SOME STUDIES ON KAEHLERIAN HOMOGENEOUS SPACES

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The present paper is devoted to the study of differential geometry of Kaehlerian homogeneous spaces. In section 1 we deal with the canonical decomposition of a simply connected complete Kaehlerian space and that of its largest connected group of automorphisms. We know that a simply connected complete Riemannian space V is the product of Riemannian spaces $V_0, V_1, \ldots, V_n$, where V_0 is a Euclidean space and $V_1, \ldots, V_n$ are not locally flat and their homogeneous holonomy groups are irreducible [2]. Moreover, if V is homogeneous, so are all V_k [10]. We shall show that if V is Kaehlerian space with real analytic metric (resp. Kaehlerian homogeneous space), each factor V_k is also Kaehlerian (resp. Kaehlerian homogeneous) and that V is the product of $V_0, V_1, \ldots, V_n$ as Kaehlerian space. We call this decomposition the de Rham decomposition of the Kaehlerian space V. Although this result is supposedly known, there is no published proof as yet. Using this decomposition theorem we shall show that the largest connected group of automorphisms of a simply connected complete Kaehlerian space with real analytic metric is the direct product of those of the factors of the de Rham decomposition. In the Riemannian case this result has be been established in [3] by one of the authors of the present paper.

On the other hand, a Kaehlerian homogeneous space G/B of a reductive Lie group G is the direct product of Kaehlerian homogeneous spaces $W_0, W_1, \ldots, W_m$, where W_0 is the center of G with an invariant Kaehlerian structure and where $W_1, W_2, \ldots, W_m$ are simply connected Kaehlerian homogeneous spaces of simple Lie groups ([1], [7], [8]). In section 2 we shall show that this decomposition of G/B is equal to the de Rham decomposition of G/B. We shall prove in fact a theorem that the homogeneous holonomy group of a Kaehlerian homogeneous space of a simple Lie group is irreducible. To prove

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this theorem we shall use, in addition to the results established in section 1, a lemma on the root system of a complex simple Lie algebra which is of some interest for itself. The arguments used in the proof of this theorem give a proof of a theorem that the restricted homogeneous holonomy group of a Riemannian homogeneous space of a compact simple Lie group with non vanishing Euler characteristic is irreducible.

Let G/B be a reductive homogeneous space of a Lie group G [9]. There exists a decomposition of the Lie algebra $\mathfrak{g}$ of G into a direct sum of two subspaces $\mathfrak{m}$ and $\mathfrak{b}$, $\mathfrak{b}$ being the Lie algebra of B, such that $\mathrm{ad}(x)\cdot\mathfrak{m}=\mathfrak{m}$ for all $x \in \mathrm{B}$. The notion of the canonical affine connection of the first kind with respect to such a decomposition of $\mathfrak{g}$ has been defined in [9]. In section 3 we shall first remark that if G/B is a Kaehlerian homogeneous space of a semi-simple Lie group G, then the decomposition $\mathfrak{g}=\mathfrak{m}+\mathfrak{b}$ having the above mentioned properties is unique. Therefore we can speak of the canonical affine connection of the first kind of G/B. We shall then prove that G/B is hermitian symmetric if the Riemannian connection induced by the invariant Kaehlerian metric is the canonical affine connection of the first kind.[1)] If G is a reductive Lie group with non-discrete center the decomposition $\mathfrak{g}=\mathfrak{m}+\mathfrak{b}$ such that $\mathrm{ad}(x)\cdot\mathfrak{m}=\mathfrak{m}$ for all $x \in B$ is not unique, but we can prove, also in this case, a theorem analogous to the one mentioned above.

1. Let V be a simply connected complete Riemannian space of class C^∞ and let p_0 be a point of V. We denote by S_{p_0} the homogeneous holonomy group of V at the point p_0. The tangent space $T(p_0)$ of V at the point p_0 decomposes into the direct sum of mutually orthogonal subspaces $T_0(p_0)$, $T_1(p_0)$, , . . . , $T_n(p_0)$, where $T_0(p_0)$ is the subspace of all vectors fixed by the operations of the elements of S_{p_0} and $T_1(p_0)$, $T_2(p_0)$, . . . , $T_n(p_0)$ are irreducible S_{p_0}-stable subspaces. By the parallel displacement of $T_0(p_0)$, $T_1(p_0)$, . . . , $T_n(p_0)$ we can define the completely integrable distributions T_0, T_1, . . . , T_n on V. We denote by V_0, V_1, . . . , V_n the maximal integral manifolds of the distributions T_0, T_1, . . . , T_n respectively passing through the point p_0. With respect to the induced Rie-

1) It should be noted here that Nomizu has anounced in a C. R. note a theorem that the restricted homogeneous holonomy group of a Riemannian homogeneous space G/B of a simple Lie group G is irreducible. But he has told us that his arguments are exact only in the case where the invariant Riemannian connection on G/B is canonical of the first kind with respect to a certain decomposition $\mathfrak{g}=\mathfrak{m}+\mathfrak{b}$. See also [5].

mannian metric the homogeneous holonomy group of V_k at the point p_0 is equal to the representation of S_{p_0} in $T_k(p_0)$ and hence it is irreducible for $k>0$ and equal to the identity group for $k=0$. It has been proved by de Rham [2] that there exists an isometry f of the Riemannian space V onto the Riemannian space $V_0 \times V_1 \times \ldots \times V_n$. Moreover, if V is real analytic, that is, if the underlying manifold and the Riemannian metric of V are real analytic, so are the $V_0, V_1, \ldots, V_n$ and the isometry f.

