Can. J. Math., Vol. XXII, No. 4, 1970, pp. 713–718

A MAPPING PROBLEM AND J_p-INDEX. II

MASAMI WAKAE AND OMA HAMARA

1. Introduction. In [**9**], indices for equivariant mappings have been defined in the case that the transformation groups are cyclic. Thus a question will naturally arise as to the generalization of [**4**, Theorem 2] or [**8**, § IV, Theorem 2.8]. In this paper we will generalize the above result when the transformation groups are of order $p^a q^b$, p, q are odd prime numbers. The method used here can be used directly for more general cyclic groups, say, of order $n = p_1^{\alpha_1} \ldots p_m^{\alpha_m}$. However, the results are too complicated to be of interest.

2. The index of a Euclidean space. Let $\alpha = p^a$ and $\beta = q^b$, where p and q are odd prime numbers. Throughout this paper let $S = \{1, s, \ldots, s^{\alpha\beta-1}\}$ act on $R^{\alpha+\beta}$ and $\mathrm{SO}(\alpha + \beta)$ as follows:

$$s(x_1, \ldots, x_\alpha, y_1, \ldots, y_\beta) = (x_2, \ldots, x_\alpha, x_1, y_2, \ldots, y_\beta, y_1).$$

Let $F(R^{\alpha+\beta}) = \{\bar{x} | \bar{x} \in R^{\alpha+\beta}$ and $s^i(\bar{x}) = \bar{x}$ for some i, $0 < i < \alpha\beta\}$. Then S acts on $R_*^{\alpha+\beta} = R^{\alpha+\beta} - F(R^{\alpha+\beta})$ and on $\mathrm{SO}(\alpha+\beta)$ properly discontinuously. It is clear that $F(R^{\alpha+\beta}) = F_1(R^\alpha) \cup F_2(R^\beta)$, where $F_1 = F(R^\alpha) \times R^\beta$, $F_2 = R^\alpha \times F(R^\beta)$ and F is the generalized (or fat) diagonal defined in [**8**, p. 411].

Let $I^{\alpha+\beta}$ be the unit $(\alpha+\beta)$-cube; then we may assume that $R^{\alpha+\beta} = \mathrm{int}\, I^{\alpha+\beta}$ so that $R_*^{\alpha+\beta} = \mathrm{int}(I^{\alpha+\beta} - F(I^{\alpha+\beta})) \subset I^{\alpha+\beta} - F(I^{\alpha+\beta}) = I_*^{\alpha+\beta}$. The inclusion is clearly an equivariant map so that $\nu(R_*^{\alpha+\beta}) \leqq \nu(I_*^{\alpha+\beta})$, where $\nu(X)$ is the J_p-index of X [**9**, (4.5)]. Now we have $I_*^{\alpha+\beta} = I_*^\alpha \times I_*^\beta$. Let K be a simplicial complex of I with the usual subdivision and $K^{\alpha+\beta}$ the cell complex $K \times \ldots \times K$ ($\alpha+\beta$ factors). Let $S_p = \{1, s^{\beta p^{a-1}}, \ldots, s^{(p-1)\beta p^{a-1}}\}$. Let $(K^{p^{a-1}})_*{}^p$ be the subcomplex of K^α which consists of all cells $\sigma_1 \times \ldots \times \sigma_p$ ($\sigma_i \in K^{p^{a-1}}$) with no vertex of K common to all these σ_i. It may be shown that $|(K^{p^{a-1}})_*{}^p|$ is a deformation retract of I_*^α as mentioned in [**10**, Theorem 1].

$(K^{p^{a-1}})_*{}^p$ is of dimension $(p-1)p^{a-1} - 1$. Also it may be shown that $|(K^{q^{b-1}})_*{}^q|$ is a deformation retract of I_*^β. Hence $|(K^{p^{a-1}})_*{}^p| \times |(K^{q^{b-1}})_*{}^q|$ is a deformation retract of $I_*^{\alpha+\beta} = I_*^\alpha \times I_*^\beta$. Therefore $H^i(I_*^{\alpha+\beta}) = 0$ for $i \geqq (p-1)p^{a-1} + (q-1)q^{b-1} - 1$. By [**3**, p. 44] we have the following result.

