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OSCILLATIONS OF INTERCONNECTED SYSTEMS WITH C^0 NONLINEARITIES

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Abstract

In this paper we establish conditions which ensure the existence of self-excited oscillations in complex dynamical systems with nondifferentiable nonlinearities, by considering those types of systems which can be viewed as an interconnection of several simpler subsystems. We find that the nonlinear terms of the system in which we are interested do not need to satisfy the Lipschitz condition.

0. Introduction

In recent years, many researchers have concerned themselves with the qualitative analysis of large-scale dynamical systems. The analysis is in terms of the qualitative properties of the free subsystems and of the structure of the interconnecting system. Examples of this method can be found in [2,6,7,8,10,11]. However these results are not applicable to some systems, for example, when the nonlinearity does not satisfy the Lipschitz condition. In this paper, we improve upon the old results and present new results, by providing conditions for the existence of limit cycles in interconnected systems with continuous nonlinearities which do not necessarily satisfy the Lipschitz condition. Using the method described in this paper, we are able to improve the oscillation result in [3] and discuss the existence of periodic solutions of second order difference equations.

Of particular interest to the present discussion are some results in [1] and [9]. In this paper, we extend their results to a large class of interconnected systems.

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1. Preliminaries

We call an $\ell \times \ell$ matrix $A = [a_{ij}]$ an M-matrix if $a_{ij} \le 0$ for all $i \neq j$ and if the successive principal minors of A are all positive. All M-matrices are, of course, nonsingular.

Define $H(\omega)$ to be the set of all square integrable functions $\phi : \left[0, \frac{2\pi}{\omega}\right] \to R$ which satisfy the conditions

$$\begin{cases} \phi(t + \frac{2\pi}{\omega}) = \phi(t) & \text{on } R, \\ \phi(t + \frac{\pi}{\omega}) = -\phi(t) & \text{on } R. \end{cases} \tag{1.1}$$

The above definition of $H(\omega)$ is easily extended to a set $H_\ell(\omega)$ of vector-valued functions $\phi : R \to R^\ell$ for which each component satisfies (1.1) above.

For $\phi \in H(\omega)$ we let

$$\phi(t) \sim \frac{1}{2} \sum_{n \text{ odd}} \hat{\phi}_n \exp(in\omega t). \tag{1.2}$$

Note also that $\|\phi\|^2 = \frac{1}{2} \sum_{n \text{ odd}} |\hat{\phi}_n|^2 = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} |\phi(t)|^2 \, dt$ determines a norm for $H(\omega)$. We define projections P and P^* onto $H(\omega)$ by $P\phi(t) = \frac{1}{2}\hat{\phi}_1 e^{i\omega t} + \frac{1}{2}\hat{\phi}_{-1} e^{-i\omega t}$ and $P^* = I - P$, for each $\phi \in H(\omega)$. For a continuous function $n : R \to R$ we define the describing function N of n by

$$N(a) = \frac{1}{\pi a} \int_0^{2\pi} e^{-i\theta} n(a \cos\theta) \, d\theta = \frac{1}{\pi a} \int_0^{2\pi} \cos\theta \, n(a \cos\theta) \, d\theta$$

for $a > 0$. Consider the qth order differential equation given by

$$L(D)x + n(x) = 0, \qquad n(-x) = -n(x) \tag{1.3}$$

where $L(D) = \sum_{j=0}^{q} a_j D^j$, $D = \frac{d}{dt}$. Now, since our purpose is to find a periodic solution $x(t)$ of this equation, with $x \in H(\omega)$, we can use (1.2) to obtain $L(D)x(t) \sim (1/2) \sum_{k \text{ odd}} \left(\sum_{j=0}^{q} a_j (ik\omega)^j\right) \hat{x}_k e^{ik\omega t}$. So (1.3) above is equivalent to

$$\frac{1}{2} \sum_{k \text{ odd}} \left(\sum_{j=0}^{q} a_j (ik\omega)^j \right) \hat{x}_k e^{ik\omega t} + \frac{1}{2} \sum_{k \text{ odd}} \hat{n}_k e^{ik\omega t} = 0, \tag{1.4}$$

where $\hat{n}_k = (\omega/\pi) \int_0^{\frac{2\pi}{\omega}} e^{-ik\omega t} n(x(t)) \, dt$. Equation (1.4) is equivalent to

$$\hat{x}_k + \frac{1}{\sum_{j=0}^{q} a_j (ik\omega)^j} \hat{n}_k = 0, \quad k = \pm 1, \pm 3, \ldots \tag{1.5}$$

for any $\omega > 0$ for which $\sum_{j=0}^{q} a_j(ik\omega)^j \neq 0$. Hence, if we define an operator g on $H(\omega)$ by

$$g\phi \sim \frac{1}{2}\sum_{k\,\mathrm{odd}}\left[1/\sum_{j=0}^{q} a_j(ik\omega)^j\right]\hat{\phi}_k e^{ik\omega t},$$

then (1.5) is equivalent to the operator equation $x + gn(x) = 0$, on $H(\omega)$.

2. Interconnected systems

We now consider systems which can be described by equations of the form

$$x_k + g_k n_k(x_k) = g_k \sum_{\substack{m=1\\ m\neq k}}^{\ell} b_{km}x_m, \qquad n_k(-x_k) = -n_k(x_k), \tag{2.1}$$

$k = 1, \ldots, \ell$, which can be written in matrix-vector form as

$$x + gn(x) = gbx, \qquad n(-x) = -n(x), \tag{2.2}$$

where all symbols in (2.2) are defined in the obvious way and $n_k : R \to R$. We can view (2.2) as an interconnection of ℓ free subsystems

$$x_k + g_k n_k(x_k) = 0. \tag{2.3}$$

The terms $g_k b_{km}$ given in (2.1) comprise the interconnecting structure of composite system (2.2). In Figures 2.1 and 2.2, the free subsystem (2.3) and composite system (2.2) with interconnecting structure (2.1) are depicted in the form of block diagrams.

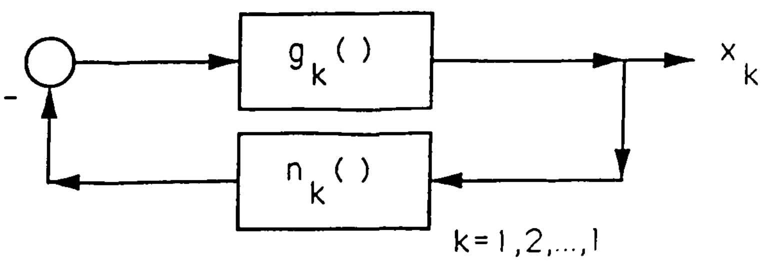

FIGURE 2.1. Free subsystems (2.3).

ASSUMPTION A_1. *For $k, m = 1, 2, \ldots, \ell$, $k \neq m$ and for all $\omega > 0$, g_k and b_{km} are continuous linear operators on $H(\omega)$. There exist continuous complex-valued functions $G_k(i\omega) = \overline{G_k(-i\omega)} \neq 0$ and $B_{km}(i\omega) = \overline{B_{km}(-i\omega)}$ such that if $u \in H(\omega)$, $w = g_k u$, $v = b_{km}u$, then $\hat{w}_n = G_k(in\omega)\hat{u}_n$ and $\hat{v}_n = B_{km}(in\omega)\hat{u}_n$ for every integer n. Furthermore,* $\lim_{\omega\to\infty} G_k(i\omega) = 0$.

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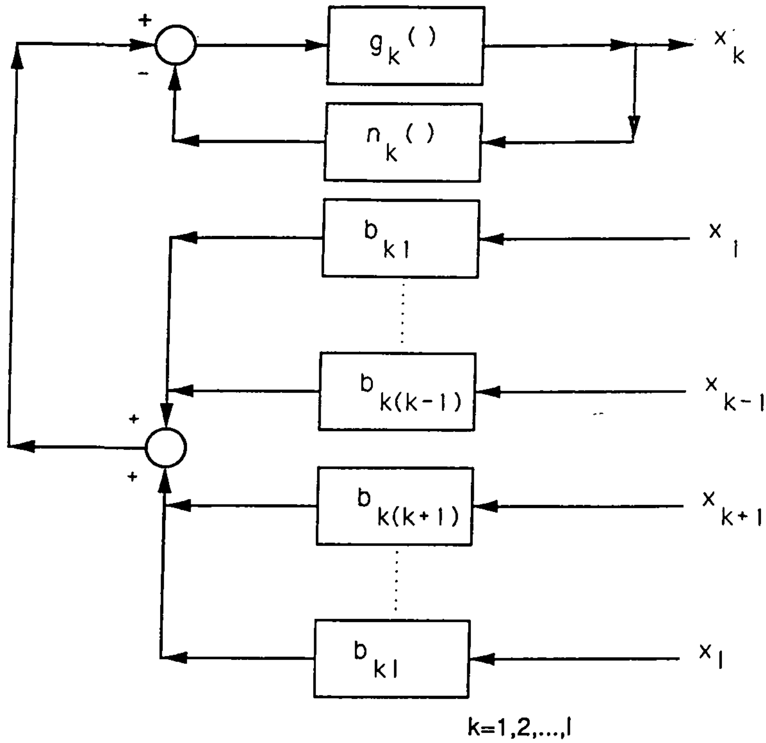

FIGURE 2.2. Interconnected system (2.2) with decomposition (2.1).

