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C^*-ALGEBRAS THAT ARE ISOMORPHIC AFTER TENSORING AND FULL PROJECTIONS

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Abstract Let A be a unital C^*-algebra and for each $n \in \mathbb{N}$ let M_n be the $n \times n$ matrix algebra over $\mathbb{C}$. In this paper we shall give a necessary and sufficient condition that there is a unital C^*-algebra B satisfying $A \ncong B$ but for which $A \otimes M_n \cong B \otimes M_n$ for some $n \in \mathbb{N} \setminus \{1\}$. Also, we shall give some examples of unital C^*-algebras satisfying the above property.

Keywords: cancellation; full projection; Murray–von Neumann equivalence; strong Morita equivalence; tensor product

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1. Introduction

Let A be a unital C^*-algebra and for each $n \in \mathbb{N}$ let M_n be the $n \times n$ matrix algebra over $\mathbb{C}$. Let $M_n(A)$ be the $n \times n$ matrix algebra over A and we identify $M_n(A)$ with $A \otimes M_n$.

In [**13**], Plastiras gave an example of a pair of unital C^*-algebras A and B satisfying $A \ncong B$ but $M_2(A) \cong M_2(B)$. Also, in [**7**], Cuntz showed that $O_3 \ncong B$ but $M_2(O_3) \cong M_2(B)$, where O_3 is the Cuntz algebra generated by three isometries with pairwise orthogonal ranges and $B = M_2(O_3)$.

In this note, we shall give a necessary and sufficient condition that for a unital C^*-algebra A there is a unital C^*-algebra B satisfying $A \ncong B$ but $M_n(A) \cong M_n(B)$ for some $n \in \mathbb{N} \setminus \{1\}$. We shall refer to these conditions as 'property $(*)$'. Also, we shall give some examples of unital C^*-algebras satisfying property $(*)$.

2. Preliminaries

Let A be a C^*-algebra, $M(A)$ its multiplier algebra and $\tilde{A}$ its unitization. Let id_A be the identity map of A and let 1_A be the unit element in A if A is unital. We denote them by id and 1 if no confusion can arise.

Let p be a projection in $M(A)$. Then we call p a *full* projection in A if $\overline{ApA} = A$ (see [**4**] or [**5**]). Let p, q be projections in A. Then p is *equivalent* to q in A, written $p \sim q$, if p is

Murray–von Neumann equivalent to q in A. We denote by (p) the equivalent class of p in A. Also, p is *subordinate* to q, written $p \precsim q$, if p is equivalent to a subprojection of q.

For every $n \in \mathbb{N}$ let $\{f_{ij}\}_{i,j=1}^n$ be matrix units of M_n and let I_n be the unit element in M_n. For any $n, m \in \mathbb{N}$ with $n \leqslant m$, we regard M_n as a left corner C^*-subalgebra of M_m and I_n as a projection in M_m. Let $\mathbb{K}$ be the C^*-algebra of all compact operators on a countably infinite-dimensional Hilbert space. We regard M_n as a C^*-subalgebra of $\mathbb{K}$ for each $n \in \mathbb{N}$ in the usual way. Let $\{e_{ij}\}_{i,j\in\mathbb{Z}}$ and $\{\overline{e_{ij}}\}_{i,j=0}^{\infty}$ be two families of matrix units of $\mathbb{K}$.

Lemma 2.1. *Let A be a unital C^*-algebra. Then the following hold.*

(i) *Let p, q be projections in $A \otimes \mathbb{K}$ with $p \leqslant q$. If p is a full projection in $A \otimes \mathbb{K}$, then so is q.*

(ii) *Let q be a projection in $A \otimes \mathbb{K}$. Suppose that q is full in $A \otimes \mathbb{K}$. Let $p \in q(A \otimes \mathbb{K})q$ be a full projection in $q(A \otimes \mathbb{K})q$. Then p is full in $A \otimes \mathbb{K}$.*

Proof. (i)

$$A \otimes \mathbb{K} = \overline{(A \otimes \mathbb{K})p(A \otimes \mathbb{K})} \subset \overline{(A \otimes \mathbb{K})q(A \otimes \mathbb{K})} \subset A \otimes \mathbb{K},$$

so that $\overline{(A \otimes \mathbb{K})q(A \otimes \mathbb{K})} = A \otimes \mathbb{K}$, i.e. q is full in $A \otimes \mathbb{K}$.

(ii) We note that

$$\overline{q(A \otimes \mathbb{K})p(A \otimes \mathbb{K})q} = q(A \otimes \mathbb{K})q$$

since p is full in $q(A \otimes \mathbb{K})q$. Then

$$\overline{(A \otimes \mathbb{K})p(A \otimes \mathbb{K})} \supset \overline{(A \otimes \mathbb{K})q(A \otimes \mathbb{K})p(A \otimes \mathbb{K})q(A \otimes \mathbb{K})} = A \otimes \mathbb{K}.$$

Therefore, p is full in $A \otimes \mathbb{K}$. □

For a unital C^*-algebra A and each $n \in \mathbb{N}$, let $\mathrm{FP}_n(A)$ be the set of all full projections p in $A \otimes \mathbb{K}$ with $p(A \otimes \mathbb{K})p \cong M_n(A)$ and let $\mathrm{FP}_n(A)/\sim = \{(p) \mid p \in \mathrm{FP}_n(A)\}$. We denote $\mathrm{FP}_1(A)$ and $\mathrm{FP}_1(A)/\sim$ by $\mathrm{FP}(A)$ and $\mathrm{FP}(A)/\sim$, respectively.