Suppose now that V is a Kaehlerian space with real analytic metric. Then V is a real analytic Riemannian space. Let I be the tensor field of tye $(1,1)$ defining the underlying complex structure of V. Now let I' be the tensor field of type $(1,1)$ on $V_0 \times V_1 \times \ldots \times V_n$ such that $I'(f(X)) = f(I(X))$ for all vector field X on V. Since the tensor field I and the mapping f is real analytic, so is I' and it defines an almost complex structure on $V_0 \times V_1 \times \ldots \times V_n$. Since I is integrable, so is I' and since $V_0 \times V_1 \times \ldots \times V_n$ is real analytic, I' defines a complex structure on $V_0 \times V_1 \times \ldots \times V_n$. It follows from the definition of I' that the mapping f is complex analytic. Moreover, since f is an isometry, we can see that the Riemannian metric $V_0 \times V_1 \times \ldots \times V_n$ is Kaehlerian. Thus we have shown that $V_0 \times V_1 \times \ldots \times V_n$ is Kaehlerian and f is an isomorphism of V onto $V_0 \times V_1 \times \ldots \times V_n$ with respect to the Kaehlerian structure. Now the tangent space $T'(q)$ of $V_0 \times V_1 \times \ldots \times V_n$ at the point $q = (q_0, q_1, \ldots, q_n)$ may be identified with the product $T_0(q_0) \times \ldots \times T_n(q_n)$, where $T_k(q_k)$ denotes the tangent space of V_k at the point q_k. We denote by $T'_k(q)$ the subspace of $T'(q)$ composed of all the vectors of the form $(0, \ldots, X_k, 0, \ldots, 0)$, where $X_k \in T_k(q_k)$. Then $T'_0(q), T'_1(q), \ldots, T'_n(q)$ are mutually orthogonal and $T'(q) = \sum_k T'_k(q)$. The assignment $q \to T'_k(q)$ defines a completely integrable analytic distribution T'_k. Let S'_q be the homogenous holonomy group of $V_0 \times V_1 \times \ldots \times V_n$ at the point q. Then $T'_0(q)$ is the subspace of all the vectors fixed by the operations of the elements of S'_q and $T'_k(q)$ for $k>0$ is an irreducible S'_q-stable subspace. Since $V_0 \times V_1 \times \ldots \times V_n$ is Kaehlerian, the covariant derivatives of the tensor field I' are zero. Therefore the value I'_q of I' at the point q, which is an orthogonal transformation of $T'(q)$, commutes with the elements of the homogeneous holonomy group S'_q. We shall show that the distributions T'_k are invariant by I', that is, $I'_q \cdot T'_k(q) = T'_k(q)$ for all point q. This is clear for $k=0$, since $T'_0(q)$ is the subspace of all vectors fixed by

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the operations of the elements of S'_q. As I'_q commutes with the elements of S'_q, so does $\exp t\cdot I'_q$ and hence $(\exp t\cdot I'_q)T'_k(q)$ is an irreducible S'_q-stable subspace of $T'(q)$ for eaeh $k=1, 2, \dots, n$. Since an irreducible S'_q-stable subspace of $T'(q)$ is equal either to a 1-dimensional subspace of $T'_0(q)$ or to some $T'_i(q)$ with $i>0$ (see [2]) and since $(\exp t\cdot I'_q)\cdot T'_k(q)$ can not be a subspace of $T'_0(q)$, it coincides with some $T_i(q)$. Now we show that $(\exp t\cdot I'_q)\cdot T'_k(q) = T'_k(q)$ for sufficiently small t. Let X be an element $\neq 0$ of $T'_k(q)$ and consider the inner product $(X, (\exp t\cdot I'_q)\cdot X)$. Since $(X, (\exp t\cdot I'_q)\cdot X)$ is continuous in t and is not zero for $t=0$, it is not zero for sufficiently small t. For such t, $(\exp t\cdot I'_q)X$ can not belong to $T'_i(q)$ $(i\neq k)$, since $T'_i(q)$ is orthogonal to $T'_k(q)$. Hence $(\exp t\cdot I'_q)\cdot T'_k(q) = T'_k(q)$ for sufficiently small t. Then $I'_q\cdot X = \lim_{t\to 0}\frac{1}{t}(\exp t\cdot I'_q - 1)\cdot X \in T'_k(q)$ for all $X\in T'_k(q)$. Thus we have seen that $I'_q\cdot T'_k(q) = T'_k(q)$ for $k=1, 2, \dots, n$. Then I' defines a tensor field $I'^{(k)}$ of type $(1,1)$ on the integral manifold V'_k of the distribution T'_k passing through the point $(p_0, p_0, \dots, p_0)\in V_0\times V_1\times \dots \times V_n$ such that $I'_q\cdot X = I'^{(k)}_q\cdot X$ for all $X\in T'_k(q)$ and $q\in V'_k$. We can see that $I'^{(k)}$ defines a complex structure on V'_k. Let α_k be the real analytic homeomorphism of V_k onto V'_k such that $\alpha_k(x) = (p_0, \dots, p_0, x, p_0, \dots, p_0)$ and let $I^{(k)}$ be a tensor field of type $(1,1)$ on V_k such that $\alpha_k(I^{(k)}(X)) = I'^{(k)}(\alpha_k(X))$ for all vector field X on V_k. $I^{(k)}$ defines a complex structure on V_k. For each tangent vector $X=(X_0, X_1, \dots, X_n)$ of $V_0\times V_1\times \dots \times V_n$ at the point $q=(q_0, q_1, \dots, q_n)$, we have $I'_q\cdot X = (I^{(0)}_{q_0}\cdot X_0, I^{(1)}_{q_1}\cdot X_1, \dots, I^{(n)}_{q_n}\cdot X_n)$. It follows that V_k is Kaehlerian and that the Kaehlerian structure of V is equal to the one which can be defined naturally by the Kaehlerian structures of the factors V_k. Thus we have proved the following

THEOREM 1. *A simply connected complete Kaehlerian space with real analytic metric V is the product of the Kaehlerian spaces $V_0, V_1, \dots, V_n$, where V_0 is a unitary space and $V_1, \dots, V_n$ are such that the homogeneous holonomy groups are irreducible.*

We call this decomposition the de Rham decomposition of the Kaehlerian space V.

Let V be a Kaehlerian space and let $I(V)$ and $K(V)$ be the group of isometries and the group of automorphisms of V respectively. $I(V)$ is a Lie group with respect to the compact-open topology. An element $g\in I(V)$ belongs to

$K(V)$ if and only if g is complex analytic. It is easily verified that $K(V)$ is a closed subgroup of $I(V)$. Therefore $K(V)$ is a Lie group.

THEOREM 2. *Let V be a simply connected complete Kaehlerian space with real analytic metric and let $V = V_0 \times V_1 \times \ldots \times V_n$ be the de Rhan decomposition of V. Let $K_0(V)$ and $K_0(V_k)$ be the largest connected groups of automorphisms of V and V_k respectively. Then $K_0(V) \cong K_0(V_0) \times K_0(V_1) \times \ldots \times K_0(V_n)$.*