For each free subsystem (2.3), we can determine the describing function $N_k(a)$ of n_k. Now define $f_k(\omega, a) = |N_k(a) + G_k(i\omega)^{-1}|$. We choose the interval $I = [\mu, \nu]$ such that $f_k(\omega, a)$ is not too small for $a > 0$, $\mu \leq \omega \leq \nu$ and all values of $k = 1, \ldots, \ell$ except at most for one value of k. We relabel the equations in (2.3) [and the corresponding (2.1)] so that if $k \leq m$ and $\mu \leq \omega \leq \nu$ then $\min_{a \geq 0} f_k(\omega, a) \leq \min_{a \geq 0} f_m(\omega, a)$. Next, we choose an integer p with $1 \leq p \leq \ell$, such that if $k > p$, then $f_k(\omega, a)$ is extremely large for $\mu \leq \omega \leq \nu$ and $a > 0$. In general, we like to choose p as small as possible, because the smaller p is, the more easily assumptions A_3, A_4 below will be satisfied. Note that, if $p = \ell$, no $f_k(\omega, a)$ for $k = 1, \ldots, \ell$ need be extremely large. We define the functions

$$\rho_k(\omega, r_k) = \inf_{\substack{n \text{ odd} \\ |n| > \delta_k}} \left|G_k(in\omega)^{-1} + r_k\right| \quad k = 1, 2, \ldots, \ell,$$

$$\xi_{km}(\omega) = \sup_{\substack{|n| > \delta_k \\ n \text{ odd}}} |B_{km}(in\,\omega)| \qquad k = 1, 2, \ldots, \ell,\ m = 1, \ldots, \ell,\ k \neq m, \tag{2.4}$$

where $\delta_k = 1$ if $k \in \{1, 2, \ldots, p\}$ and $\delta_k = 0$ if $k \in \{p + 1, \ldots, \ell\}$, $r_k \in R^+$.

ASSUMPTION A_2. *For $k = 1, 2, \ldots, \ell$, there exist constants $r_{k0} \geq 0$ and $S_{k0} > 0$ such that $|n_k(\tau) - r_{k0}\tau| < S_{k0}$ for all $\tau \in R$.*

The results of this paper make use of a *test matrix* $R(\omega) = [r_{km}(\omega)]$ defined by

$$r_{km}(\omega) = \begin{cases} \rho_k(\omega, r_{k0}) & k = m \\ -\xi_{km}(\omega) S_{m0}/S_{k0} & k \neq m. \end{cases}$$

Let $\Gamma = \{\omega > 0 \mid R(\omega)$ is an M-matrix$\}$. Then it follows that for $\omega \in \Gamma$ we can find ℓ-vectors $d(\omega) > 0$ and $e(\omega) > 0$ such that $R(\omega)e(\omega) = d(\omega)$ [5]. From now on, we assume that $0 < a_1 < a_2$, $0 < \omega_1 < \omega_2$ and $[\omega_1, \omega_2] \subset \Gamma$. By the definition of $H(\omega)$ in Section 1, it is obvious that, for each $\omega \in R$, if u is in $H(\omega)$ then $r_{k0}u$ is in $H(\omega)$ and so is $n_k(u)$ which is defined by $n_k(u)(t) = n_k(u(t))$, for each $t \in R$.

ASSUMPTION A_3. *For any given $\omega \in [\omega_1, \omega_2] \subset \Gamma$, $u \in H(\omega)$, $k = 1, 2, \ldots, p$, we have $\|r_{k0}u - n_k(u)\| < S_{k0}\sqrt{2}d_k(\omega)/d_1(\omega)$ and for $k = p+1, \ldots, \ell$, we have*

$$\|r_{k0}u - n_k(u)\| + a \sum_{m=1}^{p} \xi_{km}(\omega).d_m(\omega)/d_1(\omega) < \sqrt{2} S_{k0} d_k(\omega)/d_1(\omega),$$

where $a_1 \leq a \leq a_2$, $u \in H(\omega)$ and $\|\cdot\|$ is the norm in $H(\omega)$.

Next, let ℓ_2 be the set of sequences $\tilde{y} = \{\hat{y}_m\}_{m=-\infty}^{\infty}$ for which $\hat{y}_m = 0$ if m is even, $\hat{y}_m = \bar{\hat{y}}_{-m}$, and $\|\tilde{y}\|_{\ell_2}^2 = \frac{1}{2}\sum_{m=-\infty}^{\infty} |\hat{y}_m|^2 < \infty$. Then ℓ_2 is isometrically isomorphic to $H(\omega)$ and for any $x \in H(\omega)$ we have $\|x\| = \|\tilde{x}\|_{\ell_2}$, where $\tilde{x} = \{\hat{x}_m\}_{m=-\infty}^{\infty}$ is the sequence of modified Fourier coefficients for x. Let ℓ_1 be the subset of ℓ_2 such that for any $\tilde{y} \in \ell_1$, $\|\tilde{y}\|_{\ell_1} = \frac{1}{2}\sum |\hat{y}_m| < \infty$. Define $H_1(\omega)$ as the corresponding subset of $H(\omega)$, and for any $x \in H_1(\omega)$, $\|x\|_1 = \|\tilde{x}\|_{\ell_1}$. Let $\Omega_2(\omega)$ be the set of all elements $V \in P^* H_1(\omega)$ such that

$$\|V\|_1 < \frac{1}{\rho_1(\omega, r_{10})}\left(S_{10} + \sum_{m=2}^{\ell} S_{m0}\xi_{1m}(\omega)\frac{e_m(\omega)}{d_1(\omega)}\right).$$

Next, we define the functions $\eta_k(\omega, a)$ such that if $(a, \omega) \in [a_1, a_2] \times [\omega_1, \omega_2]$, then (i)

$$\eta_1(\omega, a) = \sup\left\{\left|\frac{\omega}{a\pi}\int_0^{\frac{2\pi}{\omega}} e^{-i\omega t}\left\{\left[n_1(a\cos\omega t) - n_1(a\cos\omega t + V_1(t))\right] + r_{10}V_1(t)\right\} dt\right| d_1(\omega) + \sum_{m=2}^{p} |B_{1m}(i\omega)| d_m(\omega) + \sum_{m=p+1}^{\ell} \frac{\sqrt{2}}{a}|B_{1m}(i\omega)| e_m(\omega) S_{m0} : V_1(t) \in \Omega_2(\omega)\right\}.$$

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(ii) For $k = 2, \dots, p$,

$$\eta_k(\omega, a) = \sup \Bigg\{ \Bigg| \frac{\omega}{a\pi} \int_0^{\frac{2\pi}{\omega}} e^{-i\omega t} \big\{ \big[n_k(u_k a \cos \omega t) - n_k(u_k a \cos \omega t + V_k(t)) \big] + r_{k0} V_k(t) \big\} \, dt \Bigg| d_1(\omega) + \sum_{\substack{m=1 \\ m \neq k}}^{p} |B_{km}(i\omega)| d_m(\omega) + \frac{\sqrt{2}}{a} \sum_{m=p+1}^{\ell} |B_{km}(i\omega)| e_m(\omega) S_{m0} : V_k \in P^* H(\omega), \|V_k(t)\| < \frac{\sqrt{2} S_{k0} e_k(\omega)}{d_1(\omega)} \text{ and } u_k \in R, |u_k| < \frac{d_k(\omega)}{d_1(\omega)} \Bigg\}.$$

Define

$$\sigma_k(\omega, a) = \frac{\eta_k(\omega, a)}{d_k(\omega)}, \quad k = 1, 2, \dots, p.$$

ASSUMPTION A_4.

(1) $0 < a_1 < a_0 < a_2$ *and* $0 < \omega_1 < \omega_0 < \omega_2$,
(2) $[\omega_1, \omega_2] \subset \Gamma$,
(3) $f_1(\omega_0, a_0) = 0$,
(4) $f_1(\omega_1, a) > \sigma_1(\omega_1, a)$ *and* $f_1(\omega_2, a) > \sigma_1(\omega_2, a)$ *for* $a_1 \leq a \leq a_2$,
(5) $f_1(\omega, a_1) > \sigma_1(\omega, a_1)$ *and* $f_1(\omega, a_2) > \sigma_1(\omega, a_2)$ *for* $\omega_1 \leq \omega \leq \omega_2$,
(6) $N_1(a)$ *and* $G_1(i\omega)^{-1}$ *are continuous and* $N_1(a)$ *and* $\mathrm{Im}[G_1(i\omega)^{-1}]$ *are one-to-one for* $a_1 \leq a \leq a_2$, $\omega_1 \leq \omega \leq \omega_2$,
(7) *For* $k = 2, \dots, p, \omega_1 \leq \omega \leq \omega_2$, *and* $0 \leq a \leq a_2 \frac{d_k(\omega)}{d_1(\omega)}$ *we have* $f_k(\omega, a) > \max_{a_1 \leq a \leq a_2} \sigma_k(\omega, a)$,
(8) $e_k(\omega)$, $d_k(\omega)$ *are continuous for* $\omega_1 \leq \omega \leq \omega_2$.