THEOREM 2.2. $\nu(R_*^{\alpha+\beta}) \leqq (p-1)p^{a-1} + (q-1)q^{b-1} - 1.$

3. The index of $\mathrm{SO}(\alpha+\beta)$**.** The following results are known [**1**, *proposition* 10.2 and théorème 19.1], where $B_X{}^* = H^*(B_X, J_p)$, B_X is the classifying space of

Received April 24, 1969.

a compact topological group X, and $J_p(\ )$ and $\bigwedge(\ )$ refer to the polynomial and to the exterior algebra over J_p, the integers modulo p, respectively:

$$H^*(\mathrm{SO}(\alpha+\beta), J_p) \cong \bigwedge(u_{\alpha+\beta-1}, u_3, u_7, \ldots, u_{2(\alpha+\beta)-5}), \quad \dim u_i = i,$$
$$B_{\mathrm{SO}(\alpha+\beta)} \cong J_p(v_{\alpha+\beta}, v_4, v_8, \ldots, v_{2(\alpha+\beta)-4}), \quad \dim v_i = i.$$

Let T be the maximal torus in $\mathrm{SO}(\alpha+\beta)$, G the subgroup of T of elements of order $\alpha\beta$. $S = \{1, s, \ldots, s^{\alpha\beta-1}\}$ may be embedded in G, since

$$s = \begin{bmatrix} \overbrace{\begin{matrix} 010\ldots0 \\ 0010\ldots0 \\ \cdots\cdots \\ 00\ldots1 \\ 10\ldots0 \end{matrix}}^{\alpha} & \overbrace{\begin{matrix} \cdots\cdots \\ \cdots\cdots \\ \mathbf{0} \end{matrix}}^{\beta} \\ \mathbf{0} & \begin{matrix} 010\ldots0 \\ 0010\ldots0 \\ \cdots\cdots \\ 0\ldots1 \\ 10\ldots0 \end{matrix} \end{bmatrix} \begin{matrix} \}\alpha \\ \\ \}\beta \end{matrix}$$

is orthogonal and similar to

$$\begin{bmatrix} D_1 & & & & & & & & & \\ & D_2 & & & & & & \mathbf{0} & & \\ & & \ddots & & & & & & & \\ & & & D_{(\alpha-1)/2} & & & & & & \\ & & & & {}_1D & & & & & \\ & & & & & \ddots & & & & \\ & \mathbf{0} & & & & & {}_{(\beta-1)/2}D & & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix} = D,$$

where

$$D_i = \begin{bmatrix} \cos\dfrac{2i\pi}{\alpha} & -\sin\dfrac{2i\pi}{\alpha} \\ \sin\dfrac{2i\pi}{\alpha} & \cos\dfrac{2i\pi}{\alpha} \end{bmatrix} \quad \text{and} \quad {}_jD = \begin{bmatrix} \cos\dfrac{2j\pi}{\beta} & -\sin\dfrac{2j\pi}{\beta} \\ \sin\dfrac{2j\pi}{\beta} & \cos\dfrac{2j\pi}{\beta} \end{bmatrix}.$$

Let $n = \frac{1}{2}(\alpha+\beta)$. We have:

$$B_T^* \cong J_p(t_1, t_2, \ldots, t_n), \quad \dim t_i = 2,$$
$$B_G^* \cong \bigwedge(a_1, \ldots, a_n) \otimes J_p(b_1, b_2, \ldots, b_n), \quad \dim a_i = 1, \dim b_i = 2,$$
$$B_S^* \cong \bigwedge(a) \otimes J_p(b), \quad \dim a = 1, \dim b = 2.$$

https://doi.org/10.4153/CJM-1970-081-8 Published online by Cambridge University Press

Let M be a compact topological group, L a subgroup of M, and E_M the N-universal space of M for a sufficiently large N. The projection $\rho(L, M)$ of $B_L = E_M/L$ onto $B_M = E_M/M$ induces $\rho^*(L, M)\colon B_M^* \to B_L^*$.