3. A necessary and sufficient condition

Suppose that A is a unital C^*-algebra with property $(*)$. Then there is a unital C^*-algebra B satisfying $A \ncong B$ but $M_n(A) \cong M_n(B)$ for some $n \in \mathbb{N} \setminus \{1\}$. Since A and B are strongly Morita equivalent, by Rieffel [**14**, Proposition 2.1] there is a full projection $q \in A \otimes \mathbb{K}$ such that $B \cong q(A \otimes \mathbb{K})q$. Then

$$M_n(A) \cong M_n(B) \cong (q \otimes I_n)(A \otimes \mathbb{K} \otimes M_n)(q \otimes I_n).$$

Let χ be an isomorphism of $M_n(A)$ onto $(q \otimes I_n)(A \otimes \mathbb{K} \otimes M_n)(q \otimes I_n)$ and let $p_1 = \chi(1_A \otimes f_{11})$.

Lemma 3.1. *With the above notation, p_1 is a full projection in $A \otimes \mathbb{K} \otimes M_n$.*

Proof. Since $1_A \otimes f_{11}$ is full in $A \otimes M_n$, p_1 is full in $(q \otimes I_n)(A \otimes \mathbb{K} \otimes M_n)(q \otimes I_n)$. Since $q \otimes I_n$ is full in $A \otimes \mathbb{K} \otimes M_n$, by Lemma 2.1 (ii) p_1 is full in $A \otimes \mathbb{K} \otimes M_n$. □

For any $n \in \mathbb{N}$ let ψ_n be an isomorphism of $\mathbb{K} \otimes M_n$ onto $\mathbb{K}$ with $\psi_{n*} = \mathrm{id}$, the identity map of $K_0(\mathbb{K} \otimes M_n)$ onto $K_0(\mathbb{K})$. Let $p = (\mathrm{id}_A \otimes \psi_n)(p_1) \in A \otimes \mathbb{K}$.

Lemma 3.2. *With the above notation, $p \in \mathrm{FP}(A)$.*

Proof. Since p_1 is a full projection in $A \otimes \mathbb{K} \otimes M_n$ by Lemma 3.1, p is a full projection in $A \otimes \mathbb{K}$. Also,

$$\begin{aligned} p(A \otimes \mathbb{K})p &\cong (\mathrm{id}_A \otimes \psi_n)(p_1(A \otimes \mathbb{K} \otimes M_n)p_1) \cong p_1(A \otimes \mathbb{K} \otimes M_n)p_1 \\ &= \chi((1_A \otimes f_{11})(A \otimes M_n)(1_A \otimes f_{11})) \cong A. \end{aligned}$$

Therefore, we obtain the conclusion. □

We shall show that $p \otimes I_n \sim q \otimes I_n$ in $A \otimes \mathbb{K} \otimes M_n$. To do this, we need lemmas.

Lemma 3.3. *With the above notation, for any $N \in \mathbb{N}$ there is a partial isometry $v \in \mathbb{K} \otimes M_n$ such that*

$$v^*v = \sum_{j=-N}^{N} e_{jj} \otimes I_n, \qquad vv^* = \sum_{j=-N}^{N} \psi_n(e_{jj} \otimes I_n) \otimes f_{11}$$

$$v(e_{ij} \otimes f_{kl})v^* = \psi_n(e_{ij} \otimes f_{kl}) \otimes f_{11} \quad \text{for } i, j = -N, \dots, 0, \dots, N \text{ and } k, l = 1, 2, \dots, n.$$

Proof. Since $(e_{00} \otimes f_{11}) \otimes f_{11}$ is a minimal projection in $(\mathbb{K} \otimes M_n) \otimes M_n$, $\psi_n(e_{00} \otimes f_{11}) \otimes f_{11}$ is a minimal projection in $\mathbb{K} \otimes M_n$. Since all minimal projections are equivalent, there is a partial isometry $w \in \mathbb{K} \otimes M_n$ such that

$$w^*w = e_{00} \otimes f_{11}, \qquad ww^* = \psi_n(e_{00} \otimes f_{11}) \otimes f_{11}.$$

Let

$$v = \sum_{k=1}^{n} \sum_{j=-N}^{N} (\psi_n(e_{j0} \otimes f_{k1}) \otimes f_{11}) w (e_{0j} \otimes f_{1k}).$$

By routine computations, we can see that v is the required partial isometry in $\mathbb{K} \otimes M_n$. □

Lemma 3.4. *With the above notation, $p \otimes f_{11} \sim p_1$ in $A \otimes \mathbb{K} \otimes M_n$.*