3. Main result

We now state and prove the main result.

THEOREM 3.1. *Suppose that for the interconnected system* (2.2), *assumptions* A_1, A_2, A_3 *and* A_4 *are true,* $p > 0$ *and the functions* $n_k(\tau)$ *are continuous for all* $\tau \in R$ *and* $k = 1, 2, \dots, \ell$. *Then there exists a solution* $x \in H_\ell(\omega)$ *with* $x \neq 0$ *and* $\omega_1 \leq \omega \leq \omega_2$. *Furthermore*

$$0 < a_1 \leq a = |\hat{x}_1| \leq a_2,$$

$$\|x_1\| \le a + \left[\sqrt{2}S_{10} + \sum_{m=2}^{\ell} \xi_{1m}(\omega)\frac{\sqrt{2}e_m(\omega)}{d_1(\omega)}S_{m0}\right]\frac{1}{\rho_1(\omega, r_{10})},$$
$$\|x_k\|^2 \le a^2\frac{d_k(\omega)^2}{d_1(\omega)^2} + 2\frac{e_k(\omega)^2}{d_1(\omega)^2}S_{k0}^2 \quad k = 2, \dots, p,$$
$$\|x_k\| \le \frac{\sqrt{2}e_k(\omega)}{d_1(\omega)}S_{k0} \qquad k = p+1, \dots, \ell. \tag{3.1}$$

PROOF. In the proof of this theorem we make use of the Leray-Schauder fixed point theorem for Banach spaces. This reads: let Z be a Banach space and Ω be a bounded open subset of Z containing the origin. Let K be a compact operator on Z. Suppose that for any $z \in \partial\Omega$ and for any real $\lambda > 1$ we have $\lambda z \neq Kz$. Then there is a $z^0 \in \bar{\Omega}$ such that $z^0 = Kz^0$.

Step 1. We define the set Z as follows. An element $z = (z_1, z_2, \dots z_p, \tilde{z}_{p+1}, \dots, \tilde{z}_{p+\ell})$ is in Z if and only if z_k is a complex number for $k = 1, 2, \dots, p$, $\tilde{z}_{p+1} \in \ell_1$ and $\tilde{z}_k \in \ell_2$ for $k = p+2, \dots, p+\ell$. We define a norm on Z by

$$\|z\|^2 = \sum_{m=1}^{p} |z_m|^2 + \|\tilde{z}_{p+1}\|_{\ell_1}^2 + \sum_{m=p+2}^{p+\ell} \|\tilde{z}_m\|_{\ell_2}^2.$$

Then Z is a Banach space.

Step 2. Define

$$\begin{aligned} &h_k = P^*g_k, \quad C_{km} = P^*b_{km}, \quad y_k = P^*x_k, \quad \underline{x}_k = Px_k \\ &\qquad \text{for } k = 1, 2, \dots, p,\ m = 1, \dots, \ell; \text{ and} \\ &h_k = g_k, \quad C_{km} = b_{km}, \quad y_k = x_k, \quad \underline{x}_k = 0 \\ &\qquad \text{for } k = p+1, \dots, \ell,\ m = 1, \dots, \ell, \end{aligned} \tag{3.2}$$

where $k \neq m$. The two sets of equations

$$\underline{x}_k = Pg_k\left[-n_k(\underline{x}_k + y_k) + \sum_{\substack{m=1\\m\neq k}}^{p} b_{km}\underline{x}_m + \sum_{m=p+1}^{\ell} b_{km}y_m\right] \tag{3.3}$$

for $k = 1, \dots, p$ and $y_k = -h_k n_k(\underline{x}_k + y_k) + h_k \sum_{\substack{m=1\\m\neq k}}^{\ell} C_{km}(y_m + \underline{x}_m)$ for $k = 1, \dots, \ell$ are equivalent to (2.1). Define $\underline{x} = (\underline{x}_1, \dots, \underline{x}_\ell)$ and $y = (y_1, \dots, y_\ell)$.

Step 3. We are now going to construct operators $\tilde{F}_k$ and $\tilde{E}_k$ in the following and estimate them.

Since $\rho_k(\omega, r_{k0}) > 0$, $I + r_{k0}h_k$ has a continuous inverse on $H(\omega)$ [11] and

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$$\begin{aligned} y_k &= F_k(\omega, \underline{x}, y) \\ &= [I + r_{k0}h_k]^{-1}h_k \left[r_{k0}(y_k + \underline{x}_k) - n_k(y_k + \underline{x}_k) + \sum_{\substack{m=1 \\ m \neq k}}^{\ell} C_{km}(y_m + \underline{x}_m) \right] \end{aligned} \tag{3.4}$$

for $k = 1, \dots, \ell$. For $k = 1, \dots, p$, we know that $h_k \underline{x}_m = 0$ for $m = 1, \dots, \ell$ from (3.2). Also,

$$\begin{aligned} \|y_k\| &= \|F_k(\omega, \underline{x}, y)\| \\ &\leq \left\| [I + r_{k0}h_k]^{-1}h_k \right\| \\ &\quad \cdot \left[\|r_{k0}(y_k + \underline{x}_k) - n_k(y_k + \underline{x}_k)\| + \sum_{\substack{m=1 \\ m \neq k}}^{\ell} \|C_{km}\| \cdot \|y_m\| \right]. \end{aligned} \tag{3.5}$$

For $k = p + 1, \dots, \ell$, since $\underline{x}_k = 0$ it follows from (3.4) that

$$\begin{aligned} \|y_k\| \leq &\left\| [I + r_{k0}h_k]^{-1}h_k \right\| \\ &\cdot \left[\|r_{k0}y_k - n_k(y_k)\| + \sum_{\substack{m=1 \\ m \neq k}}^{\ell} \|C_{km}\| \|y_m\| + \sum_{m=1}^{p} \|C_{km}\| \cdot \|\underline{x}_m\| \right]. \end{aligned} \tag{3.6}$$

Equation (3.3) is confined to the subspace $PH(\omega)$. On this space, the operator Pg_k is invertible for $k = 1, \dots, \ell$. Thus, we may write (3.3) as

$$\begin{aligned} &(Pg_k)^{-1}\underline{x}_k + Pn_k(\underline{x}_k) \\ &\quad = P\left[n_k(\underline{x}_k) - n_k(\underline{x}_k + y_k) + r_{k0}y_k + \sum_{\substack{m=1 \\ m \neq k}}^{p} b_{km}\underline{x}_m + \sum_{m=p+1}^{\ell} b_{km}y_m \right] \end{aligned} \tag{3.7}$$

for $k = 1, \dots, p$. Since $\underline{x}_k$ can have only $\pm$ Fourier coefficients which are complex conjugates and since all other Fourier coefficients are zero, we may solve (3.7) by finding the first modified Fourier coefficient $\hat{x}_k$ for $\underline{x}_k$. The first modified Fourier coefficient of (3.7) is $G_k(i\omega)^{-1}\hat{x}_k + N_k(|\hat{x}_k|)\hat{x}_k = E_k(\omega, \underline{x}, y)$, where $\hat{x}_k = \hat{\underline{x}}_{k1} = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} e^{-i\omega t}\underline{x}_k(t)\,dt$ and

$$\begin{aligned} E_k(\omega, \underline{x}, y) = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} e^{-i\omega t} &\Bigg[n_k(\underline{x}_k(t)) - n_k(\underline{x}_k(t) + y_k(t)) + r_{k0} \cdot y_k(t) \\ &+ \sum_{\substack{m=1 \\ m \neq k}}^{p} b_{km}\underline{x}_m(t) + \sum_{m=p+1}^{\ell} b_{km}y_m(t) \Bigg] dt. \end{aligned}$$

Hence

$$|E_k(\omega, \underline{x}, y)| = \left\| P\left[n_k(\underline{x}_k) - n_k(\underline{x}_k + y_k) + r_{k0} y_k + \sum_{\substack{m=1\\m\neq k}}^{p} b_{km}\underline{x}_m + \sum_{m=p+1}^{\ell} b_{km} y_m \right] \right\|$$
$$\leq \left| \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} e^{-i\omega t}\left[n_k(\underline{x}_k(t)) - n_k(\underline{x}_k(t) + y_k(t)) + r_{k0}\cdot y_k(t) \right] dt \right|$$
$$+ \sum_{\substack{m=1\\m\neq k}}^{p} |B_{km}(i\omega)| \|\underline{x}_m\| + \sum_{m=p+1}^{\ell} |B_{km}(i\omega)| \|y_m\|. \tag{3.8}$$

Since $H(\omega)$ is isometrically isomorphic to ℓ_2, we may represent each $\underline{x}_k$ and y_k uniquely by $\hat{x}_k$ and $\tilde{y}_k \in \ell_2$, respectively, where $\hat{x}_k$ is the modified first Fourier coefficient of $\underline{x}_k$ and $\tilde{y}_k$ is the sequence of modified Fourier coefficients for $y_k \in H(\omega)$. Since F_k and E_m are operators on $H(\omega)$, there are corresponding operators $\tilde{F}_k$ and $\tilde{E}_m$ on ℓ_2. Thus, we can write

$$\tilde{F}_k(\omega, \hat{x}_1, \dots, \hat{x}_p, \tilde{y}_1, \dots, \tilde{y}_\ell) = F_k(\omega, \underline{x}, y)$$

and

$$\tilde{E}_m(\omega, \hat{x}_1, \dots, \hat{x}_p, \tilde{y}_1, \dots, \tilde{y}_\ell) = E_m(\omega, \underline{x}, y)$$

for $k = 1, \dots, \ell$ and $m = 1, \dots, p$.