By [**1**, p. 200] we have:

$$\rho^*(T, \mathrm{SO}(\alpha+\beta))v_{4i} = \sigma_i(t_1^2, \ldots, t_{n-1}^2) \quad \text{for } i = 1, 2, \ldots, n-1,$$

where $\sigma_i(t_1^2, \ldots, t_{n-1}^2)$ is the ith symmetric function in the arguments $t_1^2, \ldots, t_{n-1}^2$. We have

$$\rho^*(T, \mathrm{SO}(\alpha+\beta))v_{\alpha+\beta} = t_1 t_2 \ldots t_n \quad \text{and} \quad \rho^*(G, T)t_i = b_i.$$

Since G is generated by $G_1, G_2, \ldots, G_n$, where

$$G_1 = \begin{bmatrix} \cos\frac{2\pi}{\alpha\beta} & -\sin\frac{2\pi}{\alpha\beta} & & & & \\ \sin\frac{2\pi}{\alpha\beta} & \cos\frac{2\pi}{\alpha\beta} & & & \mathbf{0} & \\ & & 1 & & & \\ & & & \cdot & & \\ & \mathbf{0} & & & \cdot & \\ & & & & & 1 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & \mathbf{0} & \\ & & \cos\frac{2\pi}{\alpha\beta} & -\sin\frac{2\pi}{\alpha\beta} & & & \\ & & \sin\frac{2\pi}{\alpha\beta} & \cos\frac{2\pi}{\alpha\beta} & & & \\ & & & & 1 & & \\ & \mathbf{0} & & & & \cdot & \\ & & & & & & 1 \end{bmatrix}, \ldots,$$

$$G_n = \begin{bmatrix} 1 & & & & & \\ & \cdot & & & \mathbf{0} & \\ & & \cdot & & & \\ & & & 1 & & \\ & \mathbf{0} & & & \cos\frac{2\pi}{\alpha\beta} & -\sin\frac{2\pi}{\alpha\beta} \\ & & & & \sin\frac{2\pi}{\alpha\beta} & \cos\frac{2\pi}{\alpha\beta} \end{bmatrix},$$

and since s is orthogonal and similar to

$$\begin{bmatrix} D_1 & & & & & & & & & \\ & D_2 & & & & & \mathbf{0} & & & \\ & & \cdot & & & & & & & \\ & & & \cdot & & & & & & \\ & & & & \cdot & & & & & \\ & & & & & D_{(\alpha-1)/2} & & & & \\ & & & & & & {}_1D & & & \\ & & & & & & & \cdot & & \\ & & & & & & & & \cdot & \\ & & & & & & & & & \cdot \\ & \mathbf{0} & & & & & & & & {}_{(\beta-1)/2}D \\ & & & & & & & & & & 1 \\ & & & & & & & & & & & 1 \end{bmatrix},$$

we see that

$\rho^*(S, G)b_i = i\beta b$, for $i = 1, \ldots, \frac{1}{2}(\alpha - 1)$, $\rho^*(S, G)b_i = (i - \frac{1}{2}(\alpha - 1))\alpha b$, for $i = \frac{1}{2}(\alpha - 1) + 1, \ldots, n - 1$, and $\rho^*(S, G)b_n = \alpha\beta b$. Hence

$$\rho^*(S, G)b_i \equiv 0 \pmod p$$

for $i > \frac{1}{2}(\alpha - 1)$. This generalizes [**5**, (a)].

We have the following lemma [**4**; **5**].