Proof. There are an $N \in \mathbb{N}$ and a projection $p_0 \in A \otimes M_{2N+1} \otimes M_n \subset A \otimes \mathbb{K} \otimes M_n$ such that $p_0 \sim p_1$ in $A \otimes \mathbb{K} \otimes M_n$. Since $(\mathrm{id}_A \otimes \psi_n)(p_0) \otimes f_{11} \sim (\mathrm{id}_A \otimes \psi_n)(p_1) \otimes f_{11} = p \otimes f_{11}$ in $A \otimes \mathbb{K} \otimes M_n$, we have only to show that $(\mathrm{id}_A \otimes \psi_n)(p_0) \otimes f_{11} \sim p_0$ in $A \otimes \mathbb{K} \otimes M_n$. We write

$$p_0 = \sum_{k,l=1}^{n} \sum_{i,j=-N}^{N} a_{ijkl} \otimes e_{ij} \otimes f_{kl}, \quad \text{where } a_{ijkl} \in A.$$

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Then, by routine computations,

$$(1 \otimes v)p_0(1 \otimes v)^* = \sum_{k,l=1}^{n} \sum_{i,j=-N}^{N} a_{ijkl} \otimes v(e_{ij} \otimes f_{kl})v^* = (\mathrm{id}_A \otimes \psi_n)(p_0) \otimes f_{11},$$

$$p_0(1 \otimes v)^*(1 \otimes v)p_0 = \sum_{k,l=1}^{n} \sum_{i,j=-N}^{N} (a_{ijkl} \otimes e_{ij} \otimes f_{kl}) \left(1 \otimes \sum_{j=-N}^{N} e_{jj} \otimes I_n\right) p_0 = p_0.$$

Therefore, $p_0 \sim (\mathrm{id}_A \otimes \psi_n)(p_0) \otimes f_{11}$ in $A \otimes \mathbb{K} \otimes M_n$. □

Proposition 3.5. *With the above notation, $p \otimes I_n \sim q \otimes I_n$ in $A \otimes \mathbb{K} \otimes M_n$.*

Proof. By Lemma 3.4, $\chi(1 \otimes f_{jj}) \sim \chi(1 \otimes f_{11}) = p_1 \sim p \otimes f_{11} \sim p \otimes f_{jj}$ in $A \otimes \mathbb{K} \otimes M_n$ for $j = 1, 2, \ldots, n$. Thus in $A \otimes \mathbb{K} \otimes M_n$,

$$q \otimes I_n = \chi(1 \otimes I_n) = \sum_{j=1}^{n} \chi(1 \otimes f_{jj}) \sim \sum_{j=1}^{n} p \otimes f_{jj} = p \otimes I_n.$$

Therefore, we obtain the conclusion. □

Theorem 3.6. *Let A be a unital C^*-algebra. Suppose that there is a unital C^*-algebra B satisfying $A \ncong B$ but $M_n(A) \cong M_n(B)$ for some $n \in \mathbb{N} \setminus \{1\}$. Then there are full projections p, q in $A \otimes \mathbb{K}$ with $p \in \mathrm{FP}(A)$, $q \notin \mathrm{FP}(A)$ such that $q(A \otimes \mathbb{K})q \cong B$, $p \otimes I_n \sim q \otimes I_n$ in $A \otimes \mathbb{K} \otimes M_n$.*

Proof. Since A and B are strongly Morita equivalent by Rieffel [**14**, Proposition 2.1], there is a full projection q in $A \otimes \mathbb{K}$ such that $q(A \otimes \mathbb{K})q \cong B$. If $q \in \mathrm{FP}(A)$, $A \cong q(A \otimes \mathbb{K})q \cong B$. This is a contradiction. Thus $q \notin \mathrm{FP}(A)$. Furthermore, by Proposition 3.5 there is a $p \in \mathrm{FP}(A)$ such that $p \otimes I_n \sim q \otimes I_n$ in $A \otimes \mathbb{K} \otimes M_n$. □

Corollary 3.7. *Let A be a unital C^*-algebra. Then the following conditions are equivalent.*

(i) *There is a unital C^*-algebra B satisfying $A \ncong B$ but $M_n(A) \cong M_n(B)$ for some $n \in \mathbb{N} \setminus \{1\}$.*

(ii) *There is a full projection q in $A \otimes \mathbb{K}$ with $q \notin \mathrm{FP}(A)$ satisfying that there is a $p \in \mathrm{FP}(A)$ such that $p \otimes I_n \sim q \otimes I_n$ in $A \otimes \mathbb{K} \otimes M_n$ for some $n \in \mathbb{N} \setminus \{1\}$.*

Proof. (i) ⇒ (ii). This is clear by Theorem 3.6.

(ii) ⇒ (i). Put $B = q(A \otimes \mathbb{K})q$. Then

$$B \otimes M_n \cong (q \otimes I_n)(A \otimes \mathbb{K} \otimes M_n)(q \otimes I_n) \cong (p \otimes I_n)(A \otimes \mathbb{K} \otimes M_n)(p \otimes I_n) \cong A \otimes M_n$$

since $p \otimes I_n \sim q \otimes I_n$ in $A \otimes \mathbb{K} \otimes M_n$ and $p \in \mathrm{FP}(A)$. Also, $A \ncong B$. Indeed, if $A \cong B$, $q(A \otimes \mathbb{K})q \cong A$. Thus $q \in \mathrm{FP}(A)$. This is a contradiction. Therefore, we obtain the conclusion. □

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Corollary 3.8. *Let A be a unital C^*-algebra. Suppose that A has cancellation or A is purely infinite simple and that $K_0(A)$ is torsion free. Then A does not satisfy property $(*)$.*

Proof. Suppose that there is a unital C^*-algebra B satisfying $A \ncong B$ but $M_n(A) \cong M_n(B)$ for some $n \in \mathbb{N} \setminus \{1\}$. Then there are full projections $p, q \in A \otimes \mathbb{K}$ with $p \in \mathrm{FP}(A)$ and $q \notin \mathrm{FP}(A)$ such that $p \otimes I_n \sim q \otimes I_n$ in $A \otimes \mathbb{K} \otimes M_n$ by Corollary 3.7. Hence $n[p] = n[q]$ in $K_0(A)$. Since $K_0(A)$ is torsion free and A has cancellation or A is purely infinite simple, $p \sim q$ in $A \otimes \mathbb{K}$. This is a contradiction. Therefore, we obtain the conclusion. □

4. Examples

In this section, we shall give some examples of unital C^*-algebras with property $(*)$.