Step 4. Let us construct an open subset Ω of Z and an operator K defined on a subset of Z.

In the following, we define the map $z_1 = J(\omega, a) = G_1(i\omega)^{-1} + N_1(a)$ which is continuous on a compact set (see A_4) $\Phi = \{(\omega, a) : \omega_1 \leq \omega \leq \omega_2, a_1 \leq a \leq a_2\}$. Let $\Psi = J(\Phi)$. Then the inverse function $J^{-1} : \Psi \to \Phi$ or $J^{-1} : z_1 \to (\omega(z_1), a(z_1))$ for $z_1 \in \Psi$ is continuous and $0 = J(\omega_0, a_0) \in \operatorname{Int}\Psi$. For each $z \in Z$ with $z_1 \in \Psi$, we have $\omega(z_1)$ and $a(z_1)$ defined above, and furthermore we define the vector operator $K = (K_1, \dots, K_{\ell+p})$ by

$$K_1(z) = \frac{1}{a(z_1)} \tilde{E}_1\left[\omega(z_1), a(z_1), z_2 a(z_1), \dots, z_p a(z_1), \tilde{z}_{1+p}, \dots, \tilde{z}_{\ell+p}\right];$$
$$K_m(z) = \frac{\tilde{E}_m\left[\omega(z_1), a(z_1), z_2 a(z_1), \dots, z_p a(z_1), \tilde{z}_{1+p}, \dots, \tilde{z}_{\ell+p}\right]}{a(z_1)\left[G_m(\omega(z_1))^{-1} + N_m(|z_m a(z_1)|)\right]},$$
$$m = 2, \dots, p;$$
$$K_m(z) = \tilde{F}_{m-p}\left[\omega(z_1), a(z_1), z_2 a(z_1), \dots, z_p a(z_1), \tilde{z}_{1+p}, \dots, \tilde{z}_{\ell+p}\right],$$
$$m = p+1, \dots, p+\ell.$$

Next, we define $\tilde{\Omega}_2(\omega)$ as a subset of ℓ_2 corresponding $\omega_2(\omega)$ in $H(\omega)$. Let

$$\Omega = \Big\{z \in Z : z_1 \in \operatorname{Int}\Psi, |z_m| < \frac{d_m(\omega(z_1))}{d_1(\omega(z_1))}, \quad m = 2, \dots, p$$
$$\tilde{z}_{p+1} \in \tilde{\Omega}_2(\omega(z_1)),$$
$$\|\tilde{z}_m\|_{\ell_2} < \frac{\sqrt{2}e_{m-p}(\omega(z_1))}{d_1(\omega(z_1))} S_{(m-p)0}, \quad m = p+2, \dots, p+\ell\Big\}.$$

It is easy to prove that Ω is open by the definition of $\tilde{\Omega}_2(\omega)$ and the continuity of $d_m(\omega), e_m(\omega)$ $(m = 1, 2, \dots, \ell)$ and ρ_1 defined in (2.4).

Step 5. In this step, we prove that K is bounded on $\bar{\Omega}$ and hence show that K can be extended to a compact operator on the whole space Z.

When $\tilde{z}_{p+1} \in \tilde{\Omega}_2(\omega)$, from the definition of $\Omega_2(\omega)$ we have

$$\|\tilde{z}_{p+1}\|_{\ell_2} \le \|\tilde{z}_{p+1}\|_{\ell_1} \cdot \sqrt{2} < \frac{1}{\rho_1(\omega, r_{10})}\sqrt{2}\left(S_{10} + \sum_{m=2}^{\ell} \xi_{1m}(\omega)\frac{e_m(\omega)}{d_1(\omega)} S_{m0}\right).$$

On the other hand, from the definitions of $e(\omega)$ and $d(\omega)$, we have

$$\frac{1}{\rho_1(\omega, r_{10})}\left(S_{10} + \sum_{m=2}^{\ell} \xi_{1m}\frac{e_m(\omega)}{d_1(\omega)} S_{m0}\right) = \frac{e_1(\omega)}{d_1(\omega)} S_{10}.$$

So, for any $\tilde{z}_{p+1} \in \tilde{\Omega}_2(\omega)$, we have $\|\tilde{z}_{p+1}\|_{\ell_2} < \sqrt{2}\frac{e_1(\omega)}{d_1(\omega)} S_{10}$.

To satisfy the condition for boundary points of Ω, we let z be in the closure of Ω. From (3.8) and the definition of $\sigma_k(\omega, a)$,

$$\begin{aligned}
&\frac{1}{|a(z_1)|}\left|\tilde{E}_1(\omega(z_1), a(z_1), z_2 a(z_1), \dots, z_p a(z_1), \tilde{z}_{p+1}, \dots, \tilde{z}_{p+\ell})\right| \\
&\quad \le \frac{1}{|a(z_1)|}\left|\frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} e^{-i\omega t}\big[n_1(a(z_1)\cos\omega t) - n_1(a(z_1)\cos\omega t + z_{p+1}(t)) \right. \\
&\qquad\qquad \left. + r_{10} z_{p+1}(t)\big]\, dt\right| \\
&\qquad + \frac{1}{|a(z_1)|}\sum_{m=2}^{p} |B_{1m}(i\omega)||z_m a(z_1)| + \frac{1}{|a(z_1)|}\sum_{m=p+1}^{\ell} |B_{1m}(i\omega)|\|\tilde{z}_{p+m}\|_{\ell} \\
&\quad \le \left|\frac{1}{a(z_1)}\frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} e^{-i\omega t}\big[n_1(a(z_1)\cos\omega t) - n_1(a(z_1)\cos\omega t + z_{p+1}(t)) \right. \\
&\qquad \left. + r_{10} z_{p+1}(t)\big]\, dt\right|
\end{aligned}$$

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$$+\sum_{m=2}^{p}|B_{1m}(i\omega)|\frac{d_m(\omega)}{d_1(\omega)}+\frac{\sqrt{2}}{|a(z_1)|}\sum_{m=p+1}^{\ell}|B_{1m}(i\omega)|\frac{e_m(\omega)}{d_1(\omega)}S_{m0}$$
$$\le \sigma_1(\omega(z_1),a(z_1)). \tag{3.9}$$

For $k=2,\dots,p$,

$$\frac{1}{|a(z_1)|}\left|\tilde{E}_k(\omega(z_1),a(z_1),z_2a(z_1),\dots,z_pa(z_1),\tilde{z}_{p+1},\dots,\tilde{z}_{p+\ell})\right|$$
$$\le\left|\frac{1}{a(z_1)}\frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}}e^{-i\omega t}\big[n_k(a(z_1)z_k\cos\omega t)-n_k(a(z_1)z_k\cos\omega t+z_{p+k}(t))\right.$$
$$\left.+r_{k0}z_{p+k}(t)\big]\,dt\right|$$
$$+\sum_{\substack{m=2\\ m\neq k}}^{p}|B_{km}(i\omega)|\frac{d_m(\omega)}{d_1(\omega)}+|B_{k1}(i\omega)|+\frac{\sqrt{2}}{|a(z_1)|}\sum_{m=p+1}^{\ell}|B_{km}(i\omega)|\frac{e_m(\omega)}{d_1(\omega)}S_{m0}$$
$$\le\frac{\eta_k(\omega(z_1),a(z_1))}{d_1(\omega(z_1))}. \tag{3.10}$$

Furthermore, since

$$\left\|r_{10}(y_1+\underline{x}_1)-n_1(y_1+\underline{x}_1)+\sum_{m=2}^{\ell}C_{1m}(y_m+\underline{x}_m)\right\|$$
$$\le\left\|r_{10}(y_1+\underline{x}_1)-n_1(y_1+\underline{x}_1)\right\|+\sum_{m=2}^{\ell}\|C_{1m}\|\cdot\|y_m\|$$
$$\le\sqrt{2}S_{10}+\sum_{m=2}^{\ell}\xi_{1m}(\omega)\|y_m\|$$

and $\|[I+r_{k0}h_k]^{-1}h_k\|\le 1/\rho_k(\omega,r_{k0})$, we have from (3.4) and the definition of $\tilde{\Omega}_2(\omega)$ that

$$\tilde{F}_1\big[\omega(z_1),a(z_1),z_2a(z_1),\dots,z_pa(z_1),\tilde{z}_{1+p},\dots,\tilde{z}_{\ell+p}\big]\in\tilde{\Omega}_2(\omega). \tag{3.11}$$