Let $I_0(V_0)$ and $I_0(V_k)$ be the largest connected groups of isometries of V and V_k. Then there exists an isomorphism $g \to (g_0, g_1, \ldots, g_n)$ of $I_0(V)$ onto $I_0(V_0) \times I_0(V_1) \times \ldots \times I_0(V_n)$ such that $g(p_0, p_1, \ldots, p_n) = (g_0(p_0), g_1(p_1), \ldots, g_n(p_n))$ for all point $(p_0, p_1, \ldots, p_n) \in V$ [3]. Let $g \in K_0(V)$. Then $g \in I_0(V)$ and g is complex analytic. Let I and $I^{(k)}$ be the tensor fields defining the complex structures of V and V_k and let $X = (X_0, X_1, \ldots, X_n)$ be the tangent vector of V at the point $p = (p_0, p_1, \ldots, p_n)$, where X_k is a tangent vector of V_k at the point p_k. Then $I_p \cdot X = (I^{(0)}_{p_0} \cdot X_0, I^{(1)}_{p_1} \cdot X_1, \ldots, I^{(n)}_{p_n} \cdot X_n)$ and since g is complex analytic, we have $g(I_p \cdot X) = I_{g(p)} \cdot g(X)$. As $g(I_p \cdot X) = (g_0(I^{(0)}_{p_0} \cdot X_0), \ldots, g_n(I^{(n)}_{p_n} \cdot X_n))$ and $I_{g(p)} \cdot g(X) = (I^{(0)}_{g_0(p_0)}(g_0(X_0)), \ldots, I^{(n)}_{g_n(p_n)}(g_n(X_n)))$, we get $g_k(I^{(k)}_{p_k} X_k) = I^{(k)}_{g_k(p_k)}(g_k(X_k))$, which shows that g_k is complex analytic. It follows that $g_k \in K_0(V_k)$. Conversely, if $g_k \in K_0(V_k)$, then the transformation g of V defined by $g(p_0, \ldots, p_n) = (g_0(p_0), \ldots, g_n(p_n))$ belongs to $K_0(V)$. Theorem 2 is thus proved.

In the following we call the element g_k of $K_0(V_k)$ associated to the element $g \in K_0(V)$ by the isomorphism $K_0(V) \cong K_0(V_0) \times \ldots \times K_0(V_n)$ the $K(V_k)$-component of g.

A Kaehlerian space V is called Kaehlerian homogeneous if the group $K_0(V)$ is transitive on V. In this case the Kaehlerian metric of V is real analytic.

THEOREM 3. *Let V be a simply connected Kaehlerian homogeneous space. Then each factor of the de Rham decomposition of V is also Kaehlerian homogeneous.*

Since V is homogeneous, V is complete and $K_0(V)$ is transitive on V. Let $V = V_0 \times V_1 \times \ldots \times V_n$ be the de Rham decomposition of V. Let p_k and q_k be two points of V_k and let p and q points of V whose V_k-componets are equal

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to p_k and q_k respectively. There exists an element $g \in K_0(V)$ such that $g(p) = q$. Let g_k be the $K_0(V_k)$-component of g. Then we have $g_k(p_k) = q_k$ and this shows that $K_0(V_k)$ is transitive on V_k and hence V_k is Kaehlerian homogeneous.

2. A connected Lie group G is called reductive if its Lie algebra $\mathfrak{g}$ is the direct sum of the center $\mathfrak{c}$ and a semi-simple ideal $\mathfrak{s}$ which is equal to the derived algebra of $\mathfrak{g}$. The subgroup C of G corresponding to $\mathfrak{c}$ is the connected center of G and the invariant subgroup S of G corresponding to the ideal $\mathfrak{s}$ will be called the semi-simple part of G. Now let G/B be a Kaehlerian homogeneous space of a reductive Lie group G and let G be effective on G/B. We know the following facts (see [1] and [8]).

1) B is compact connected and contained in S and equal to the centraliser in S of a toral subgroup of S. B contains a Cartan subgroup of S.

2) The center of S is equal to (e) and $G = C \times S$ and hence C is equal to the center of G.

3) C and S/B are Kaehlerian homogeneous and $C/B = C \times (S/B)$ as Kaehlerian space. Moreover S/B is simply connected.

4) Let $S = S_1 \times \ldots \times S_m$ be the decomposition of S into the direct product of simple Lie groups. Then S_k/B_k with $B_k = S_k \cap B$ is a simply connected Kaehlerian homogeneous space and $S/B = S_1/B_1 \times \ldots \times S_m/B_m$ as Kaehlerian space.

It follows from 3) and 4) that $G/B = C \times S_1/B_1 \times \ldots \times S_m/B_m$. Since C is an abelian Lie group with an invariant Kaehlerian structure, it is locally flat. To see that the above decomposition of G/B is equal to the de Rham decomposition, it is sufficient to prove the following theorem.

THEOREM 4. *Let G/B be a Kaehlerian homogeneous space of a simple Lie group G and let G be effective on G/B. Then the homogeneous holonomy group of G/B is irreducible.*

To prove this theorem we first prove the following

LEMMA 1. *Let $\mathfrak{g}$ be a simple non-abelian Lie algebra over the field of all complex numbers and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let Σ be the set of all non-zero roots of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$ and let Σ_1 and Σ_2 be two subsets of Σ satisfying the following conditions,*

1) *If* $\alpha \in \Sigma_k$, *then* $-\alpha \in \Sigma_k$ $(k=1, 2)$.

2) *If* $\alpha, \beta \in \Sigma_k$ *and* $\alpha+\beta \in \Sigma$, *then* $\alpha+\beta \in \Sigma_k$ $(k=1, 2)$.

If the union of Σ_1 *and* Σ_2 *is equal to* Σ, *one of* Σ_1 *and* Σ_2 *coincides with* Σ.

Let $\Sigma_2 \neq \Sigma$ and we shall prove that $\Sigma_1 = \Sigma$. For each $\alpha \in \Sigma$ we denote by E_α an element of $\mathfrak{g}$ such that $[H, E_\alpha] = \alpha(H) \cdot E_\alpha$ for all $H \in \mathfrak{h}$. Let $\mathfrak{u}$ be the subspace of $\mathfrak{g}$ spanned by E_α with $\alpha \in \Sigma_1 - \Sigma_2$. Since $\Sigma_2 \neq \Sigma$ and $\Sigma = \Sigma_1 \cup \Sigma_2$, $\Sigma_1 - \Sigma_2$ is not empty and therefore $\mathfrak{u} \neq (0)$. We show that

$$(1) \qquad [E_\alpha, \mathfrak{u}] = (0) \text{ for all } \alpha \in \Sigma_2 - \Sigma_1.$$

Let $\alpha \in \Sigma_2 - \Sigma_1$ and $\beta \in \Sigma_1 - \Sigma_2$ and suppose that $[E_\alpha, E_\beta] \neq 0$. Then $\gamma = \alpha + \beta$ is a non-zero root. As $\Sigma = \Sigma_1 \cup \Sigma_2$, γ is contained in Σ_1 or in Σ_2. If $\gamma \in \Sigma_1$, we have $\alpha = \gamma - \beta$ and since $-\beta \in \Sigma_1$, we have $\alpha \in \Sigma_1$ contrary to the hypothesis. If $\gamma \in \Sigma_2$, we have $\beta = \gamma - \alpha \in \Sigma_2$ and this is also a contradiction. Therefore we must have $[E_\alpha, E_\beta] = 0$ and hence $[E_\alpha, \mathfrak{u}] = (0)$.