Let A be a unital C^*-algebra. For any $p \in \mathrm{FP}(A)$, $((\mathrm{id}_A \otimes \psi_n)(p \otimes I_n)) \in \mathrm{FP}_n(A)/\sim$ by an easy calculation. In the same way as in the proof of [**8**, Lemma 3.1], we see that $((\mathrm{id}_A \otimes \psi_n)(p \otimes I_n))$ is independent of the choices of $p \in \mathrm{FP}(A)$ and ψ_n by easy computations. Hence we can define a map $\mu_n : \mathrm{FP}(A)/\sim \to \mathrm{FP}_n(A)/\sim$ by $\mu_n((p)) = ((\mathrm{id}_A \otimes \psi_n)(p \otimes I_n))$.

Lemma 4.1. *With the above notation, μ_n is surjective for any $n \in \mathbb{N}$.*

Proof. Let $q \in \mathrm{FP}_n(A)$. Then there is an isomorphism χ_q of $A \otimes M_n$ onto $q(A \otimes \mathbb{K})q$. Put $p = \chi_q(1_A \otimes f_{11})$. Then $p(A \otimes \mathbb{K})p \cong A$ and by Lemma 2.1 (ii) p is a full projection in $A \otimes \mathbb{K}$ since p is full in $q(A \otimes \mathbb{K})q$. Thus $p \in \mathrm{FP}(A)$. Furthermore, in the same way as in the proof of Lemma 3.4, $(\mathrm{id} \otimes \psi_n)(\chi_q(1 \otimes f_{11}) \otimes f_{jj}) \sim \chi_q(1 \otimes f_{jj})$ in $A \otimes \mathbb{K}$ for $j = 1, 2, \ldots, n$. Hence, in $A \otimes \mathbb{K}$,

$$(\mathrm{id} \otimes \psi_n)(p \otimes I_n) = \sum_{j=1}^{n} (\mathrm{id} \otimes \psi_n)(\chi_q(1 \otimes f_{11}) \otimes f_{jj}) \sim \sum_{j=1}^{n} \chi_q(1 \otimes f_{jj}) = q.$$

Therefore, we obtain the conclusion. □

Proposition 4.2. *Let A be a unital C^*-algebra such that $K_0(A)$ has a torsion element x with $nx = 0$ and $kx \neq 0$ for $k = 1, 2, \ldots, n-1$, where $n \in \mathbb{N}$ with $n \geqslant 2$. Suppose that $\mathrm{FP}(A)/\sim = \{(1_A \otimes e_{00})\}$. Then there are unital C^*-algebras A_1 and A_2 strongly Morita equivalent to A such that $M_n(A_1) \cong M_n(A_2)$, $M_k(A_1) \ncong M_k(A_2)$ for $k = 1, 2, \ldots, n-1$.*

Proof. For the $x \in K_0(A)$, there are $l, m \in \mathbb{N}$ and a projection $p \in M_l(A)$ such that $x = [p] - [1_A \otimes I_m]$ in $K_0(A)$. Since $nx = 0$ in $K_0(A)$, $[p \otimes I_n] = [1 \otimes I_{mn}]$ in $K_0(A)$. Thus there are $k, N \in \mathbb{N}$ with $N \geqslant l, m$ such that

$$(p \otimes I_n) \oplus (1_A \otimes I_k \otimes I_n) \sim (1_A \otimes I_m \otimes I_n) \oplus (1_A \otimes I_k \otimes I_n)$$

in $M_{N+k}(A) \otimes M_n$, where we regard p and I_m as projections in $M_{N+k}(A)$. Thus

$$\begin{aligned}
&(p \oplus (1 \otimes I_k)) M_{N+k}(A) (p \oplus (1 \otimes I_k)) \otimes M_n \\
&\quad \cong ((1 \otimes I_m \otimes I_n) \oplus (1 \otimes I_k \otimes I_n))(M_{N+k}(A) \otimes M_n)((1 \otimes I_m \otimes I_n) \oplus (1 \otimes I_k \otimes I_n)) \\
&\quad \cong M_{m+k}(A) \otimes M_n.
\end{aligned}$$