For $k=p+2,\dots,2p$, $n=k-p$, we have from (3.5), (3.2) and Assumption A_3,

$$\left\|\tilde{F}_n(\omega(z_1),a(z_1),z_2a(z_1),\dots,z_pa(z_1),\tilde{z}_{p+1},\dots,\tilde{z}_{p+\ell})\right\|_{\ell_2}$$
$$\le\frac{1}{\rho_n(\omega(z_1),r_{n0})}\left[\|r_{n0}u-n_n(u)\|+\sum_{\substack{m=1\\ m\neq n}}^{\ell}\xi_{nm}(\omega(z_1))\|\tilde{z}_{m+p}\|_{\ell_2}\right]$$

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$$\leq \frac{1}{\rho_n(\omega(z_1), r_{n0})}\left[\|r_{n0}u - n_n(u)\| + \sum_{\substack{m=1\\m\neq n}}^{\ell} \xi_{nm}(\omega(z_1))\frac{\sqrt{2}e_m(\omega(z_1))}{d_1(\omega(z_1))}S_{m0}\right]$$

$$= \frac{1}{\rho_n(\omega(z_1), r_{n0})}\left[\|r_{n0}u - n_n(u)\| - \sqrt{2}S_{n0}\frac{d_n(\omega)}{d_1(\omega)} + \sqrt{2}S_{n0}\rho_n(\omega, r_{n0})\frac{e_n(\omega)}{d_1(\omega)}\right]$$

$$\leq \frac{\sqrt{2}e_n(\omega(z_1))S_{n0}}{d_1(\omega(z_1))}, \tag{3.12}$$

where $u(t) \equiv z_{p+k}(t) + z_k a(z_1)\cos\omega t$ for $k = p+2, \ldots, 2p$. For $k = 2p+1, \ldots, p+\ell$, $n = k - p$, from (3.6), $\sum_{m=1}^{\ell} r_{nm}(\omega)(e_m(\omega)/d_1(\omega)) = d_n(\omega)/d_1(\omega)$ and Assumption A_3 we have

$$\left\|\tilde{F}_n(\omega(z_1), a(z_1), z_2 a(z_1), \ldots, z_p a(z_1), \tilde{z}_{p+1}, \ldots, \tilde{z}_{p+\ell}\right\|_{\ell_2}$$

$$\leq \frac{1}{\rho_n(\omega(z_1), r_{n0})}\left[\|r_{n0}(u) - n_n(u)\| + \sum_{\substack{m=1\\m\neq n}}^{\ell} \xi_{nm}(\omega(z_1))\|z_{p+m}\|_{\ell_2}\right.$$

$$\left. + \sum_{m=2}^{p} \xi_{nm}(\omega(z_1))|z_m a(z_1)| + \xi_{n1}(\omega(z_1))a(z_1)\right]$$

$$\leq \frac{1}{\rho_n(\omega(z_1), r_{n0})}\left[\sum_{\substack{m=1\\m\neq n}}^{\ell} \xi_{nm}(\omega(z_1))\frac{\sqrt{2}e_m(\omega(z_1))}{d_1(\omega(z_1))}S_{m0}\right.$$

$$\left. + \|r_{n0}(u) - n_n(u)\| + \sum_{m=1}^{p} \xi_{nm}(\omega(z_1))\frac{d_m(\omega(z_1))}{d_1(\omega(z_1))}a(z_1)\right]$$

$$= \frac{1}{\rho_n(\omega(z_1), r_{n0})}\left[\rho_n(\omega(z_1), r_{n0})\frac{\sqrt{2}e_n(\omega(z_1))}{d_1(\omega(z_1))}S_{n0} - S_{n0}\sum_{m=1}^{\ell} r_{nm}(\omega(z_1))\frac{\sqrt{2}e_m(\omega(z_1))}{d_1(\omega(z_1))}\right.$$

$$\left. + \|r_{n0}(u) - n_n(u)\| + \sum_{m=1}^{p} \xi_{nm}(\omega(z_1))\frac{d_m(\omega(z_1))}{d_1(\omega(z_1))}a\right]$$

$$\leq \frac{\sqrt{2}e_n(\omega(z_1))}{d_1(\omega(z_1))}S_{n0}, \tag{3.13}$$

where $u(t) \equiv z_n(t)$, for $n = p+1, \ldots, \ell$.

From (2.17) to (3.13) and Assumption A_4, we have that the operator K is bounded on $\bar{\Omega}$. From the definition of K, we also know that K is continuous on $\bar{\Omega} \subset Z$ and so we can extend K continuously to all of Z by the Tietze extension theorem ([4], pages 15-16) such that the operator K is continuous and bounded on Z. The components K_m, $m = 1, 2, \ldots, p$, are one-dimensional and thus compact. The components K_m for $m = p+1, \ldots, \ell$ involve the operators h_m which are defined as either $P^* g_m$ or g_m

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on $H(\omega)$. By Assumption A_1, $\lim_{k\to\infty}|G_m(ik\omega)| = 0$, $m = 1, 2, \ldots, \ell$. Therefore, the operator g_m, and hence h_m, must be compact. Thus F_m and $\tilde{F}_m$ are compact. It follows that K is a compact operator on Z.

Step 6. We now show that the boundary condition in the Leray-Schauder theorem is satisfied. Let $\lambda > 1$ and let $z \in Z$ be on the boundary of Ω. We must show that $\lambda z \neq Kz$. Treating K componentwise, we consider four cases.

(1) Suppose z_1 is on the boundary of Ψ. From $A_4(4)$ and $A_4(5)$ we see that for $k = 1, 2$ either $\omega(z_1) = \omega_k$ and

$$|z_1| = |J(\omega_k, a)| = f_1(\omega_k, a) \geq \sigma_1(\omega_k, a) = \sigma_1(\omega(z_1), a)$$

for $a_1 \leq a \leq a_2$ or $a(z_1) = a_k$ and

$$|z_1| = |J(\omega(z_1), a_k)| = f_1(\omega(z_1), a_k) \geq \sigma_1(\omega(z_1), a_k)$$

for $\omega_1 \leq \omega \leq \omega_2$. From (3.9) we see that $|K_1(z)| \leq \sigma_1(\omega(z_1), a(z_1))$ for all z in $\bar{\Omega}$. But for $z_1 \in \partial\Psi$ we have

$$\lambda|z_1| > |z_1| \geq \sigma_1(\omega(z_1), a(z_1)) \geq |K_1(z)|$$

so that in this case $\lambda z \neq Kz$.

(2) For $2 \leq m \leq p$ suppose that $|z_m| = d_m(\omega(z_1))/d_1(\omega(z_1))$. By Assumption A_4 (7) and (3.10) we have

$$\begin{aligned} |K_m(z)| &= \frac{|\tilde{E}_m(z)|}{|a(z_1)| f_m(\omega(z_1), |z_m a(z_1)|)} \\ &\leq \frac{\eta_m(\omega(z_1), a(z_1))}{d_1(\omega(z_1))\sigma_m(\omega(z_1), a(z_1))} = \frac{d_m(\omega(z_1))}{d_1(\omega(z_1))} = |z_m| < \lambda|z_m|, \end{aligned}$$

so that in this case $\lambda z \neq Kz$.

(3) Suppose $\tilde{z}_{p+1} \in \partial\tilde{\Omega}_2(\omega(z_1))$. By (3.11) and the definition of Ω_2 we have

$$\begin{aligned} \|K_{p+1}(z)\|_{\ell_2} &= \|\tilde{F}_1(z)\|_{\ell_2} \\ &\leq \frac{1}{\rho_1(\omega(z_1), r_{10})}\left(S_{10} + \sum_{m=2}^{\ell} S_{m0}\xi_{1m}(\omega(z_1))\frac{e_m(\omega(z_1))}{d_1(\omega(z_1))}\right) \\ &= \|\tilde{z}_{p+1}\|_{\ell_2} < \lambda\|\tilde{z}_{p+1}\|_{\ell_2}, \end{aligned}$$

so that in this case $\lambda z \neq Kz$.

(4) For $m = p+2, \ldots, \ell+p$, suppose that $\|\tilde{z}_m\|_{\ell_2} = \sqrt{2}e_{m-p}(\omega(z_1))S_{(m-p)0}/d_1(\omega(z_1))$. In view of (3.12) and (3.13) we have

$$\|K_m(z)\|_{\ell_2} = \|\tilde{F}_{m-p}(z)\|_{\ell_2} \leq \frac{\sqrt{2}e_{m-p}(\omega(z_1))S_{(m-p)0}}{d_1(\omega(z_1))} = \|\tilde{z}_m\|_{\ell_2} < \lambda\|\tilde{z}_m\|_{\ell_2},$$

so that in this case $\lambda z \neq Kz$.

It now follows from the Leray-Schauder theorem that there is an element z in the closure of Ω such that $z = K(z)$. We define an element $x \in H_\ell(\omega(z_1))$ by modified Fourier coefficients of $\underline{x}$ and y given by

$$\begin{aligned}
\hat{x}_1 &= a(z_1) \neq 0, \\
\hat{x}_k &= z_k a(z_1) && \text{for } k = 2, \dots, p, \\
\hat{x}_k &= 0 && \text{for } k = p+1, \dots, \ell, \\
\tilde{y}_k &= \tilde{z}_{p+k} && \text{for } k = 1, \dots, \ell.
\end{aligned}$$

Then $x = \underline{x} + y \in H_\ell(\omega(z_1))$ and x is a solution of (2.2) with $x \neq 0$.