Next we show that

$$(2) \qquad [E_\alpha, \mathfrak{u}] \subset \mathfrak{u} \text{ for all } \alpha \in \Sigma_1 \cap \Sigma_2.$$

Let $\alpha \in \Sigma_1 \cap \Sigma_2$ and $\beta \in \Sigma_1 - \Sigma_2$ and suppose that $[E_\alpha, E_\beta] \neq 0$. Then $\gamma = \alpha + \beta$ is a non-zero root and since $\alpha, \beta \in \Sigma_1$, we have $\gamma \in \Sigma_1$. If $\gamma \in \Sigma_2$, we have $\beta = \gamma - \alpha \in \Sigma_2$ contrary to the hypothesis. Therefore $\gamma \in \Sigma_1 - \Sigma_2$ and hence $[E_\alpha, E_\beta] \in \mathfrak{u}$.

Now let $\mathfrak{g}'$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{u}$. It follows from (1) and (2) and from the fact $[\mathfrak{h}, \mathfrak{u}] \subset \mathfrak{u}$ that $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$. Since $\mathfrak{u} \neq (0)$, we have $\mathfrak{g}' \neq (0)$ and as $\mathfrak{g}$ is simple, $\mathfrak{g}'$ is equal to $\mathfrak{g}$. On the other hand, let $\mathfrak{g}_1$ be the subspace of $\mathfrak{g}$ spanned by $\mathfrak{h}$ and E_α with $\alpha \in \Sigma_1$. Then $\mathfrak{g}_1$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{u}$ and therefore $\mathfrak{g}_1 \supset \mathfrak{g}'$. Thus we have $\mathfrak{g} = \mathfrak{g}_1$ and this implies that $\Sigma_1 = \Sigma$. Lemma 1 is thus proved.

Proof of Theorem 4. Let G/B be a Kaehlerian homogeneous space of a simple Lie group G and let G be effective on G/B. Then G/B is simply connected and complete. Put $V = G/B$ and let $V = V_0 \times V_1 \times \ldots \times V_n$ be the de Rham decomposition of V, where $n \geqq 0$ and $\dim V_0 \geqq 0$. We show first that $n \geqq 1$. Indeed, if $V = V_0$ then V would be a unitary space and G would be a subgroup of the group K of automorphisms of the unitary space V. Let d be the complex dimension of V and let α be the natural homomorphism of K onto the unitary group $U(d)$ (= the group of rotations of V around a fixed point). The restriction of α on G is a representation of G. As G is simple and has center reduced to (e), the kernel of this representation is equal either to G itself or to (e). The former case does not occur, because in this case G must be abelian. In the latter case, the image G' of G in $U(d)$ is a simple Lie subgroup of $U(d)$ and since every semi-simple Lie subgroup of $U(d)$ is closed in $U(d)$, G' is compact. Therefore G is also compact and so is V. This is a contradiction and hence $V \neq V_0$ and $n \geqq 1$. Now we propose to show that $\dim V_0 = 0$ and $n = 1$. For this purpose, put $W_1 = V_0 \times V_1 \times \ldots \times V_{n-1}$ and $W_2 = V_n$. By Theorem 2 there exists an isomorphism φ of $K_0(V)$ onto $K_0(W_1) \times K_0(W_2)$. Let $\varphi(g) = (\varphi_1(g), \varphi_2(g))$ for $g \in K_0(V)$. Then φ_k is a homomorphism of $K_0(V)$ onto $K_0(W_k)$. Since G is transitive on V, the image $\varphi_k(G)$ of G in $K_0(W_k)$ is transitive on W_k. Moreover, since G is simple with center reduced to (e), the homomorphism of G onto $\varphi_k(G)$ is an isomorphism. Let o be the image in $V = G/B$ of the identity e of G and let $o = (o_1, o_2)$ with $o_k \in W_k$. Let G_1 (resp. G_2) be the subgroup of G composed of all the elements $g \in G$ such that $\varphi_2(g) \cdot o_2 = o_2$ (resp. $\varphi_1(g) \cdot o_1 = o_1$). Then $\varphi_1(G_2)$ is the subgroup of $\varphi_1(G)$ of all elements which leave fixed the point o_1. Since W_1 is a Kaehlerian homogeneous space of the simple Lie group $\varphi_1(G)$ and since $\varphi_1(G_2)$ is its isotropy group, $\varphi_1(G_2)$ is compact and connected. It follows that G_2 is compact and connected. In an analogous way we can show that G_1 is compact and connected. Now let W_1' (resp. W_2') be the submanifold of V composed of all the points of the form (p_1, o_2) (resp. (o_1, p_2)). Let $g \in G_1$ and let $(p_1, o_2) \in W_1'$. Then $g \cdot (p_1, o_2) = (\varphi_1(g) \cdot p_1, \varphi_2(g) \cdot o_2) = (\varphi_1(g) \cdot p_1, o_2)$, because $\varphi_2(g) \cdot o_2 = o_2$ by the definition of G_1. Hence G_1 leaves invariant the submanifold W_1'. Now let (p_1, o_2) and (q_1, o_2) be two points of W_1'. Since G is transitive on V, there exists an element $g \in G$ such that $g \cdot (p_1, o_2) = (q_1, o_2)$. Since $g \cdot (p_1, o_2) = (\varphi_1(g) \cdot p_1, \varphi_2(g) \cdot o_2) = (q_1, o_2)$, we get $\varphi_2(g) \cdot o_2 = o_2$ and hence $g \in G_1$. Therefore G_1 is transitive on W_1'. Since G_1 is compact, W_1' is compact. Moreover the isotropy group at the point $o \in W_1'$ is equal to B. In the same way we can show that G_2 is transitive on W_2' and the isotropy subgroup at the point $o \in W_2'$ is eqal to B. Since G_2 is compact, W_2' is compact. Now W_k being homeomorphic to W_k', W_k is also compact and so is $V = W_1 \times W_2$. Since $V = G/B$ and B are compact, G is compact.