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Put $A_1 = (p \oplus (1 \otimes I_k)) M_{N+k}(A)(p \oplus (1 \otimes I_k))$, $A_2 = M_{m+k}(A)$. Then $M_n(A_1) \cong M_n(A_2)$. Let $q = p \oplus (1 \otimes I_k)$. Since $1 \otimes I_k$ is full in $M_{N+k}(A)$, by Lemma 2.1 (i) q is full in $M_{N+k}(A)$. Hence by Brown [**4**, Corollary 2.6] A_1 is strongly Morita equivalent to A. Suppose that $M_r(A_1) \cong M_r(A_2)$ for some $r \in \mathbb{N}$ with $1 \leqslant r \leqslant n-1$. Then by an easy computation, $(\mathrm{id}_A \otimes \psi_r)(q \otimes I_r) \in \mathrm{FP}_{(m+k)r}(A)$. Since $\mathrm{FP}(A)/\sim = \{(1 \otimes e_{00})\}$, by Lemma 4.1

$$\mathrm{FP}_{(m+k)r}(A)/\sim = \left\{ \left(1 \otimes \sum_{j=0}^{(m+k)r} e_{jj} \right) \right\}.$$

Hence $(\mathrm{id} \otimes \psi_r)(q \otimes I_r) \sim 1 \otimes \sum_{j=0}^{(m+k)r} e_{jj}$ in $A \otimes \mathbb{K}$. Since ψ_{r*} is the identity map of $K_0(\mathbb{K} \otimes M_r)$ onto $K_0(\mathbb{K})$, $r[q] = (m+k)r[1_A]$ in $K_0(A)$. Hence $rx = 0$ in $K_0(A)$. This is a contradiction. Therefore, we obtain the conclusion. □

The Cuntz algebra O_3 satisfies the assumptions of Proposition 4.2 since $K_0(O_3) \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{FP}(O_3)/\sim = \{(1 \otimes e_{00})\}$ by [**8**, Corollary 4.6] and [**9**, Corollary 15]. We shall give an example of a simple unital C^*-algebra with cancellation satisfying the assumptions of Proposition 4.2.

For a C^*-algebra C we denote by $\mathrm{Aut}(C)$ the group of all automorphisms of C and by $\mathrm{sr}(C)$ its stable rank.

Let θ be a non-quadratic irrational number in $(0,1)$ and let $\mathbb{Z} + \mathbb{Z}\theta$ be the ordered group with the usual total ordering. Let $\mathbb{D}$ be the group of all rational numbers and let $G = (\mathbb{Z} + \mathbb{Z}\theta) \oplus \mathbb{D}$ be the ordered group with the strict ordering from the first coordinate. We denote by G_+ its positive cone and we choose an order unit $u \in G$ by $u = (1, 0)$. Then by routine calculations, we can see that (G, G_+, u) is a simple dimension group by Blackadar [**2**, Theorem 7.4.1]. Let C be a unital AF-algebra corresponding to (G, G_+, u). Let α be an automorphism of C such that the automorphism α_* of $K_0(C)$ is defined by $\alpha_*(a, b) = (a, -2b)$ for any $(a, b) \in K_0(C)$. Let $A = C \times_\alpha \mathbb{Z}$. Then in the same way as in Blackadar [**2**, 10.11.2], we can see that A is a simple unital stably finite C^*-algebra with its scaled ordered group as follows:

$$((\mathbb{Z} + \mathbb{Z}\theta) \oplus \mathbb{Z}/3\mathbb{Z}, \{(a, b) \in (\mathbb{Z} + \mathbb{Z}\theta) \oplus \mathbb{Z}/3\mathbb{Z} | a > 0\} \cup \{(0, 0)\}, (1, 0)).$$

Example 4.3. Let A be as above. Then A has cancellation and satisfies the assumptions of Proposition 4.2. Thus there are unital C^*-algebras A_1 and A_2 strongly Morita equivalent to A such that $M_3(A_1) \cong M_3(A_2)$, $M_k(A_1) \ncong M_k(A_2)$ for $k = 1, 2$.

In fact, let $(a, [b])$ be any positive element in $(\mathbb{Z} + \mathbb{Z}\theta) \oplus \mathbb{Z}/3\mathbb{Z}$, where $a \in \mathbb{Z} + \mathbb{Z}\theta$ with $a > 0$ and $[b]$ is an equivalence class in $\mathbb{Z}/3\mathbb{Z}$ of $b \in \mathbb{Z}$ with $0 \leqslant b \leqslant 2$. Then by the Pimsner–Voiculescu exact sequence, there is a projection q in some $M_n(C)$ such that $[q] = (a, [b])$ in $K_0(A)$. Since C has cancellation, by the definition of the ordering of $K_0(C)$, there is a projection p in $M_n(C)$ such that $p \leqslant q$ and $[p] \in (\mathbb{Z} + \mathbb{Z}\theta) \oplus 0 \subset K_0(C)$. Since $qM_n(A)q$ is simple, p is full in $qM_n(A)q$.

