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A SIMPLE SOLUTION TO THE WORD PROBLEM FOR LATTICES

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1. Whitman [2] solved the word problem for lattices by giving an explicit construction of the free lattice, $FL(X)$, on a given set of generators X.

The solution is the following:

For $x, y \in X$, and $a, b, c, d \in FL(X)$,

(W1) $x \leq y$ iff $x = y$

(W2) $a \wedge b \leq x$ iff $a \leq x$ or $b \leq x$

(W3) $x \leq c \vee d$ iff $x \leq c$ or $x \leq d$

(W4) $a \wedge b \leq c \vee d$ if $\{a, b, c, d\} \cap [a \wedge b, c \vee d] \neq \varnothing$

where $[p, q] = \{x; p \leq x \leq q\}$.

The purpose of this note is to give a simple nonconstructive proof that the condition (W4) must hold in every projective (hence every free) lattice. Jonsson [1] has shown that in every equational class of lattices (W1), (W2), and (W3) hold. Therefore the combination of these results gives a complete nonconstructive solution to the word problem for lattices.

2. A lattice P is called projective if for every lattice A and every surjective homomorphism $\alpha: A \to B$, any homomorphism $\varphi: P \to B$ can be 'lifted' to a homomorphism $\bar{\varphi}: P \to A$ with $\alpha \circ \bar{\varphi} = \varphi$. Since every lattice is the homomorphic image of a free lattice, we have an equivalent condition: if $\varphi: A \to P$ is a surjective homomorphism then there exists a homomorphism $\alpha: P \to A$ with $\varphi \circ \alpha = id_P$, the identity map on P. Clearly every free lattice is projective.

THEOREM. *Every projective lattice satisfies* (W4).

Proof. Let P be a projective lattice and assume to the contrary that there exists $a, b, c, d \in P$ with $u = a \wedge b \leq c \vee d = v$ and $\{a, b, c, d\} \cap [u, v] = \varnothing$. We construct a new lattice on the set $L = (P \backslash [u, v]) \cup ([u, v] \times 2)$ (assuming of course that $P \cap ([u, v] \times 2) = \varnothing$) by: $x \leq y$ in L iff one of the following holds:

(a) $x, y \in P \backslash [u, v]$ with $x \leq y$ in P

(b) $x = (p, i), y \in P \backslash [u, v]$ with $p \leq y$ in P

(c) $x \in P \backslash [u, v], y = (q, j)$ with $x \leq q$ in P

(d) $x = (p, i), y = (q, j)$ with $p \leq q$ in P and $i \leq j$ in 2

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(2 is assumed to be the two element lattice $0 \leq 1$). With this partial order, L becomes a lattice in which:

$$a \wedge_L b = (u, 1) \quad \text{and} \quad c \vee_L d = (v, 0)$$

The map $\varphi: L \to P$ defined by:

$$\varphi(x) = \begin{cases} x, & x \in P \backslash [u, v] \\ p, & x = (p, i) \end{cases}$$

is clearly a surjective homomorphism of L onto P hence there exists $\alpha: P \to L$ with $\varphi \circ \alpha = id_P$ since P is projective. This implies that $\alpha \mid P \backslash [u, v]$ is the identity on $P \backslash [u, v]$ and that $\alpha(u) \leq \alpha(v)$ in L. But $\alpha(u) = \alpha(a \wedge_P b) = a \wedge_L b = (u, 1)$ and $\alpha(v) = \alpha(c \vee_P d) = c \vee_L d = (v, 0)$ since $\{a, b, c, d\} \cap [u, v] = \varnothing$ and by definition $(u, 1) \nless (v, 0)$ in L, a contradiction.

COROLLARY 1. *Every free lattice satisfies* (W4).

COROLLARY 2. *The word problem for lattices is solvable.*

3. The method cannot be immediately applied to the projective lattices in an arbitrary class of lattices since the given class may not be closed under this construction. It can however be applied to other 'lattice like' equational classes of algebras (e.g. bounded pseudo-complemented lattices and Heyting algebras) to obtain partial solutions to their respective word problems.

BIBLIOGRAPHY

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