We denote by small German letters the Lie algebras of the Lie groups denoted by the corresponding capital Latin letters. The tangent space of V (resp. W'_k) at the point o may be indentified with the vector space $\mathfrak{g}/\mathfrak{b}$ (resp. $\mathfrak{g}_k/\mathfrak{b}$). Since the tangent space of V at the point o is the direct sum of those of W'_1 and W'_2, we have $\mathfrak{g}/\mathfrak{b} = \mathfrak{g}_1/\mathfrak{b} + \mathfrak{g}_2/\mathfrak{b}$. It follows that $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$. Let $\mathfrak{g}^c$ be the complexificatin of $\mathfrak{g}$. Since G is compact and simple, $\mathfrak{g}^c$ is a complex simple Lie algebra. We denote by $\mathfrak{n}^c$ the complex subspace of $\mathfrak{g}^c$ spanned over C by a subspace $\mathfrak{n}$ of $\mathfrak{g}$. If $\mathfrak{n}$ is a subalgebra of $\mathfrak{g}$, $\mathfrak{n}^c$ is a complex subalgebra of $\mathfrak{g}^c$. This being said, let H be a Cartan subgroup of G contained in B. H is a maximal toral subgroup of G and $\mathfrak{h}^c$ is a Cartan subalgebra of $\mathfrak{g}^c$. Let Σ be the system of the non-zero roots of $\mathfrak{g}^c$ with respect to the Cartan subalgebra $\mathfrak{h}^c$. Now, since $G_k \supset H$, we have $\mathfrak{g}_k^c \supset \mathfrak{h}^c$ and hence $\mathfrak{g}_k^c$ is spanned by $\mathfrak{h}^c$ and those E_α such that $E_\alpha \in \mathfrak{g}_k^c$, where E_α, $\alpha \in \Sigma$, denotes an element of $\mathfrak{g}^c$ such that $[H, E_\alpha] = \alpha(H) \cdot E_\alpha$ for all $H \in \mathfrak{h}^c$. Let Σ_k be the subset of Σ of all α such that $E_\alpha \in \mathfrak{g}_k^c$. If α, $\beta \in \Sigma_k$ and if $\alpha + \beta \in \Sigma$, then E_α, $E_\beta \in \mathfrak{g}_k^c$, and since $\mathfrak{g}_k^c$ is a subalgebra, we have $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \in \mathfrak{g}_k^c$ with $N_{\alpha,\beta} \neq 0$. Therefore $\alpha + \beta \in \Sigma_k$. Since G_k is compact, $\mathfrak{g}_k^c$ is reductive. It follows from this that if $\alpha \in \Sigma_k$, then $-\alpha \in \Sigma_k$. Moreover, since $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$, we have $\mathfrak{g}^c = \mathfrak{g}_1^c + \mathfrak{g}_2^c$ and hence $\Sigma = \Sigma_1 \cup \Sigma_2$. By Lemma 1, we have either $\Sigma = \Sigma_1$ or $\Sigma = \Sigma_2$. If $\Sigma = \Sigma_1$, then $\mathfrak{g}^c = \mathfrak{g}_1^c$ and hence $\mathfrak{g} = \mathfrak{g}_1$. Then $G = G_1$ and we have $W'_1 = V$. It follows that $\dim W'_2 = \dim W_2 = \dim V_n = 0$ and this is a contradiction. Therefore $\Sigma = \Sigma_2$ and it follows that $\dim W_1 = \dim (V_0 \times V_1 \times \ . \ . \ . \ \times V_{n-1}) = 0$. Hence $\dim V_0 = 0$ and $n = 1$. Thus $V = V_1$ and V is irreducible. Theorem 4 is thus proved.

Incidentally the arguments in the proof of Theorem 4 give a proof of the following theorem.

THEOREM 5. *Let G/B be a Riemannian homogeneous space of a compact simple Lie group G with non-vanishing Euler characteristic. Then the restricted homogeneous holonomy group of G/B is irreducible.*

Since the Euler characteristic of G/B does not vanish, B contains a maximal toral subgroup of G [4]. Let $\tilde{G}$ be the universal covering group of G and let $\tilde{B}$ be the subgroup of G corresponding to the subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. Then $\tilde{G}$ is compact simple and $\tilde{B}$ is a closed subgroup of G containing a maximal toral subgroup of $\tilde{G}$. Since $\tilde{G}/\tilde{B}$ is the universal covering space of G/B, the in-

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variant Riemannian metric on G/B defines an invariant Riemannian metric on $\tilde{G}/\tilde{B}$ such that the homogeneous holonomy group of $\tilde{G}/\tilde{B}$ can be identified with the restricted homogeneous holonomy group of G/B. By the arguments used in the proof of Theorem 4 we can see that the homogeneous holonomy group of $\tilde{G}/\tilde{B}$ is irreducible.

3. In this section we shall use the following notations. The small German letters denote the Lie algebras of the Lie groups denoted by the corresponding capital Latin letters. $\mathfrak{g}^c$ denotes the complexification of a real Lie algebra $\mathfrak{g}$ and $\mathfrak{n}^c$ denotes the complex subspace of $\mathfrak{g}^c$ spanned by a subspace $\mathfrak{n}$ of $\mathfrak{g}$. If $\mathfrak{n}$ is a subalgebra of $\mathfrak{g}$, then $\mathfrak{n}^c$ is a complex subalgebra of $\mathfrak{g}^c$.

Now let G/B be a reductive homogeneous space of a connected Lie group G [9]. There exists a demposition of $\mathfrak{g}$ into the direct sum $\mathfrak{g} = \mathfrak{m} + \mathfrak{b}$ such that $\mathrm{ad}(x) \cdot \mathfrak{m} = \mathfrak{m}$ for all $x \in B$. Such a decomposition of $\mathfrak{g}$ will be called in the following a B-invariant decomposition of $\mathfrak{g}$. To each invariant affine connection on G/B there is associated a bilinear function α on $\mathfrak{m} \times \mathfrak{m}$ with values in $\mathfrak{m}$ such that $\mathrm{ad}(x) \cdot \alpha(X, Y) = \alpha(\mathrm{ad}(x) \cdot X, \mathrm{ad}(x) \cdot Y)$ for all $x \in B$ and $X, Y \in \mathfrak{m}$ ([9], Theorem 8.1). We shall call α the connection function of the invariant affine connection with respect to the B-invariant decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{b}$. We know that there exists one and only one invariant affine connection on G/B satisfying the following conditions:

1) The torsion is zero.

2) The images in G/B of the one-parameter subgroups generated by the elements of $\mathfrak{m}$ are the paths ([9], Theorem 10.1).

The invaraint affine connection of G/B satisfying these two conditions will be called the canonical affine connection of the first kind on G/B with respect to the B-invariant decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{b}$. Its connection function is given by

$$\alpha(X, Y) = \frac{1}{2}[X, Y]_{\mathfrak{m}}$$

for all $X, Y \in \mathfrak{m}$, where $[X, Y]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of the element $[X, Y] \in \mathfrak{g} = \mathfrak{m} + \mathfrak{b}$.

We denote in the following by o the image in G/B of the identity e of G. We can identify the vector space $\mathfrak{m}$ with the tangent space of G/B at the point o. Suppose now that there is defined an invariant complex structure on G/B and let I be the tensor field of type $(1, 1)$ defining this complex structure, Then the value I_0 of I at the point o may be considered as an endomorphism of the vector space $\mathfrak{m}$. I_0 satisfies the following condition [6]:

(1) $I_0^2 = -1.$

(2) $\operatorname{ad}(X) \cdot I_0 \cdot Y = I_0 \cdot \operatorname{ad}(X) \cdot Y$ for all $X \in \mathfrak{b}$ and $Y \in \mathfrak{m}$.