The conjecture in Blackadar [**1**, Remark A7] has been proved by Blackadar. This we can obtain that $\mathrm{sr}(qM_n(A)q) \leqslant \mathrm{sr}(pM_n(A)p)$. Also since $\alpha_*([p]) = [p]$ and C has cancellation, there is a unitary element $w \in M_n(C)$ such that $\alpha(p) = w^*pw$. Hence

$$\begin{aligned}\mathrm{sr}(qM_n(A)q) \leqslant \mathrm{sr}(pM_n(A)p) &= \mathrm{sr}(pM_n(C)p \times_{Ad(w)\circ(\alpha\otimes\mathrm{id}_{M_n})} \mathbb{Z}) \\ &\leqslant \mathrm{sr}(pM_n(C)p) + 1 = 2.\end{aligned}$$

Thus by Blackadar [**1**, Theorem A1], A has cancellation. Since θ is non-quadratic, by Shen [**15**, Theorem 2.1] the identity map of $\mathbb{Z}+\mathbb{Z}\theta$ is the unique order-preserving automorphism of $\mathbb{Z}+\mathbb{Z}\theta$. Thus for any $\beta \in \mathrm{Aut}(A\otimes\mathbb{K})$ there is an automorphism σ of $\mathbb{Z}/3\mathbb{Z}$ such that $\beta_* = \mathrm{id}_{\mathbb{Z}+\mathbb{Z}\theta} \oplus \sigma$ on $K_0(A\otimes\mathbb{K}) = (\mathbb{Z}+\mathbb{Z}\theta)\oplus\mathbb{Z}/3\mathbb{Z}$. Hence $\beta_*([1\otimes e_{00}]) = (1, \sigma(0)) = (1, 0) = [1\otimes e_{00}]$ in $K_0(A\otimes\mathbb{K})$. Since A has cancellation, $\beta(1\otimes e_{00}) \sim 1\otimes e_{00}$ in $A\otimes\mathbb{K}$. Therefore, $\mathrm{FP}(A)/\sim = \{(1\otimes e_{00})\}$ since $\mathrm{FP}(A)/\sim = \{(\beta(1\otimes e_{00})) \mid \beta \in \mathrm{Aut}(A\otimes\mathbb{K})\}$ by [**8**, Theorem 4.5].

Next we shall give an example of a unital C^*-algebra A with property $(*)$ whose $K_0(A)$ is torsion free.

Let $\mathbb{C}^m$ be the topological space of m-tuples of complex numbers and let S^{2m-1} be the $(2m-1)$-dimensional unit sphere of $\mathbb{C}^m$. Let $C(S^{2m-1})$ be the C^*-algebra of all complex-valued continuous functions on S^{2m-1}. Then $K_0(C(S^{2m-1})) = \mathbb{Z}[1_{C(S^{2m-1})}]$ and $K_1(C(S^{2m-1})) = \mathbb{Z}[v]$, where v is a unitary element in $M_m(C(S^{2m-1}))$. In the same way as in Clarke [**6**], we shall define the Toeplitz algebra $\tau(S^{2m-1})$ as follows: let $\mathrm{Ext}(C(S^{2m-1}))$ be the group of all stable strong equivalence classes of unital extensions of $C(S^{2m-1})$ by $\mathbb{K}$. Since there is the isomorphism γ of $\mathrm{Ext}(C(S^{2m-1}))$ onto $\mathrm{Hom}(K_1(C(S^{2m-1})), \mathbb{Z}) \cong \mathbb{Z}$ defined in Brown [**3**, Theorem] or Blackadar [**2**, 16.3.2], we define a unital extension τ as $\gamma([\tau])([v]) = 1$, where $[\tau]$ is the stable strong equivalence class in $\mathrm{Ext}(C(S^{2m-1}))$ of τ. We may assume that τ is essential by Blackadar [**2**, Proposition 15.6.5]. Let $\tau(S^{2m-1})$ be the pull-back of $(C(S^{2m-1}), M(\mathbb{K}))$ along τ and the quotient map of $M(\mathbb{K})$ onto $M(\mathbb{K})/\mathbb{K}$. We regard $\mathbb{K}$ as a C^*-subalgebra of $\tau(S^{2m-1})$. Then $K_0(\tau(S^{2m-1})) \cong \mathbb{Z}$.

Example 4.4. With the above notation, suppose that $m \geqslant 3$. Then there is a unital C^*-algebra B satisfying $M_{m-1}(\tau(S^{2m-1})) \ncong B$ but $M_{2m-2}(\tau(S^{2m-1})) \cong M_2(B)$.

In fact, since $\gamma([\tau])([v^*]) = -[\overline{e_{00}}]$ in $K_0(\mathbb{K})$, by the definition of $\gamma([\tau])$,

$$\left[V^* \begin{bmatrix} 1\otimes I_m & 0 \\ 0 & 0 \end{bmatrix} V\right] - \left[\begin{bmatrix} 1\otimes I_m & 0 \\ 0 & 0 \end{bmatrix}\right] = -[\overline{e_{00}}]$$

in $K_0(\mathbb{K})$, where V is a unitary element in $M_{2m}(\tau(S^{2m-1}))$ with

$$(\pi\otimes\mathrm{id}_{M_{2m}})(V) = \begin{bmatrix} v & 0 \\ 0 & v^* \end{bmatrix}$$

and where π is the homomorphism of $\tau(S^{2m-1})$ onto $C(S^{2m-1})$ associated with τ. Hence

$$\left[V^* \begin{bmatrix} 1\otimes I_m & 0 \\ 0 & 0 \end{bmatrix} V\right] = \left[\begin{bmatrix} (1-\overline{e_{00}})\otimes f_{11} + \sum_{j=2}^m 1\otimes f_{jj} & 0 \\ 0 & 0 \end{bmatrix}\right]$$