LEMMA $\prod_{i=1}^{m}(1 + (ib)^2) \equiv 1 + Ab^{2m} \pmod p$ *where* $m = \frac{1}{2}(p - 1)$ *and* A *is a non-vanishing constant.*

$$\begin{aligned} \rho^*(S, \mathrm{SO}(\alpha + \beta))(\overset{*}{B}_{\mathrm{SO}(\alpha+\beta)}) &= \rho^*(S, G)\rho^*(G, T)\rho^*(T, \mathrm{SO}(\alpha + \beta))B^*_{\mathrm{SO}(\alpha+\beta)} \\ &= \rho^*(S, G)\left[J_p\left(\prod_{i=1}^{n-1}(1 + b_i^2)\right) \otimes J_p(b_1, b_2, \ldots, b_n)\right] \\ &= \rho^*(S, G)\left[J_p\left(\prod_{i=1}^{\frac{1}{2}(\alpha-1)}(1 + b_i^2)\right)\right]. \end{aligned}$$

This equals, since $\alpha = p^a$, $1 + A'b^{(p-1)\alpha/p}$ [**8**, § IV, Lemma 2.5], where A' is a non-vanishing constant. Therefore, by [**2**, Proposition 10.3],

$$H^*(\mathrm{SO}(\alpha + \beta)/S) = \wedge(a) \otimes J_p(b)/\mathscr{I}(b^{(p-1)\alpha/p}) \otimes P',$$

where $\mathscr{I}(b^{(p-1)\alpha/p})$ is the ideal generated by $b^{(p-1)\alpha/p}$ and

$$P' = \wedge(u_{\alpha+\beta-1}, u_3, \ldots, \hat{u}_{(2(p-1)\alpha)/p-1}, \ldots, u_{4n-5}).$$

By an argument similar to that in [**9**], we may show the following.

THEOREM 1. $\nu(\mathrm{SO}(\alpha + \beta)) = 2(p - 1)\alpha/p = 2(p - 1)p^{a-1}$.

4. A mapping problem. When $g: S^{\alpha+\beta-1} \to \mathbf{R}$ is given, the map

$$G: \mathrm{SO}(\alpha + \beta) \to \mathbf{R}^{\alpha+\beta}$$

is defined by $G(w_1, \ldots, w_{\alpha+\beta}) = (g(w_1), \ldots, g(w_{\alpha+\beta}))$. Let $\mathbf{E} = G^{-1}(F(\mathbf{R}^{\alpha+\beta}))$ and $\mathbf{A} = \mathrm{SO}(\alpha+\beta) - \mathbf{E} = G^{-1}(\mathbf{R}_*^{\alpha+\beta})$. It is clear that $F|\mathbf{A}\colon \mathbf{A} \to \mathbf{R}_*^{\alpha+\beta}$ is free equivariant. Therefore, $\nu(\mathbf{A}) \leqq \nu(\mathbf{R}_*^{\alpha+\beta}) \leqq (p-1)p^{a-1} + (q-1)q^{b-1} - 1$.

If X is a locally Euclidean manifold, with Čech or Alexander (co)homology groups over J_p with closed supports, [**7**, (6.2)] becomes

$$\begin{array}{ccccccc} \ldots \to H^i(X', U') & \to & H^i(X') & \to & H^i(U') & \xrightarrow{\delta} & \ldots \\ \downarrow \cong & & \downarrow \cong & & \cong \downarrow & & \\ \ldots \to H_{N-i}(X'-U') & \to & H_{N-i}(X') & \to & H_{N-i}(X', X'-U') & \xrightarrow{\delta} & \ldots \end{array} \tag{a}$$

with exact rows and commutativity in the squares, where U is an open set in X, $X' = X/S$, $U' = U/S$, S is a properly discontinuous transformation group on X and U and $N = \dim X$.

(b) $\dim (X' - U') = M$ implies $H_j(X' - U') = 0$ for $j > M$.

Let $\mathscr{A}^i(X, s)$ be the ith J_p-Smith class of X in $H^i(X/S, J_p)$ [**9**, Definition 3.5]. Let $\phi\colon X \to Y$ be a free equivariant mapping. Then

$$\phi^* \mathscr{A}^i(Y, s) = \mathscr{A}^i(X, s)$$

[**9**, (3.6)].

With the above notation we have the following result.

THEOREM 1. *Let* $\mathbf{E}' = \mathbf{E}/S$. *If*

$$2(p-1)p^{a-1} > (p-1)p^{a-1} + (q-1)q^{b-1} - 1,$$

that is, if $\alpha > \beta$, *then* $H_{N-j}(\mathbf{E}') \neq 0$ *for*

$$(p-1)p^{a-1} + (q-1)q^{b-1} - 1 \leqq j \leqq 2(p-1)p^{a-1} - 1,$$

where $N = \dim \mathrm{SO}(\alpha+\beta) = \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)$.