(3) $I_0[X, Y]_{\mathfrak{m}} - [I_0 X, Y]_{\mathfrak{m}} - [X, I_0 Y]_{\mathfrak{m}} - I_0[I_0 X, I_0 Y]_{\mathfrak{m}} = 0$

for all $X, Y \in \mathfrak{m}$, where $[X, Y]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of the element $[X, Y] \in \mathfrak{g} = \mathfrak{m} + \mathfrak{b}$,

Now let us consider an invariant affine connection on G/B such that the covariant derivatives (with respect to this affine connection) of the tensor field I are zero and let α be the corresponding connection function with respect to the B-invariant decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{b}$. It follows from the definition of α (see [9], Theorem 8.1) and from the invariance of I that

$$\alpha(X, I_0 \cdot Y) = I_0 \cdot \alpha(X, Y) \tag{4}$$

for all $X, Y \in \mathfrak{m}$.

This being said, we now prove the following

LEMMA 2. *Let G/B be a Kaehlerian homogeneous space of a semi-simple Lie group G and let G be effective on G/B. Then G/B is reductive and the B-invariant decomposition of $\mathfrak{g}$ is unique, i.e. if $\mathfrak{g} = \mathfrak{m} + \mathfrak{b}$ and $\mathfrak{g} = \mathfrak{m}' + \mathfrak{b}$ are two B-invariant decompositions of $\mathfrak{g}$, then $\mathfrak{m} = \mathfrak{m}'$.*

Since B is compact, G/B is reductive and let $\mathfrak{g} = \mathfrak{m} + \mathfrak{b}$ be a B-invariant decomposition of $\mathfrak{g}$. Since $\operatorname{ad}(x) \cdot \mathfrak{m} = \mathfrak{m}$ for all $x \in B$, $\mathfrak{m}$ is a representation module of the compact connected Lie group B. Let H be a Cartan subgroup of G contained in B. Then H is a maximal total subgroup of B and $\mathfrak{m}^c$ decomposes into the direct sum $\mathfrak{m}^c = \Sigma \mathfrak{m}^c_\lambda$, where λ denotes a linear function on $\mathfrak{h}^c$ and $\mathfrak{m}^c_\lambda$ is the subspace of $\mathfrak{m}^c$ composed of all the elements $Y \in \mathfrak{m}^c$ such that $[X, Y] = \lambda(X) \cdot Y$ for all $X \in \mathfrak{h}^c$. Since $\mathfrak{h}^c$ is a Cartan subalgebra of the complex semi-simple Lie algebra $\mathfrak{g}^c$ and since $\mathfrak{m}^c \subset \mathfrak{g}^c$ and $\mathfrak{m}^c \frown \mathfrak{b}^c = (0)$, λ is a non-zero root of $\mathfrak{g}^c$ with respect to $\mathfrak{h}^c$. For each non-zero root α we denote by E_α an element of $\mathfrak{g}^c$ such that $[X, E_\alpha] = \alpha(X) \cdot E_\alpha$ for all $X \in \mathfrak{h}^c$. Then $\mathfrak{m}^c$ is spanned by those E_λ such that $E_\lambda \notin \mathfrak{b}^c$. The same arguments show that if $\mathfrak{g} = \mathfrak{m}' + \mathfrak{b}$ is an another B-invariant decomposition of $\mathfrak{g}$, $\mathfrak{m}'^c$ is also spanned by these E_λ. Hence we have $\mathfrak{m}^c = \mathfrak{m}'^c$ and since $\mathfrak{m} = \mathfrak{g} \frown \mathfrak{m}^c$ and $\mathfrak{m}' = \mathfrak{g} \frown \mathfrak{m}'^c$, we get $\mathfrak{m} = \mathfrak{m}'$, It follows from this lemma that if G/B is a Kaehlerian homogeneous space of a semi-simple Lie group G, we can speak of the canonical affine connection of the first kind of G/B, because the B-invariant decomposition of $\mathfrak{g}$ is unique.

Now a homogeneous space G/B of a connected Lie group is called hermitian symmetric, if the following conditions are satisfied:

A) G/B admits an invariant complex structure.

B) B is compact and there exists an involutive automorphism σ of G such that: 1) $\sigma(x) = x$ for all $x \in B$, 2) B contains the connected component of the identity of the closed subgroup of G composed of all $x \in G$ such that $\sigma(x) = x$.

Now we prove the following

THEORHM 6. *Let G/B be a Kaehlerian homogeneous space of a semi-simple Lie group G and let G be effective on G/B. If the Riemannian connection on G/B induced by the invariant Kaehlerian metric of G/B is the canonical affine connection of the first kind, then G/B is hermitian symmetric.*

Let $\mathfrak{g} = \mathfrak{m} + \mathfrak{b}$ be the unique B-invariant decomposition of $\mathfrak{g}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}$. We denote by $\alpha, \beta, \ldots$ the non-zero roots of $\mathfrak{g}^c$ with respect to $\mathfrak{h}^c$ and by E_α an element of $\mathfrak{g}^c$ such that $[X, E_\alpha] = \alpha(X) \cdot E_\alpha$ for all $X \in \mathfrak{h}^c$. Then $\mathfrak{m}^c$ is spanned by those E_α such that $E_\alpha \notin \mathfrak{b}^c$ (cf. the proof of Lemma 2). Let I be the tensor field of type $(1, 1)$ defining the underlying invariant complex structure of G/B. Its value I_0 at the point o is an endomorphism of the vector space $\mathfrak{m}$ satisfying the conditions (1), (2) and (3). We can extend I_0 to a complex endomorphism I_0^c of $\mathfrak{m}^c$. Then the following conditions are satisfied:

(1′) $(I_0^c)^2 = -1.$

(2′) $\mathrm{ad}(X) \cdot I_0 \cdot Y = I_0 \cdot \mathrm{ad}(X) \cdot Y$ for all $X \in \mathfrak{b}^c$ and $Y \in \mathfrak{m}^c$.

(3′) $I_0^c[X, Y]_{\mathfrak{m}^c} - [I_0^c \cdot X, Y]_{\mathfrak{m}^c} - [X, I_0^c \cdot Y]_{\mathfrak{m}^c} - I_0^c[I_0^c \cdot X, I_0^c \cdot Y]_{\mathfrak{m}^c} = 0$

for all $X, Y \in \mathfrak{m}^c$, where $[X, Y]_{\mathfrak{m}^c}$ denotes the $\mathfrak{m}^c$-component of the element $[X, Y] \in \mathfrak{g}^c = \mathfrak{m}^c + \mathfrak{b}^c$.