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in $K_0(\tilde{\mathbb{K}})$. Since $\tilde{\mathbb{K}}$ has cancellation, there is a unitary element R in $M_{2m}(\tilde{\mathbb{K}})$ such that

$$R^*V^*\begin{bmatrix}1\otimes I_m & 0\\ 0 & 0\end{bmatrix}VR=\begin{bmatrix}(1-\overline{e_{00}})\otimes f_{11}+\sum_{j=2}^m 1\otimes f_{jj} & 0\\ 0 & 0\end{bmatrix}.$$

Thus

$$(1_{\tau(S^{2m-1})}-\overline{e_{00}})\oplus I_{m-1}\sim 1_{\tau(S^{2m-1})}\oplus I_{m-1}\quad \text{in } M_m(\tau(S^{2m-1})),$$

where we regard I_{m-1} as the unit element in $M_{m-1}(\tau(S^{2m-1}))$. Let $p=(1-\overline{e_{00}})\oplus I_{m-2}\in M_{m-1}(\tau(S^{2m-1}))$. Then

$$\begin{aligned}p\oplus p&=(1-\overline{e_{00}})\oplus I_{m-2}\oplus(1-\overline{e_{00}})\oplus I_{m-2}\sim(1-\overline{e_{00}})\oplus I_{m-3}\oplus(1-\overline{e_{00}})\oplus I_{m-1}\\ &\sim(1-\overline{e_{00}})\oplus I_{m-3}\oplus 1\oplus I_{m-1}=(1-\overline{e_{00}})\oplus I_{m-1}\oplus I_{m-2}\\ &\sim I_{m-1}\oplus I_{m-1}\end{aligned}$$

in $M_{2m-2}(\tau(S^{2m-1}))$. Thus

$$pM_{m-1}(\tau(S^{2m-1}))p\otimes M_2\cong(p\oplus p)M_{2m-2}(\tau(S^{2m-1}))(p\oplus p)\cong M_{2m-2}(\tau(S^{2m-1})).$$

Put $B=pM_{m-1}(\tau(S^{2m-1}))p$. Then $M_2(B)\cong M_{2m-2}(\tau(S^{2m-1}))$. We shall prove that $B\not\cong M_{m-1}(\tau(S^{2m-1}))$. By Phillips and Raeburn [**12**, Remark 2.23], the Picard group of $C(S^{2m-1})$, $\mathrm{Pic}(C(S^{2m-1}))$, is isomorphic to the semi-direct product group $H^2(S^{2m-1},\mathbb{Z})\times_s \mathrm{Homeo}(S^{2m-1})$, where $\mathrm{Homeo}(S^{2m-1})$ is the group of all homeomorphisms on S^{2m-1} and it acts on $H^2(S^{2m-1},\mathbb{Z})$ in the natural way. Since $H^2(S^{2m-1},\mathbb{Z})=0$ by Massey [**11**, Theorem 2.14], $\mathrm{Pic}(C(S^{2m-1}))\cong\mathrm{Homeo}(S^{2m-1})$. Thus, by [**8**, Theorem 4.5], $\mathrm{FP}(C(S^{2m-1}))/\sim=\{(1\otimes e_{00})\}$. Since $(\pi\otimes\mathrm{id}_{\mathbb{K}})(q)\in\mathrm{FP}(C(S^{2m-1}))$ for any $q\in\mathrm{FP}(\tau(S^{2m-1}))$, $(\pi\otimes\mathrm{id})(q)\sim 1_{C(S^{2m-1})}\otimes e_{00}$ in $C(S^{2m-1})\otimes\mathbb{K}$. Hence, by [**10**, Lemma 4.1],

$$\begin{aligned}&\mathrm{FP}(\tau(S^{2m-1}))/\sim\\ &\qquad\subset\{(q)\mid q \text{ is a projection in } \tau(S^{2m-1})\otimes\mathbb{K} \text{ with } (\pi\otimes\mathrm{id})(q)=1\otimes e_{00}\}.\end{aligned}$$

By [**10**, Theorem 2.1], for every projection $q\in\tau(S^{2m-1})\otimes\mathbb{K}$ with $(\pi\otimes\mathrm{id})(q)=1\otimes e_{00}$, $q(\tau(S^{2m-1})\otimes\mathbb{K})q\cong\tau(S^{2m-1})$. Furthermore, since τ is essential, q is full in $\tau(S^{2m-1})\otimes\mathbb{K}$. Thus

$$\begin{aligned}&\mathrm{FP}(\tau(S^{2m-1}))/\sim\\ &\qquad=\{(q)\mid q \text{ is a projection in } \tau(S^{2m-1})\otimes\mathbb{K} \text{ with } (\pi\otimes\mathrm{id})(q)=1\otimes e_{00}\}.\end{aligned}$$

Hence, by an easy computation, we can see that

$$\begin{aligned}\mathrm{FP}(\tau(S^{2m-1}))/\sim=&\left\{\left(\left(1-\sum_{j=0}^n\overline{e_{jj}}\right)\otimes e_{00}\right)\,\middle|\, n\in\mathbb{N}\cup\{0\}\right\}\\ &\cup\left\{\left(1\otimes e_{00}+\sum_{j=1}^n\overline{e_{jj}}\otimes e_{11}\right)\,\middle|\, n\in\mathbb{N}\right\}.\end{aligned}$$

