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ON A DIRICHLET PROBLEM WITH p-LAPLACIAN AND SET-VALUED NONLINEARITY

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Abstract

The existence of solutions to a homogeneous Dirichlet problem for a p-Laplacian differential inclusion is studied via a fixed-point type theorem concerning operator inclusions in Banach spaces. Some meaningful special cases are then worked out.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, let $p \in (1, +\infty)$, and let $j : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable in $x \in \Omega$ for every $z \in \mathbb{R}$. Consider the Dirichlet problem

$$\begin{cases} -\Delta_p u = j(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian. If j is a Carathéodory's function then a number of existence and multiplicity results involving (1.1) are available in the literature; see for instance the monographs [8, 9, 15], besides the very recent paper [3]. Variational, subsupersolutions, as well as topological methods represent the most exploited technical approaches. When $j(x, \cdot)$ turns out to be locally essentially bounded only, (1.1) is usually replaced by

$$\begin{cases} -\Delta_p u \in \partial J(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with

$$J(x, \xi) := \int_0^{\xi} j(x, t)\, dt, \qquad (x, \xi) \in \Omega \times \mathbb{R}, \tag{1.3}$$

and $\partial J(x, z)$ being the Clarke generalized gradient of $J(x, \cdot)$ at the point $z \in \mathbb{R}$. Problem (1.2) has been the subject of numerous investigations, mainly based on the critical point theory for locally Lipschitz continuous functions [4, 10, 14], sometimes combined with subsupersolution arguments [2, 8]. By the way, setting

$$\underline{j}(x, z) := \lim_{\delta \to 0^+} \operatorname*{ess\,inf}_{|w-z|<\delta} j(x, w), \quad \overline{j}(x, z) := \lim_{\delta \to 0^+} \operatorname*{ess\,sup}_{|w-z|<\delta} j(x, w), \quad (x, z) \in \Omega \times \mathbb{R}, \tag{1.4}$$

the inclusion in (1.2) becomes

$$\underline{j}(x, u) \le -\Delta_p u \le \overline{j}(x, u) \quad \text{in } \Omega, \tag{1.5}$$

which reduces to $-\Delta_p u = j(x, u)$ at each point u where $j(x, \cdot)$ is continuous.

In this paper, we simply point out that Problem (1.2), with J unnecessarily of the type (1.3), can also be treated through an existence result for operator inclusions, previously established in [1], provided $p > N$. One assumes that $(x, z) \mapsto J(x, z)$, $(x, z) \in \Omega \times \mathbb{R}$, is measurable with respect to $x \in \Omega$ and locally Lipschitz continuous in $z \in \mathbb{R}$. A further condition, compatible with any growth rate of $J(x, \cdot)$, fits our purposes; see Theorem 3.1. Some meaningful special cases, namely Corollaries 3.2–3.3, are then worked out.

The recent work [7] treats p-Laplacian differential inclusions via fixed points for multifunctions in partially ordered sets. Amidst the results of [7] let us mention Proposition 4.1, which provides extremal solutions to a problem like (1.5) under hypotheses different from those employed here.

2. Preliminary results

From now on, Ω denotes a bounded domain of the real Euclidean N-space $(\mathbb{R}^N, |\cdot|)$ with a smooth boundary $\partial\Omega$, $p \in (N, +\infty)$, $p' := p/(p-1)$, $\|\cdot\|_q$ is the usual norm of $L^q(\Omega)$, $1 \le q \le +\infty$, while $W_0^{1,p}(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. On $W_0^{1,p}(\Omega)$ we introduce the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p}, \quad u \in W_0^{1,p}(\Omega).$$

It is known that $W_0^{1,p}(\Omega)$ compactly embeds in $L^p(\Omega)$ and one has

$$\|u\|_p \le \lambda_1^{-1/p} \|u\| \quad \forall u \in W_0^{1,p}(\Omega),$$

where λ_1 indicates the first Dirichlet eigenvalue of the p-Laplacian [11]. Moreover, since $p > N$, we actually get $W_0^{1,p}(\Omega) \subseteq L^\infty(\Omega)$ as well as

$$\|u\|_\infty \le a\|u\|, \quad u \in W_0^{1,p}(\Omega), \tag{2.1}$$

for suitable $a > 0$; see, for example, [5, Ch. IX]. The constant a has been estimated in [16, Formula (6b)] and, for convex Ω, in [6, Theorem 1].