Now let $E_\alpha \in \mathfrak{m}^c$. Since $\mathfrak{h}^c \subset \mathfrak{b}^c$, it follows from (2′) that $[X, I_0^c \cdot E_\alpha] = \alpha(X) \cdot I_0^c \cdot E_\alpha$ for all $X \in \mathfrak{h}^c$ and hence $I_0^c \cdot E_\alpha = aE_\alpha$, where a is a complex number. Since the eigen-values of I_0^c are $\pm i$ by (1′), we have $a = \pm i$. Thus $I_0^c \cdot E_\alpha = \pm i \cdot E_\alpha$. Let $\mathfrak{D}$ (resp. $\mathfrak{D}'$) be the set of the non-zero roots α such that $E_\alpha \in \mathfrak{m}^c$ and $I_0^c \cdot E_\alpha = i \cdot E_\alpha$ (resp. $I_0^c \cdot E_\alpha = -i \cdot E_\alpha$). We shall show that $\alpha \in \mathfrak{D}$

if and only if $-\alpha \in \mathfrak{D}'$. For this purpose, let $X \to \overline{X}$ be the real endomorphism of the vector space $\mathfrak{g}^c$ such that $\overline{X} = \sum_i \overline{a}_i \cdot X_i$, where $(X_1, \ldots, X_n)$ is a base of $\mathfrak{g}$ and $X = \sum_i a_i \cdot X_i$ with $a_i \in C$. Then $\overline{[X, E_\alpha]} = \overline{\alpha(X)} \cdot \overline{E}_\alpha$ for all $X \in \mathfrak{h}$ and since $\overline{[X, E_\alpha]} = [X, \overline{E}_\alpha]$, we have $[X, \overline{E}_\alpha] = \overline{\alpha(X)} \cdot \overline{E}_\alpha$ for all $X \in \mathfrak{h}$. As H is compact, the eigen-values $\alpha(X)$ of $\mathrm{ad}(X)$ are purely imaginary for all $X \in \mathfrak{h}$. Hence we have $[X, \overline{E}_\alpha] = -\alpha(X) \cdot \overline{E}_\alpha$ for all $X \in \mathfrak{h}$ and hence for all $X \in \mathfrak{h}^c$. It follows that $\overline{E}_\alpha = aE_{-\alpha}$, where a is a suitable complex number. Now, since I_0^c is the extension to $\mathfrak{m}^c$ of the endomorphism I_0 of $\mathfrak{m}$, we have $\overline{I_0^c \cdot X} = I_0^c \cdot \overline{X}$ for all $X \in \mathfrak{m}^c$. Let $\alpha \in \mathfrak{D}$. Then $I_0^c \cdot E_\alpha = i \cdot E_\alpha$ and $\overline{I_0^c \cdot E_\alpha} = I_0^c \cdot \overline{E}_\alpha = -i \cdot \overline{E}_\alpha$. Since $\overline{E}_\alpha = aE_{-\alpha}$, we get $I_0^c \cdot E_{-\alpha} = -i \cdot E_{-\alpha}$ and hence $-\alpha \in \mathfrak{D}'$. In the same way we can show that if $-\alpha \in \mathfrak{D}'$, then $\alpha \in \mathfrak{D}$. Thus we have shown that $\alpha \in \mathfrak{D}$ if and only if $-\alpha \in \mathfrak{D}'$. Using the relation (3′), we can show that if α, $\beta \in \mathfrak{D}$ (resp. α, $\beta \in \mathfrak{D}'$) and if $\alpha + \beta$ is a root, then $\alpha + \beta \in \mathfrak{D}$ (resp. $\alpha + \beta \in \mathfrak{D}'$) (cf. [6], p. 574).

Suppose that the Riemannian connection on G/B induced by the invariant Kaehlerian metric is the canonical affine connection of the first kined. Then the connection function α is given by

$$\alpha(X, Y) = \frac{1}{2}[X, Y]_{\mathfrak{m}} \tag{5}$$

for all $X, Y \in \mathfrak{m}$. Since the covariant derivatives of the tensor field I are zero, it follows from (4) and (5) that

$$I_0[X, Y]_{\mathfrak{m}} = [X, I_0 Y]_{\mathfrak{m}} \tag{6}$$

for all $X, Y \in \mathfrak{m}$ and hence

$$I_0^c[X, Y]_{\mathfrak{m}^c} = [X, I_0^c Y]_{\mathfrak{m}^c} \tag{6'}$$

for all $X, Y \in \mathfrak{m}^c$.

Now let $\alpha \in \mathfrak{D}$ and $\beta \in \mathfrak{D}'$ and let $\alpha + \beta$ be a root. Let us show that $[E_\alpha, E_\beta] \in \mathfrak{b}^c$. If $\alpha + \beta = 0$, then $[E_\alpha, E_\beta] \in \mathfrak{h}^c \subset \mathfrak{b}^c$. Let now $\alpha + \beta \neq 0$ and suppose that $[E_\alpha, E_\beta] \notin \mathfrak{b}^c$. Then we have $[E_\alpha, E_\beta]_{\mathfrak{m}^c} = [E_\alpha, E_\beta]$ and it follows from (6′) that $I_0^c[E_\alpha, E_\beta] = -i[E_\alpha, E_\beta]$, because $\beta \in \mathfrak{D}'$. In the same way we can show that $I_0^c[E_\beta, E_\alpha] = i \cdot [E_\beta, E_\alpha]$ and hence $I_0^c[E_\alpha, E_\beta] = i[E_\alpha, E_\beta]$. Therefore we get $I_0^c[E_\alpha, E_\beta] = 0$ and this contradicts the hypothesis that $[E_\alpha, E_\beta] \notin \mathfrak{b}^c$. Hence $[E_\alpha, E_\beta] \in \mathfrak{b}^c$.

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Now let $\alpha, \beta \in \mathfrak{D}$ and suppose that $\alpha+\beta$ be a root. Then we have $\gamma=\alpha+\beta \in \mathfrak{D}$ and $\beta=\gamma+(-\alpha)$ with $\gamma \in \mathfrak{D}$ and $-\alpha \in \mathfrak{D}'$. It follows from what we have proved above that $[E_\gamma, E_{-\alpha}] \in \mathfrak{b}^c$ and hence $E_\beta \in \mathfrak{b}^c$. This is a contradiction, because $\beta \in \mathfrak{D}$. Thus we have shown that if $\alpha, \beta \in \mathfrak{D}$, then $\alpha+\beta$ can not be a root. In the same way we can show that if $\alpha, \beta \in \mathfrak{D}'$, then $\alpha+\beta$ can not be a root. It follows from what we have proved that $[\mathfrak{m}^c, \mathfrak{m}^c] \subset \mathfrak{b}^c$ and hence $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{b}$. The endomorphism σ of $\mathfrak{g}$ such that $\sigma X=-X$ for $X \in \mathfrak{m}$ and $\sigma X=X$ for $X \in \mathfrak{b}$ is an involutive automorphism of the Lie algebra $\mathfrak{g}$. σ defines an involutive automorphism $\tilde{\sigma}$ of the simply connected Lie group $\tilde{G}$ corresponding to the Lie algebra $\mathfrak{g}$. $\tilde{G}$ is the universal covering group of G and since the center of G is equal to (e), we have $G=\tilde{G}/\tilde{Z}$, where $\tilde{Z}$ denotes the discrete center of $\tilde{G}$. Since we have $\tilde{\sigma}\cdot\tilde{Z}=\tilde{Z}$, $\tilde{\sigma}$ induces an involutive automorphism of G which we denote again by $\tilde{\sigma}$. Since the involutive automorphism of $\mathfrak{g}$ induced by $\tilde{\sigma}$ is clearly equal to σ, we can see that G/B is hermitian symmetric. Theorem 6 is thus proved.