Step 7. Now let us verify that (3.1) is true. Since z is in the closure of Ω, the bounds on the z_k $(k = 1, 2 \dots, p)$ and $\tilde{z}_m$ $(m = p+1, p+2, \dots, p+\ell)$ are satisfied. In the following we let $\omega \triangleq \omega(z_1)$ and $a \triangleq a(z_1)$. Since $\hat{x}_1 = a(z_1) = a$, we have

$$\begin{aligned}
&0 < a_1 \leq a = |\hat{x}_1| \leq a_2, \\
&\|x_1\| = \sqrt{\|\underline{x}_1\|^2 + \|y_1\|^2} = \sqrt{|\hat{x}_1|^2 + \|z_{p+1}\|^2} = \sqrt{|\hat{x}_1|^2 + \|\tilde{z}_{p+1}\|^2_{\ell_2}} \\
&\qquad \leq \left(|\hat{x}_1| + \|\tilde{z}_{p+1}\|_{\ell_2}\right) \\
&\qquad \leq \left(|\hat{x}_1| + \sqrt{2}\|\tilde{z}_{p+1}\|_{\ell_1}\right) \\
&\qquad \leq a + \left[\sqrt{2}S_{10} + \sum_{m=2}^{\ell} \xi_{1m}(\omega)\frac{\sqrt{2}e_m(\omega)}{d_1(\omega)}S_{m0}\right]\frac{1}{\rho_1(\omega, r_{10})}, \qquad \text{(see Step 5)} \\
&\|x_k\|^2 = \|\underline{x}_k\|^2 + \|y_k\|^2 = |z_k a(z_1)|^2 + \|\tilde{z}_{p+k}\|^2_{\ell_2} \\
&\qquad \leq a^2\frac{d_k(\omega)^2}{d_1(\omega)^2} + 2\frac{e_k(\omega)^2}{d_1(\omega)^2}S_{k0}^2 \qquad\qquad k = 2, \dots, p, \\
&\|x_k\| = \|y_k\|_{\ell_2} = \|z_{p+k}\|_{\ell_2} \leq \frac{\sqrt{2}e_k(\omega)}{d_1(\omega)}S_{k0} \quad k = p+1, \dots, \ell \quad \text{(see Step 5)}.
\end{aligned}$$

This completes the proof of Theorem 3.1.

4. An example

To demonstrate the applicability of the present results, we consider a specific composite system consisting of two subsystems, described by

$$\begin{cases} \dfrac{d^3x_1}{dt^3} + 2\dfrac{d^2x_1}{dt^2} + 4\dfrac{dx_1}{dt} + 6x_1 + f(x_1) = \dfrac{x_2}{3000}, \\ 100\dfrac{d^3x_2}{dt^3} + 200\dfrac{d^2x_2}{dt^2} - 400\dfrac{dx_2}{dt} - 600x_2 + f(x_2) = x_1, \end{cases} \tag{4.1}$$

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where

$$f(x) = \begin{cases} -1, & x < -1, \\ \sqrt[3]{x}, & -1 \le x \le 1, \\ 1, & x > 1. \end{cases} \tag{4.2}$$

It is easy to see that $f(x)$ is continuous but does not satisfy the Lipschitz condition when x is small. System (4.1) is a special case of (2.1) with $p = \ell = 2, n_1 = n_2, S_{10} = S_{20} \equiv 1, r_{10} = r_{20} = 0$. The system (4.1) is equivalent to the system

$$\begin{cases} x_1 + g_1 f(x_1) = g_1\left(\dfrac{x_2}{3000}\right), \\ x_2 + g_2 f(x_2) = g_2(x_1), \end{cases} \tag{4.3}$$

where g_1 and g_2 are two operators on $H(\omega)$, such that for $k = 1, 2$, and $\phi \in H(\omega)$,

$$(g_k\phi)(t) \sim \frac{1}{2}\sum_{n \text{ odd}} G_k(in\omega)\hat{\phi}_n \exp(in\omega t).$$

Also $G_1(S)^{-1} = S^3 + 2S^2 + 4S + 6$ and $G_2(S)^{-1} = 100S^3 + 200S^2 - 400S - 600$. Next, since $n_1(x) = n_2(x) = f(x)$, the describing functions $N_1(a)$ and $N_2(a)$ of n_1 and n_2 when $0 < a < 1$ are

$$\begin{aligned} N_1(a) = N_2(a) &= \frac{1}{\pi a}\int_0^{2\pi} (\cos\theta) f(a\cos\theta)\, d\theta \\ &= \frac{\sqrt[3]{a}}{\pi a}\int_0^{2\pi} \sqrt[3]{(\cos\theta)^4}\, d\theta = \frac{4}{\pi\sqrt[3]{a^2}}\int_0^{\pi/2} \sqrt[3]{(\cos\theta)^4}\, d\theta. \end{aligned}$$

It is evident that

$$-G_1(i\omega)^{-1} = (2\omega^2 - 6) + i(\omega^3 - 4\omega)$$

and

$$-G_2(i\omega)^{-1} = (200\omega^2 + 600) + i(100\omega^3 + 400\omega).$$

Now, from $f_1(\omega, a) = |G_1(i\omega)^{-1} + N_1(a)| = 0$, we obtain the solution $\omega_0 = 2$ and

$$N_1(a_0) - 2 = 0. \tag{4.4}$$

Note that (see Figure 4.1)

$$\int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{4}{3}} d\theta < \int_0^{\frac{\pi}{3}} \cos\theta\, d\theta + (\text{Area of } \Delta\text{CDE})$$

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$$= \sin\frac{\pi}{3} + \frac{1}{2}\cdot\left(\frac{\pi}{2} - \frac{\pi}{3}\right)\cdot\frac{1}{\sqrt[3]{16}} \approx 0.97$$

and

$$\int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{4}{3}}\, d\theta > \text{(Area of Trapezoid OABF)}$$

$$+ \text{(Area of Trapezoid BCEF)} + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\cdot\left(\frac{1}{\sqrt[3]{4}} + 1\right)\cdot\frac{\pi}{4} + \frac{1}{2}\cdot\left(\frac{1}{\sqrt[3]{16}} + \frac{1}{\sqrt[3]{4}}\right)\cdot\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + \left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right)$$

$$\approx 0.64 + 0.1344 + 0.0453 \approx 0.82.$$

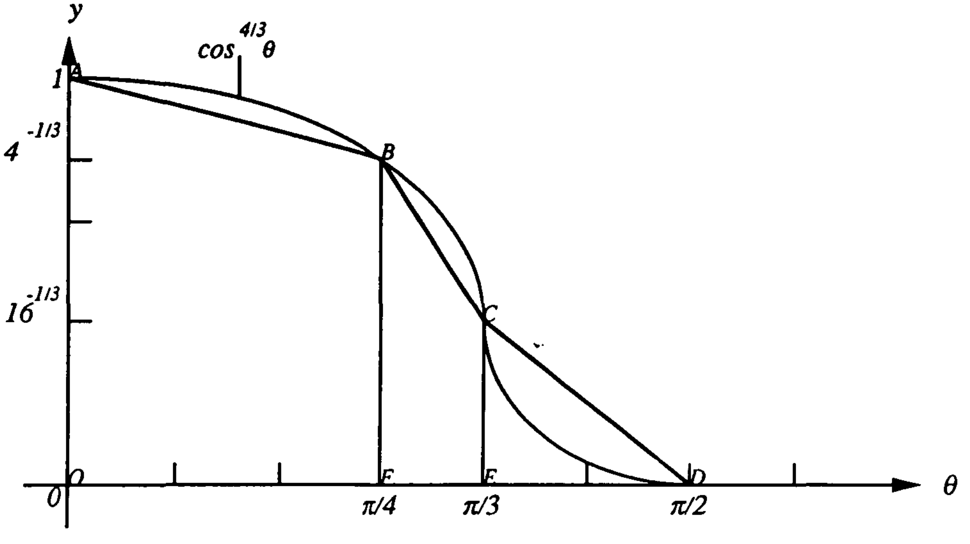

FIGURE 4.1.

So

$$\frac{4}{\pi\sqrt[3]{a^2}}\cdot 0.82 < N_1(a) < \frac{4}{\pi\sqrt[3]{a^2}}\cdot 0.97.$$

Hence, using (4.4), we have that $0.377 < a_0 < 0.48$.