Proof. Substitute $X' = \mathrm{SO}(\alpha+\beta)/S$, $U' = \mathbf{A}'$ in the diagram (a). Since $\nu(\mathbf{R}_*^{\alpha+\beta}) \leqq (p-1)p^{a-1} + (q-1)q^{b-1} - 1$, $\mathscr{A}^i(\mathbf{A}, s) = 0$ for

$$i \geqq (p-1)p^{a-1} + (q-1)q^{b-1} - 1.$$

On the other hand, $\mathscr{A}^i(\mathrm{SO}(\alpha+\beta), s) \neq 0$ for $i \leqq 2(p-1)p^{a-1} - 1$. The exactness and commutativity of the diagram (a) will yield the desired result.

Remarks. (1) If $\beta > \alpha$, that is, if

$$2(q-1)q^{b-1} > (p-1)p^{a-1} + (q-1)q^{b-1} - 1,$$

then we may use the J_q-indices to obtain a similar result.

(2) $\dim \mathbf{E} \geqq \frac{1}{2}(\alpha+\beta-1)(\alpha+\beta) - (p-1)p^{a-1} - (q-1)q^{b-1} + 1$.

(3) By [**6**, p. 41] and the technique in [**4**, Corollary 4 or **8**, § IV, Corollary 2.10], we may show that there exists a point $\bar{x} \in F(\mathbf{R}^{\alpha+\beta})$ such that if $E_0 = G^{-1}(\bar{x})$, then $\dim E_0 \geqq \dim \mathbf{E} - \dim F(\mathbf{R}^{\alpha+\beta})$.

(4) A $(p + q)$-dimensional rectangular parallelotope has $(p + q)$ possible edge lengths. If q or more edge lengths are the same, the $(p + q)$-dimensional rectangular parallelotope is called a q-semicube.

If $a = b = 1$ and $p > q$, then by the method used in [**4**, Corollary 5] we may show that: If K is a compact convex body in $\mathbf{R}^{p+q}$, then the dimension of the set of circumscribing q-semicubes is at least $\frac{1}{2}(p + q)(p + q - 3) + 3$.

There is a set of dimension at least

$$\tfrac{1}{2}(p + q)(p + q - 3) + 1 = \tfrac{1}{2}(p + q - 1)(p + q - 2)$$

of circumscribing q-semicubes with the same edge length.

References

1. A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2) *57* (1953), 115–207.

2. ——— *Sur l'homologie et la cohomologie des groupes de Lie compacts connexes*, Amer. J. Math. *76* (1954), 273–342.

3. A. Borel (with contributions by G. Bredon, E. E. Floyd, D. Montgomery, and R. Palais), *Seminar on transformation groups*, Ann. of Math. Studies, No. 46 (Princeton Univ. Press, Princeton, N. J., 1960).

4. D. G. Bourgin, *Multiplicity of solutions in frame mappings*, Illinois J. Math. *9* (1965), 169–177.

5. ——— *Multiplicity of solutions in frame mappings*. II, Illinois J. Math. *10* (1966), 557–562.

6. W. Hurewicz and H. Wallman, *Dimension theory* (Princeton Univ. Press, Princeton, N. J., 1941).

7. F. Raymond, *Local cohomology groups with closed supports*, Math. Z. *76* (1961), 31–41.

8. M. Wakae, *Some results on multiplicity of solutions in frame mappings*, Math. Z. *98* (1967), 407–421.

9. M. Wakae and O. Hamara, *A mapping problem and J_p-index*. I, Can. J. Math. *22* (1970), 705-712.

10. Wen-Tsün Wu, *On the $\Phi(p)$-classes of a topological space*, Sci. Record (N.S.) *1* (1957), 377–380.

University of Manitoba,
Winnipeg, Manitoba;
University of Arizona,
Tucson, Arizona

https://doi.org/10.4153/CJM-1970-081-8 Published online by Cambridge University Press