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Let $W^{-1,p'}(\Omega)$ be the dual space of $W_0^{1,p}(\Omega)$. By [5, Theorem VI.4] the space $L^{p'}(\Omega)$ compactly embeds in $W^{-1,p'}(\Omega)$. Thus, there exists $b > 0$ satisfying

$$\|v\|_{W^{-1,p'}(\Omega)} \le b\|v\|_{p'}, \quad v \in L^{p'}(\Omega). \tag{2.2}$$

REMARK 2.1. The constant b can be evaluated through λ_1. In fact,

$$\|v\|_{W^{-1,p'}(\Omega)} := \sup_{\|u\|\le 1}\left|\int_\Omega u(x)v(x)\,dx\right| \le \sup_{\|u\|\le 1}\|u\|_p\|v\|_{p'} \le \lambda_1^{-1/p}\|v\|_{p'}$$

for all $v \in L^{p'}(\Omega)$, whence $b \le \lambda_1^{-1/p}$.

Let $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be the nonlinear operator stemming from the negative p-Laplacian, that is,

$$\langle A(u), v\rangle := \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)\,dx, \quad u, v \in W_0^{1,p}(\Omega). \tag{2.3}$$

Theorem A.0.6 in [15] and an elementary argument ensure the following properties.
(p_1) A is bijective and uniformly continuous on bounded sets.
(p_2) Its inverse A^{-1} turns out to be continuous.
(p_3) $\|A(u)\|_{W^{-1,p'}(\Omega)} = \|u\|^{p-1}$ in $W_0^{1,p}(\Omega)$.
Let U be a nonempty set and let $\Phi : U \to W_0^{1,p}(\Omega)$, $\Psi : U \to L^{p'}(\Omega)$ be two operators such that the following conditions (i_1) hold true.
(i_1) Ψ is bijective and for any $v_h \rightharpoonup v$ in $L^{p'}(\Omega)$ there is a subsequence of $\{\Phi(\Psi^{-1}(v_h))\}$ which converges to $\Phi(\Psi^{-1}(v))$ almost everywhere in Ω. Furthermore, a nondecreasing function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \cup \{+\infty\}$ can be defined in such a way that

$$\|\Phi(u)\|_\infty \le \varphi(\|\Psi(u)\|_{p'}) \quad \forall u \in U. \tag{2.4}$$

Finally, let $F : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ be a convex closed-valued multifunction. Theorem 3.1 of [1] directly yields the next result.

THEOREM 2.2. *Suppose (i_1) holds true and, moreover, suppose that the following conditions hold true.*
(i_2) *$F(\cdot, z)$ is measurable for all $z \in \mathbb{R}$.*
(i_3) *$F(x, \cdot)$ has a closed graph for almost every $x \in \Omega$.*
(i_4) *There exists $r > 0$ such that the function $m(x) := \sup_{|z|\le\varphi(r)} \inf\{|y| : y \in F(x, z)\}$, $x \in \Omega$, belongs to $L^{p'}(\Omega)$ and $\|m\|_{p'} \le r$.*
Then the problem $\Psi(u) \in F(x, \Phi(u))$ in Ω possesses at least one solution $u \in U$ satisfying $|\Psi(u)(x)| \le m(x)$ for almost every $x \in \Omega$.

For the notions on multifunctions (respectively, nonsmooth analysis) exploited in the paper, we simply refer the reader to [1] (respectively, [12]), measurable always means Lebesgue measurable, while the symbol $m(E)$ will indicate the Lebesgue measure of E.

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3. Existence of solutions

Keep the same notation of Section 2 and define, for every $t \in \mathbb{R}_0^+$,

$$\varphi(t) := a(bt)^{1/(p-1)}. \tag{3.1}$$

The function φ turns out to be monotone increasing in $\mathbb{R}_0^+$. Let $J : \Omega \times \mathbb{R} \to \mathbb{R}$. We shall make the following assumptions.

