then Γ has an equilibrium.

Proof: For each $i \in I$, let $G_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ and for each $x \in X$, let $I(x) = \{i \in I : A_i(x) \cap P_i(x) \neq \emptyset\}$. For each $i \in I$, we define the correspondences $S_i, T_i : X = \prod_{i \in I} X_i \to 2^{D_i}$ by

$$S_i(x) = \begin{cases} co\,P_i(x) \cap A_i(x), & \text{if } i \in I(x), \\ A_i(x), & \text{if } i \notin I(x), \end{cases}$$

$$T_i(x) = \begin{cases} co\,P_i(x) \cap B_i(x), & \text{if } i \in I(x), \\ B_i(x), & \text{if } i \notin I(x). \end{cases}$$

Then we have the following properties:

(i) for each $i \in I$ and $x \in X$, $co\,S_i(x) \subset T_i(x)$ and $S_i(x) \neq \emptyset$,
(ii) for each $i \in I$ and $y_i \in D_i$,

$$\begin{aligned} S_i^{-1}(y_i) &= \{[(co\,P_i)^{-1}(y_i) \cap A_i^{-1}(y_i)] \cap G_i\} \cup [A_i^{-1}(y_i) \cap F_i] \\ &= [(co\,P_i)^{-1}(y_i) \cap A_i^{-1}(y_i)] \cup [A_i^{-1}(y_i) \cap F_i] \\ &= [(co\,P_i)^{-1}(y_i) \cup F_i] \cap A_i^{-1}(y_i) \end{aligned}$$

is open in X by (3).

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By Theorem 2, there exists $\widehat{x} \in D$ such that $\widehat{x}_i \in T_i(\widehat{x})$ for all $i \in I$. By (4) and the definition of T_i, we have $\widehat{x}_i \in B_i(\widehat{x})$ and $A_i(\widehat{x}) \cap P_i(\widehat{x}) = \emptyset$ for all $i \in I$. This completes the proof. $\square$

Finally we remark that Theorems 4 and 5 are closely related to those results of Shafer-Sonnenschein [**11**, p.347], Borglin-Keiding [**1**, p.315] and Tarafdar [**14**, Theorem 3.1].

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