Moreover, by Lemma 4.1,

$$\mathrm{FP}_{m-1}(\tau(S^{2m-1}))/\sim = \left\{ \left(\left(1 - \sum_{j=0}^{n} \overline{e_{jj}} \right) \otimes \sum_{k=0}^{m-2} e_{kk} \right) \;\middle|\; n \in \mathbb{N} \cup \{0\} \right\}$$
$$\cup \left\{ \left(1 \otimes \sum_{k=0}^{m-2} e_{kk} + \sum_{j=1}^{n} \overline{e_{jj}} \otimes \sum_{k=m-1}^{2m-3} e_{kk} \right) \;\middle|\; n \in \mathbb{N} \right\}.$$

Since $M_{m-1}(\tau(S^{2m-1}))$ is finite by Blackadar [**2**, 6.10.1], $(p) \notin \mathrm{FP}_{m-1}(\tau(S^{2m-1}))/\sim$. Therefore, $B \ncong M_{m-1}(\tau(S^{2m-1}))$.

Finally, we shall give an example of a unital C^*-algebra A without property $(*)$ whose $K_0(A)$ has a torsion element.

For every $k \in \mathbb{N} \setminus \{1\}$ let τ_k be an essential unital extension of $C(S^1)$ by $\mathbb{K}$ with $\gamma([\tau_k])([v]) = k$. Let E_k be the pull-back of $(C(S^1), M(\mathbb{K}))$ along τ_k and the quotient map of $M(\mathbb{K})$ onto $M(\mathbb{K})/\mathbb{K}$. Then $K_0(E_k) \cong \mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$ and, in the same way as in Example 4.4, we can see that

$$\mathrm{FP}(E_k)/\sim = \{(p) \mid p \text{ is a projection in } E_k \otimes \mathbb{K} \text{ with } (\pi \otimes \mathrm{id})(p) = 1_{C(S^1)} \otimes e_{00}\},$$

where π is the homomorphism of E_k onto $C(S^1)$ associated with τ_k.

Example 4.5. With the above notation, E_k does not satisfy property $(*)$.

In fact, by Corollary 3.7 it suffices to show that for any projection $q \in E_k \otimes \mathbb{K}$ satisfying that there is a projection $p \in \mathrm{FP}(E_k)$ with $p \otimes I_n \sim q \otimes I_n$ in $E_k \otimes \mathbb{K} \otimes M_n$ for some $n \in \mathbb{N} \setminus \{1\}$, $q \in \mathrm{FP}(E_k)$. Suppose that q is such a projection. Then $n\pi_*([q]) = n[1]$ in $K_0(C(S^1))$. Since $K_0(C(S^1)) \cong \mathbb{Z}$ and $C(S^1)$ has cancellation, $(\pi \otimes \mathrm{id}_{\mathbb{K}})(q) \sim 1 \otimes e_{00}$ in $C(S^1) \otimes \mathbb{K}$. Thus, by [**10**, Lemma 4.1], there is a projection $q_0 \in E_k \otimes \mathbb{K}$ such that $q_0 \sim q$ in $E_k \otimes \mathbb{K}$ and $(\pi \otimes \mathrm{id})(q_0) = 1 \otimes e_{00}$. Therefore, $q \in \mathrm{FP}(E_k)$.

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References

1. B. Blackadar, A stable cancellation theorem from simple C^*-algebras, *Proc. Lond. Math. Soc.* **47** (1983), 303–305.
2. B. Blackadar, *K-theory for operator algebras*, 2nd edn, MSRI Publication 5 (Cambridge University Press, 1998).
3. L. G. Brown, Operator algebras and algebraic K-theory, *Bull. Am. Math. Soc.* **81** (1975), 1119–1121.
4. L. G. Brown, Stable isomorphism of hereditary subalgebras of C^*-algebras, *Pac. J. Math.* **71** (1977), 349–363.
5. L. G. Brown, P. Green and M. A. Rieffel, Stable isomorphism and strong Morita equivalence of C^*-algebras, *Pac. J. Math.* **71** (1977), 349–363.

6. N. P. Clarke, A finite but not stably finite C^*-algebra, *Proc. Am. Math. Soc.* **96** (1986), 85–88.
7. J. Cuntz, K-theory for certain C^*-algebras, *Ann. Math.* **113** (1981), 181–197.
8. K. Kodaka, Full projections, equivalence bimodules and automorphisms of stable algebras of unital C^*-algebras, *J. Operat. Theory* **37** (1997), 357–369.
9. K. Kodaka, Picard groups of irrational rotation C^*-algebras, *J. Lond. Math. Soc.* **56** (1997), 179–188.
10. K. Kodaka, A lifting problem of full projections in stable algebras of separable unital nuclear C^*-algebras to their unital extensions, preprint, 2001.
11. W. S. Massey, *Homology and cohomology theory* (Marcel Dekker, New York, 1978).
12. J. Phillips and I. Raeburn, Automorphisms of C^*-algebras and second Čech cohomology, *Indiana Univ. Math. J.* **29** (1980), 799–822.
13. J. Plastiras, C^*-algebras isomorphic after tensoring, *Proc. Am. Math. Soc.* **66** (1977), 276–278.
14. M. A. Rieffel, C^*-algebras associated with irrational rotations, *Pac. J. Math.* **93** (1981), 415–429.
15. C. L. Shen, A note on the automorphism groups of simple dimension groups, *Pac. J. Math.* **89** (1980), 199–207.