Now let G/B be a Kaehlerian homogeneous space of a reductive Lie group with non-discrete center and let G be effective on G/B. Then S/B is also Kaehlerian, where S denotes the semi-simple part of G. We prove now the following

LEMMA 3. *Let G/B be a Kaehlerian homogeneous space of a reductive Lie group G with non-discrete center C. Let $\mathfrak{s}=\mathfrak{n}+\mathfrak{b}$ be the unique B-invariant decomposition of $\mathfrak{s}$ and let $\mathfrak{z}$ be the center of $\mathfrak{b}$. Let $\mathfrak{g}=\mathfrak{m}+\mathfrak{b}$ be a B-invariant decomposition of $\mathfrak{g}$. Then $\mathfrak{m}=\mathfrak{k}+\mathfrak{n}$, where $\mathfrak{k}$ is a subspace of $\mathfrak{c}+\mathfrak{z}$ such that $\rho(\mathfrak{k})=\mathfrak{c}$, ρ being the projection of $\mathfrak{c}+\mathfrak{z}$ onto $\mathfrak{c}$.*

Let H be a Cartan subgroup of S contained in B. Then H is a maximal toral subgroup of B and $\mathfrak{m}^c$ decomposes into the direct sum $\mathfrak{m}^c=\Sigma\mathfrak{m}_\lambda^c$, where λ denotes a linear function on $\mathfrak{h}^c$ and $\mathfrak{m}_\lambda^c$ is the subspace of all $Y \in \mathfrak{m}^c$ such that $[X, Y]=\lambda(X)\cdot Y$ for all $X \in \mathfrak{h}^c$. If $\lambda \neq 0$, then $\mathfrak{m}_\lambda^c \subset \mathfrak{s}^c$, because $[\mathfrak{h}^c, \mathfrak{m}_\lambda^c]=\mathfrak{m}_\lambda^c$ and $[\mathfrak{g}^c, \mathfrak{g}^c]=\mathfrak{s}^c$. It is easily verified that $\mathfrak{n}^c=\sum_{\lambda\neq 0}\mathfrak{m}_\lambda^c$. Then we get $\mathfrak{m}=\mathfrak{m}\frown\mathfrak{m}_0^c+\mathfrak{n}$. Let $X \in \mathfrak{m}\frown\mathfrak{m}_0^c$ and let $X=X_1+X_2$, where $X_1 \in \mathfrak{c}$ and $X_2 \in \mathfrak{s}$. We shall show that $X_2 \in \mathfrak{z}$. As $X \in \mathfrak{m}\frown\mathfrak{m}_0^c$ and $X_1 \in \mathfrak{c}$, we have $[W, X]=[W, X_2]=0$ for all $W \in \mathfrak{h}$. Since $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{s}$, the normalisor of $\mathfrak{h}$ in $\mathfrak{s}$ is equal to $\mathfrak{h}$ and hence $X_2 \in \mathfrak{h}$. Now let $Y \in \mathfrak{b}$. Then we have $[Y, X]=[Y, X_2]$, because $X_1 \in \mathfrak{c}$. As $X \in \mathfrak{m}$ and $[\mathfrak{b}, \mathfrak{m}] \subset \mathfrak{m}$, we have $[Y, X] \in \mathfrak{m}$. On the other hand, since $X_2 \in \mathfrak{h}$ and $\mathfrak{h} \subset \mathfrak{b}$, we have $[Y, X_2] \in \mathfrak{b}$. Therefore $[Y, X] = [Y, X_2] = 0$, because $\mathfrak{m} \frown \mathfrak{b} = (0)$. It follows that X_2 is an element of the center $\mathfrak{z}$ of $\mathfrak{b}$. Therefore $\mathfrak{k} = \mathfrak{m} \frown \mathfrak{m}_0^c$ is a subspace of $\mathfrak{c} + \mathfrak{z}$. It is easily verified that $\rho(\mathfrak{k}) = \mathfrak{c}$.

THEOREM 7. *Let G/B be a Kaehlerian homogeneous space of a reductive Lie group G and let G be effective on G/B. If the Riemannian connection on G/B induced by the invariant Kaehlerian metric is the canonical affine connection of the first kind with respect to a certain B-invariant decomposition of $\mathfrak{g}$, then G/B is hermitian symmetric.*

Let C and S be the center and the semi-simple part of G respectively. Then C and S/B admit the homogeneous structures such that $G/B = C \times S/B$. Let $\mathfrak{g} = \mathfrak{m} + \mathfrak{b}$ be the B-invariant decomposition of $\mathfrak{g}$ with respect to which the Riemannian connection on G/B is canonical of the first kind. Let $\mathfrak{s} = \mathfrak{n} + \mathfrak{b}$ be the unique B-invariant decomposition of $\mathfrak{s}$. Then we have $\mathfrak{m} \supset \mathfrak{n}$ by Lemma 3. Let $x^*(t)$ be the image in G/B of the one parameter subgroup of G generated by an element of $\mathfrak{n}$. Then it is a path. Since $x^*(t) \subset S/B$, it is a path of S/B. It follows that the Riemannian connection on S/B induced by the invariant Kaehlerian metric is canonical of the first kind. Hence S/B is hermitian symmetric by Theorem 6. Therefore there exists an involutive automorphism σ of S such that B is equal to the connected component of the identity of the closed subgroup of S of all $x \in S$ such that $\sigma(x) = x$. Now let σ' be the mapping of $G = C \times S$ onto itself such that $\sigma'(x, y) = (x^{-1}, \sigma(y))$, where $x \in C$ and $y \in S$. Then σ' is an involutive automorphism of G and B is equal to the connected component of the identity of the closed subgroup of G of all $w \in G$ such that $\sigma'(w) = w$. Therefore G/B is hermitian symmetric.

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