(a_1) $J(\cdot, z)$, $z \in \mathbb{R}$, is measurable.

(a_2) To every $M > 0$ there corresponds $k(M) > 0$ such that

$$|J(x, z_1) - J(x, z_2)| \le k(M)|z_1 - z_2| \quad \text{almost everywhere in } \Omega \text{ and } \forall z_1, z_2 \in [-M, M].$$

(a_3) For suitable $\varepsilon, r > 0$ one has $m(\Omega)^{1-1/p} k(a(br)^{1/(p-1)} + \varepsilon) \le r$.

By (a_2) it makes sense to consider the generalized Clarke gradient $\partial J(x, z)$ of $J(x, \cdot)$ at the point $z \in \mathbb{R}$.

Theorem 3.1. *If $p > N$ and (a_1)–(a_3) hold true then there exists $u \in W_0^{1,p}(\Omega)$ satisfying $-\Delta_p u(x) \in \partial J(x, u(x))$ almost everywhere in Ω.*

Proof. Set $U := A^{-1}(L^{p'}(\Omega))$, $\Phi(u) := u$, and $\Psi(u) := A(u)$ for all $u \in U$. Property (p_1) ensures that the operator $\Psi : U \to L^{p'}(\Omega)$ is bijective. Let $v_h \rightharpoonup v$ in $L^{p'}(\Omega)$. Because of the compact embedding $L^{p'}(\Omega) \subseteq W^{-1,p'}(\Omega)$ and (p_2) we obtain, up to subsequences, $\Phi(\Psi^{-1}(v_h)) \to \Phi(\Psi^{-1}(v))$ almost everywhere in Ω. Hence, (i_1) is verified once we prove (2.4). Since $p > N$, gathering (2.1), (2.2), and (p_3) together, one has

$$\|\Phi(u)\|_\infty \le a\|u\| = a\|\Psi(u)\|_{W^{-1,p'}(\Omega)}^{1/(p-1)} \le a(b\|\Psi(u)\|_{p'})^{1/(p-1)} = \varphi(\|\Psi(u)\|_{p'}), \quad u \in U,$$

with φ given by (3.1), and (i_1) follows.

Now define $F(x, z) := \partial J(x, z)$, $(x, z) \in \Omega \times \mathbb{R}$. A simple computation shows that

$$F(x, z) = [-J^0(x, z; -1), J^0(x, z; +1)], \tag{3.2}$$

where, as usual,

$$J^0(x, z; \pm 1) := \limsup_{w \to z, t \to 0^+} \frac{J(x, w \pm t) - J(x, w)}{t}.$$

Thanks to (a_1) the functions $x \mapsto J^0(x, z; \pm 1)$ are measurable in Ω for every $z \in \mathbb{R}$. So, taking account of [13, Proposition 1.1], condition (i_2) of Theorem 2.2 holds.

Let us next verify (i_3). Pick $\{z_h\}, \{y_h\} \subseteq \mathbb{R}$ fulfilling

$$z_h \to z, \quad y_h \to y, \quad y_h \in F(x, z_h) \quad \forall h \in \mathbb{N}.$$

The upper semicontinuity of $\zeta \mapsto J^0(x, \zeta; \pm 1)$, combined with (3.2), yield, as $h \to +\infty$,

$$-J^0(x, z; -1) \le y \le J^0(x, z; +1), \quad \text{namely } y \in F(x, z),$$

which represents the desired conclusion.