Now we choose $\omega_1 = 1.98$, $\omega_2 = 2.02$, $a_1 = 0.30$ and $a_2 = 0.58$. We have defined the functions $\rho_1(\omega, r_{10})$ and $\rho_2(\omega, r_{20})$ in (2.4). For this example,

$$\rho_1(\omega, r_{10}) = \rho_1(\omega, 0) = \inf_{\substack{n \text{ odd} \\ |n|>1}} |G_1(in\omega)^{-1}|$$

$$= \inf_{\substack{n \text{ odd} \\ |n|>1}} \sqrt{(n\omega)^6 - 4(n\omega)^4 - 8(n\omega)^2 + 36}$$
$$= \sqrt{(3\omega)^6 - 4(3\omega)^4 - 8(3\omega)^2 + 36} \tag{4.5}$$

and

$$\rho_2(\omega, r_{20}) = \rho_2(\omega, 0) = \inf_{\substack{n \text{ odd} \\ |n|>1}} |G_2(in\omega)^{-1}|$$
$$= \inf_{\substack{n \text{ odd} \\ |n|>1}} \sqrt{[100(n\omega)^3 + 400n\omega]^2 + [200(n\omega)^2 + 600]^2}$$
$$= \sqrt{[100(3\omega)^3 + 400 \times 300]^2 + [200(3\omega)^2 + 600]^2}$$
$$= 100\sqrt{[(3\omega)^3 + 12\omega]^2 + [18\omega^2 + 6]^2}, \tag{4.6}$$

for $\omega \in [\omega_1, \omega_2]$.

It is easy to verify that $\rho_1(\omega, 0)$ and $\rho_2(\omega, 0)$ are increasing functions on $[\omega_1, \omega_2]$. So, for $\omega \in [\omega_1, \omega_2]$,

$$\rho_1(\omega, 0) \geq \sqrt{(3\omega_1)^6 - 4(3\omega_1)^4 - 8(3\omega_1)^2 + 36}$$
$$= \sqrt{(3 \cdot 1.98)^6 - 4(3 \cdot 1.98)^4 - 8(3 \cdot 1.98)^2 + 36} \approx 196.72$$

and

$$\rho_2(\omega, 0) \geq 100\sqrt{[(3\omega_1)^3 + 12\omega_1]^2 + [18\omega_1^2 + 6]^2}$$
$$= 100\sqrt{[(3 \cdot 1.98)^3 + 12 \cdot 1.98]^2 + [18 \cdot 1.98^2 + 6]^2} \approx 24558.5.$$

Therefore, from

$$R(\omega) = \begin{bmatrix} \rho_1(\omega, 0) & -\frac{1}{3000} \\ -1 & \rho_2(\omega, 0) \end{bmatrix},$$

we have $|R(\omega)| = \rho_1(\omega, 0)\rho_2(\omega, 0) - \frac{1}{3000} > 0$ for $\omega \in [\omega_1, \omega_2]$. Hence $R(\omega)$ is an M-matrix.

From the properties of M-matrices it follows that for $\omega \in \Gamma$, we can find 2-vectors

$$d(\omega) = \begin{pmatrix} d_1(\omega) \\ d_2(\omega) \end{pmatrix} > 0, \qquad e(\omega) = \begin{pmatrix} e_1(\omega) \\ e_2(\omega) \end{pmatrix} > 0$$

such that $R(\omega)e(\omega) = d(\omega)$.

For this example, it is evident that we can let $e_1(\omega) = e_2(\omega) = 1$, and thus

$$d_1(\omega) = \rho_1(\omega, 0)e_1(\omega) + e_2(\omega)\left(-\frac{1}{3000}\right) = \rho_1(\omega, 0) - \frac{1}{3000} > 0,$$

$$d_2(\omega) = -e_1(\omega) + \rho_2(\omega,0)e_2(\omega) = \rho_2(\omega,0) - 1 > 0$$

for $\omega \in [\omega_1, \omega_2]$.

Next for any $V_1(t) \in \Omega_2(\omega)$, from the definition of $\Omega_2(\omega)$, we have

$$|V_1(t)| < \frac{1}{\rho_1(\omega,0)} \cdot \frac{\frac{1}{3000} + d_1(\omega)}{d_1(\omega)} = \frac{1}{\rho_1(\omega,0) - \frac{1}{3000}} \leq \frac{1}{196.72 - \frac{1}{3000}} \approx 0.005. \tag{4.7}$$

So, when $V_1(t) \in \Omega_2(\omega)$ and $a \in [a_1, a_2]$,

$$\begin{aligned}
&\left| \frac{\omega}{a\pi} \int_0^{\frac{2\pi}{\omega}} e^{-i\omega t}\big[f(a\cos\ \omega t) - f(a\cos\ \omega t + V_1(t))\big]\,dt \right| \\
&= \frac{1}{a\pi} \left| \int_0^{2\pi} e^{-i\theta} \left[\sqrt[3]{a\cos\theta} - \sqrt[3]{a\cos\theta + V_1(\frac{\theta}{\omega})} \right] d\theta \right| \\
&\leq \frac{4}{a\pi} \int_0^{\frac{\pi}{2}} \cos\theta \left[\sqrt[3]{a\cos\theta + 0.005} - \sqrt[3]{a\cos\theta} \right] d\theta \\
&\quad + \frac{4}{a\pi} \int_0^{\frac{\pi}{2}} \sin\theta \left[\sqrt[3]{a\cos\theta + 0.005} - \sqrt[3]{a\cos\theta} \right] d\theta \\
&= \frac{4}{a^2\pi} \int_0^{\frac{\pi}{2}} \left[(a\cos\theta + 0.005)\sqrt[3]{a\cos\theta + 0.005} - a\cos\theta\sqrt[3]{a\cos\theta} \right] d\theta \\
&\quad - \frac{4 \cdot 0.005}{a^2\pi} \int_0^{\frac{\pi}{2}} \sqrt[3]{a\cos\theta + 0.005}\,d\theta \\
&\quad + \frac{4}{a^2\pi} \left[-\sqrt[3]{(a\cos\theta + 0.005)^4} + \sqrt[3]{(a\cos\theta)^4} \right]_0^{\frac{\pi}{2}} \\
&\leq \frac{4}{a^2\pi} \int_0^{\frac{\pi}{2}} \left[\sqrt[3]{(a+0.005)^4} - \sqrt[3]{a^4} \right] d\theta - \frac{4 \cdot 0.005}{a^2\pi} \int_0^{\frac{\pi}{2}} \sqrt[3]{a\cos^3\theta}\,d\theta \\
&\quad + \frac{4}{a^2\pi} \left[-\sqrt[3]{(0.005)^4} + \sqrt[3]{(a+0.005)^4} - \sqrt[3]{a^4} \right] \\
&= \frac{2}{a^2} \left[\sqrt[3]{(a+0.005)^4} - \sqrt[3]{a^4} \right] - \frac{4 \cdot 0.005}{a^2\pi}\sqrt[3]{a} \\
&\quad + \frac{4}{a^2\pi} \left[-\sqrt[3]{(0.005)^4} + \sqrt[3]{(a+0.005)^4} - \sqrt[3]{a^4} \right] \\
&\leq \frac{2}{a^2} \left[\sqrt[3]{(a+0.005)^4} - \sqrt[3]{a^4} \right] \\
&\quad + \frac{4}{a^2\pi} \left[\sqrt[3]{(a+0.005)^4} - \sqrt[3]{a^4} - \sqrt[3]{(0.005)^4} - 0.005\sqrt[3]{a} \right] \\
&\leq \frac{2}{a^2} \left[\sqrt[3]{(a+0.005)^4} - \sqrt[3]{a^4} \right]
\end{aligned}$$

$$+\frac{4}{a^2\pi}\left[\sqrt[3]{(a+0.005)^4}-\sqrt[3]{a^4}-\sqrt[3]{(0.005)^4}-0.005\sqrt[3]{a_1}\right]$$
$$\triangleq h(a).$$

Therefore, by the definition of η_1,

$$\sigma_1(\omega,a)=\frac{\eta_1(\omega,a)}{d_1(\omega)}\le h(a)+\frac{1}{3000}\frac{d_2(\omega)}{d_1(\omega)}. \tag{4.8}$$

Since $\rho_2(\omega_2,0)\approx 25925.8$,

$$\frac{1}{3000}\frac{d_2(\omega)}{d_1(\omega)}\le\frac{1}{3000}\frac{\rho_2(\omega_2,0)-1}{\rho_1(\omega_1,0)-\frac{1}{3000}}\approx\frac{1}{3000}\frac{25924.8}{196.72}\approx 0.044, \tag{4.9}$$

for $\omega\in[\omega_1,\omega_2]$. Thus we have that

$$\begin{aligned}\sigma_1(\omega,a_1)\le h(a_1)+0.044 &= \frac{2}{a_1^2}\left(1+\frac{2}{\pi}\right)\left[\sqrt[3]{(a_1+0.005)^4}-\sqrt[3]{a_1^4}\right]\\ &\quad-\frac{4\cdot 0.005}{a_1^2\pi}\left[\sqrt[3]{0.005}+\sqrt[3]{a_1}\right]+0.044\\ \approx 0.1032+0.044 &= 0.1472,\\ \sigma_1(\omega,a_2)\le h(a_2)+0.044 &= \frac{2}{a_2^2}\left(1+\frac{2}{\pi}\right)\left[\sqrt[3]{(a_2+0.005)^4}-\sqrt[3]{a_2^4}\right]\\ &\quad-\frac{4\cdot 0.005}{a_2^2\pi}\left[\sqrt[3]{0.005}+\sqrt[3]{a_1}\right]+0.044\\ \approx 0.038+0.044 &= 0.082.\end{aligned}$$