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Finally, to prove (i_4) observe at first that

$$|J^0(x, z; \pm 1)| \leq k(M) \quad \forall M > 0, z \in (-M, M).$$

This implies

$$m(x) := \sup_{|z|\leq\varphi(r)} \inf\{|y| : y \in F(x, z)\} \leq \sup_{|z|<\varphi(r)+\epsilon} \inf\{|y| : y \in F(x, z)\} \leq k(\varphi(r) + \epsilon)$$

almost everywhere in Ω. Consequently, by (a_3),

$$\|m\|_{p'} \leq m(\Omega)^{1-1/p} k(\varphi(r) + \epsilon) \leq r.$$

Now Theorem 2.2 can be applied, and we obtain $u \in U \subseteq W_0^{1,p}(\Omega)$ such that

$$-\Delta_p u(x) = \Psi(u)(x) \in F(x, u(x)) = \partial J(x, u(x))$$

for almost all $x \in \Omega$. □

A meaningful special case occurs when J is given by (1.3), where $j : \Omega \times \mathbb{R} \to \mathbb{R}$ fulfils the following hypotheses.

(a_4) j turns out to be measurable in each variable separately.

(a_5) To every $M > 0$ there corresponds $k(M) > 0$ such that $|j(x, z)| \leq k(M)$ almost everywhere in Ω and for all $z \in [-M, M]$.

Indeed, under (a_4)–(a_5), the function J satisfies (a_1), (a_2), and we get

$$\partial J(x, z) = [\underline{j}(x, z), \overline{j}(x, z)],$$

with $\underline{j}$, $\overline{j}$ being as in (1.4); see [12, Example 1]. Hence, Theorem 3.1 directly leads to the following corollary.

Corollary 3.2. *If (a_4)–(a_5), besides (a_3), hold true then there exists $u \in W_0^{1,p}(\Omega)$ such that $\underline{j}(x, u(x)) \leq -\Delta_p u(x) \leq \overline{j}(x, u(x))$ for almost every $x \in \Omega$.*

In particular, when

$$|j(x, z)| \leq c_1 + c_2|z|^{p-1} \quad \forall (x, z) \in \Omega \times \mathbb{R}, \tag{3.3}$$

where $c_1, c_2 > 0$, from the above result we deduce the following corollary.

Corollary 3.3. *Let the function j comply with (a_4) and (3.3). Assume also that*

$$m(\Omega)^{1-1/p} a^{p-1} b c_2 < 1.$$

Then the conclusion of Corollary 3.2 holds.

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Remark 3.4. Applications of Theorem 3.1 and its consequences can basically be constructed only if one knows explicit estimates of constants a and b. As already observed in Section 2, thanks to [16, Formula (6b)] we get

$$a \leq \frac{N^{-1/p}}{\sqrt{\pi}}\left(\frac{p-1}{p-N}\right)^{1-1/p}\left(\Gamma\left(1+\frac{N}{2}\right)\right)^{1/N} m(\Omega)^{1/N-1/p},$$

with Γ being the gamma function. Since, for every $u \in W_0^{1,p}(\Omega)$,

$$\|u\|_p \leq m(\Omega)^{1/p}\|u\|_\infty \leq m(\Omega)^{1/p} a\|u\|,$$

Remark 2.1 provides

$$b \leq \lambda_1^{-1/p} \leq m(\Omega)^{1/p} a \leq \frac{N^{-1/p}}{\sqrt{\pi}}\left(\frac{p-1}{p-N}\right)^{1-1/p}\left(m(\Omega)\Gamma\left(1+\frac{N}{2}\right)\right)^{1/N}.$$

Remark 3.5. Condition (3.3), with $c_2 < \lambda_1$, appears also in [7, Proposition 4.1]. It is a simple matter to realize that this result and Corollary 3.3 are mutually independent.

Remark 3.6. The main difficulty in treating the case $\Omega := \mathbb{R}^N$ is to verify (i_1). However, if the operator $A : W^{1,p}(\mathbb{R}^N) \to W^{-1,p'}(\mathbb{R}^N)$ given by

$$\langle A(u), v\rangle := \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \cdot \nabla v + c(x)|u|^{p-2}uv)\, dx \quad \forall u, v \in W^{1,p}(\mathbb{R}^N),$$

where $c \in L^\infty(\mathbb{R}^N)$ and $\operatorname{ess\,inf}_{x\in\Omega} c(x) > 0$, takes the place of the one defined in (2.3), it can be done, as we shall see in a future work.

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