Note that

$$N_1(a_1)=2\left(\frac{a_0}{a_1}\right)^{\frac{2}{3}}\ge 2\left(\frac{0.377}{0.30}\right)^{\frac{2}{3}}\approx 2.329$$

and

$$N_1(a_2)=2\left(\frac{a_0}{a_2}\right)^{\frac{2}{3}}\le 2\left(\frac{0.48}{0.58}\right)^{\frac{2}{3}}\approx 1.763.$$

Put $j_1=N_1(a_1)$, $j_2=2.329$, $k_2=N_1(a_2)$ and $k_1=1.763$. Then we have

$$\begin{aligned}f_1(\omega,a_1)^2 &= |(N_1(a_1)+6-2\omega^2)+i(4\omega-\omega^3)|^2\\ &= (j_1-j_2)^2+2(j_1-j_2)[6-2\omega^2+j_2]+[6-2\omega^2+j_2]^2+[4\omega-\omega^3]^2\\ &\ge (j_1-j_2)^2+2(j_1-j_2)[6-2\omega_2^2+j_2]+[6-2\omega^2+j_2]^2+[4\omega-\omega^3]^2\end{aligned}$$

$$> [6 - 2\omega^2 + j_2]^2 + [4\omega - \omega^3]^2,$$

since $[6 - 2\omega_2^2 + j_2] > 0$. It can be verified numerically that the minimum of the right-hand side occurs at $\omega = 2.02$ and the minimum value is 0.0283+0.0264=0.0547. So,

$$f_1(\omega, a_1) \geq \sqrt{0.0547} \approx 0.2339 > 0.1472 \geq \sigma_1(\omega, a_1).$$

Similarly,

$$\begin{aligned} f_1(\omega, a_2)^2 &= |(N_1(a_2) + 6 - 2\omega^2) + i(4\omega - \omega^3)|^2 \\ &= (k_2 - k_1)^2 + 2(k_2 - k_1)[6 - 2\omega^2 + k_1] + [6 - 2\omega^2 + k_1]^2 + [4\omega - \omega^3]^2 \\ &\geq (k_2 - k_1)^2 + 2(k_1 - k_2)[2\omega_1^2 - k_1 - 6] + [6 - 2\omega^2 + k_1]^2 + [4\omega - \omega^3]^2 \\ &> [6 - 2\omega^2 + k_1]^2 + [4\omega - \omega^3]^2, \end{aligned}$$

since $[2\omega_1^2 - k_1 - 6] > 0$. The minimum of the right-hand side occurs at $\omega = 1.985$ and the minimum value is 0.0278. So,

$$f_1(\omega, a_2) \geq \sqrt{0.0278} \approx 0.1667 > 0.082 \geq \sigma_1(\omega, a_2).$$

Now, when $a \in [a_1, a_2]$,

$$\begin{aligned} h'(a) &= \frac{2}{a^2}\left[\frac{4}{3}(a + 0.005)^{\frac{1}{3}} - \frac{4}{3}a^{\frac{1}{3}} - \frac{2}{a}(a + 0.005)^{\frac{4}{3}} + \frac{2}{a}a^{\frac{4}{3}}\right] \\ &\quad + \frac{4}{a^2\pi}\left[\frac{4}{3}(a + 0.005)^{\frac{1}{3}} - \frac{4}{3}a^{\frac{1}{3}} - \frac{2}{a}(a + 0.005)^{\frac{4}{3}} + \frac{2}{a}a^{\frac{4}{3}}\right. \\ &\qquad \left. + \frac{2}{a}0.005^{\frac{4}{3}} + \frac{2}{a}0.005\sqrt[3]{a_1}\right] \\ &= \frac{2}{a^2}\left\{(a + 0.005)^{\frac{1}{3}}\left[\frac{4}{3} - \frac{2}{a}(a + 0.005)\right] - a^{\frac{1}{3}}\left(\frac{4}{3} - \frac{2}{a}a\right)\right\} \\ &\quad + \frac{4}{a^2\pi}\left\{(a + 0.005)^{\frac{1}{3}}\left[\frac{4}{3} - \frac{2}{a}(a + 0.005)\right] - a^{\frac{1}{3}}\left(\frac{4}{3} - \frac{2}{a}a\right)\right. \\ &\qquad \left. + \frac{2}{a}0.005\left(\sqrt[3]{0.005} + \sqrt[3]{a_1}\right)\right\} \\ &= \frac{2}{a^2}\left[(a + 0.005)^{\frac{1}{3}}\left(-\frac{2}{3} - \frac{2 \cdot 0.005}{a}\right) - a^{\frac{1}{3}}\left(-\frac{2}{3}\right)\right] \\ &\quad + \frac{4}{a^2\pi}\left[(a + 0.005)^{\frac{1}{3}}\left(-\frac{2}{3} - \frac{2 \cdot 0.005}{a}\right) - a^{\frac{1}{3}}\left(-\frac{2}{3}\right)\right. \\ &\qquad \left. + \frac{2 \cdot 0.005}{a}\left(\sqrt[3]{0.005} + \sqrt[3]{a_1}\right)\right] \end{aligned}$$

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$$
\begin{aligned}
&= -\frac{4}{3a^2}\left[(a+0.005)^{\frac{1}{3}} - a^{\frac{1}{3}}\right] - \frac{0.02}{a^3}(a+0.005)^{\frac{1}{3}} \\
&\quad - \frac{8}{3a^2\pi}\left[(a+0.005)^{\frac{1}{3}} - a^{\frac{1}{3}}\right] - \frac{0.04}{a^3\pi}\left[(a+0.005)^{\frac{1}{3}} - \sqrt[3]{0.005} - \sqrt[3]{a_1}\right] \\
&< -\frac{0.02}{a^3}(a+0.005)^{\frac{1}{3}} - \frac{0.04}{a^3\pi}\left[(a+0.005)^{\frac{1}{3}} - \sqrt[3]{0.005} - \sqrt[3]{a_1}\right] \\
&= -\frac{0.04}{a^3\pi}\left[\frac{\pi}{2}(a+0.005)^{\frac{1}{3}} + (a+0.005)^{\frac{1}{3}} - \sqrt[3]{0.005} - \sqrt[3]{a_1}\right] \\
&\le -\frac{0.04}{a^3\pi}\left[\frac{\pi}{2}(a_1+0.005)^{\frac{1}{3}} + (a_1+0.005)^{\frac{1}{3}} - \sqrt[3]{0.005} - \sqrt[3]{a_1}\right] \\
&\approx -\frac{0.04}{a^3\pi}(1.057 + 0.673 - 0.17 - 0.669) < 0.
\end{aligned}
$$

Hence $h(a)$ is a decreasing function on $[a_1, a_2]$. Thus, when $a \in [a_1, a_2]$ and $\omega \in [\omega_1, \omega_2]$,

$$\sigma_1(\omega, a) \le h(a_1) + 0.044 = 0.1472$$

by (4.8) and (4.9). Next,

$$
\begin{aligned}
f_1(\omega_1, a) &\ge |\operatorname{Im}[G_1(i\omega_1)^{-1} + N_1(a)]| = |4\omega_1 - \omega_1^3| = 0.157608 \\
f_1(\omega_2, a) &\ge |\operatorname{Im}[G_1(i\omega_2)^{-1} + N_1(a)]| = |4\omega_2 - \omega_2^3| = 0.162408.
\end{aligned}
$$

Hence,

$$f_1(\omega_k, a) > 0.1472 \ge \sigma_1(\omega_k, a) \quad \text{for } k = 1, 2,\ a_1 \le a \le a_2.$$

To verify $A_4(7)$, we note that

$$
\begin{aligned}
&\left|\frac{\omega}{\pi a}\int_0^{\frac{2\pi}{\omega}} e^{-i\omega t}\left\{f(u_2 a\cos\omega t) - f(u_2 a \cos\omega t + V_2(t))\right\} dt\right| d_1(\omega) + d_1(\omega) \\
&\qquad \le \left(\frac{4}{a} + 1\right) d_1(\omega) \le \left(\frac{4}{a_1} + 1\right) d_1(\omega) \approx 14.333 \cdot d_1(\omega).
\end{aligned}
$$

This means that $\eta_2(\omega, a) \le 14.333 \cdot d_1(\omega)$ and hence

$$\sigma_2(\omega, a) \le 14.333\frac{d_1(\omega)}{d_2(\omega)} < 14.333.$$

But

$$f_2(\omega, a) = \left|G_2(i\omega)^{-1} + \frac{4}{\pi a}\right| \ge \left|\operatorname{Im}\left(G_2(i\omega)^{-1} + \frac{4}{\pi a}\right)\right|$$

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$$> |100\omega^3 + 400\omega| > 14.333,$$

for $\omega_1 \leq \omega \leq \omega_2$. Thus, $f_2(\omega, a) > \sigma_2(\omega, a)$. So, $A_4(7)$ is satisfied. Assumptions A_1, A_2, A_3 and $A_4(1)$, (2), (6) and (8) are satisfied evidently. Therefore, all of A_1 through A_4 are satisfied. Hence, by Theorem 3.1, there is a nontrivial solution, $(x_1, x_2) \in H_2(\omega)$ with $\omega_1 \leq \omega \leq \omega_2$, of the system (4.1).

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