Canad. Math. Bull. Vol. **32** (4), 1989

ON THE DAVISON CONVOLUTION OF ARITHMETICAL FUNCTIONS

BY
PENTTI HAUKKANEN

ABSTRACT. The Davison convolution of arithmetical functions f and g is defined by $(f \circ g)(n) = \sum_{d|n} f(d)g(n/d)K(n,d)$, where K is a complex-valued function on the set of all ordered pairs (n, d) such that n is a positive integer and d is a positive divisor of n. In this paper we shall consider the arithmetical equations $f^{(r)} = g$, $f^{(r)} = fg$, $f \circ g = h$ in f and the congruence $(f \circ g)(n) \equiv 0 \pmod{n}$, where $f^{(r)}$ is the iterate of f with respect to the Davison convolution.

1. **Introduction.** Let K be a complex-valued function on the set of all ordered pairs (n, d) such that n is a positive integer and d is a positive divisor of n. Then the K-convolution of arithmetical functions f and g is defined by

$$(f \circ g)(n) = \sum_{d|n} f(d)g(n/d)K(n,d).$$

The concept of the K-convolution originates to Davison [3]. In the case in which $K(n, d)$ depends only on the g.c.d. $(d, n/d)$ the concept is due to Gioia and Subbarao ([9], see also [8]). For further study of K-convolutions we refer to [4], [5], [7] and [14].

An arithmetical function f is said to be quasi-multiplicative [12] if $f(1) \neq 0$ and

$$f(1)f(mn) = f(m)f(n) \text{ whenever } (m,n) = 1.$$

A quasi-multiplicative function is said to be multiplicative if $f(1) = 1$. It is easy to see that an arithmetical function f with $f(1) \neq 0$ is quasi-multiplicative if, and only if, $f/f(1)$ is multiplicative. Rearick [16] defined an arithmetical function f to be semi-multiplicative if there exist a non-zero complex-number c_f, a positive integer a_f and a multiplicative function f' such that

$$f(n) = c_f f'(n/a_f).$$

Clearly semi-multiplicative functions with $a_f = 1$ are quasi-multiplicative.

Received by the editors June 23, 1988.
1980 AMS Mathematics Subject Classification. 10A20.

https://doi.org/10.4153/CMB-1989-067-4 Published online by Cambridge University Press

It is known ([3], [7], [14]) that the set of multiplicative functions forms an Abelian group with identity with respect to the K-convolution if, and only if,
(a) $K(n,n) = K(n,1) = 1$ for all n,
(b) $K(mn, de) = K(m,d)K(n,e)$ for all m,n,d,e such that $d|m, e|n, (m,n) = 1$,
(c) $K(n,d)K(d,e) = K(n,e)K(n/e, d/e)$ for all n,d,e such that $d|n, e|d$,
(d) $K(n,d) = K(n, n/d)$ for all n, d with $d|n$.
For example, a regular convolution due to Narkiewicz [15] satisfies (a)–(d). If $K \equiv 1$, we obtain the well-known Dirichlet convolution, which is regular and satisfies (a)–(d). Further, if $K = U$, defined by $U(n,d) = 1$ for $d|n$ with $(d, n/d) = 1$, and 0 otherwise, then we obtain the unitary convolution [2], which is also regular and satisfies (a)–(d).

Throughout this paper K is an arbitrary but fixed convolution satisfying (a)–(d).

The rth K-iterate of an arithmetical function f is defined by

$$f^{(r)} = f \circ \cdots \circ f \ (r \text{ factors }).$$

Clearly

$$f^{(r)}(n) = \sum_{a_1 a_2 \ldots a_r = n} f(a_1)f(a_2)\ldots f(a_r)K(n,a_1)K(a_2\ldots a_r, a_2)\ldots K(a_{r-1}a_r, a_{r-1}).$$

The inverse of an arithmetical function f with respect to the K-convolution is defined by

$$f \circ f^{(-1)} = f^{(-1)} \circ f = E_0,$$

where $E_0(1) = 1$, $E_0(n) = 0$ for $n > 1$. The inverse exists and is unique if, and only if, $f(1) \neq 0$ (see [3]).

In this paper we consider the arithmetical equations $f^{(r)} = g$, $f^{(r)} = fg$, $f \circ g = h$ in f and the congruence $(f \circ g)(n) \equiv 0 \pmod{n}$. For the arithmetical equations we need the concepts given in the following preliminaries.

2. **Preliminaries.** We define an arithmetical function f to be quasi-K-multiplicative if $f(1) \neq 0$ and

$$f(d)f(n/d)K(n,d) = f(1)f(n)K(n,d) \text{ for all } d|n.$$

If, in particular, $f(1) = 1$, we say f to be K-multiplicative. It is easy to see that an arithmetical function f with $f(1) \neq 0$ is quasi-K-multiplicative if, and only if, $f/f(1)$ is K-multiplicative. If K is the Dirichlet convolution, then K-multiplicative functions are completely multiplicative functions. Moreover, if K is a regular convolution due to Narkiewicz [15], we obtain the concept of multiplicativity due to Yocom [19].

For an arithmetical function f with $f(1) = 1$ we define (cf. [6]) a logarithm operator by

$$(\log f)(n) = \left(\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} (f - E_0)^{(r)} \right) (n).$$

https://doi.org/10.4153/CMB-1989-067-4 Published online by Cambridge University Press

Further, for an arithmetical function f with $f(1) = 0$ we define (cf. [6]) an exponential operator by

$$(\exp f)(n) = \left(E_0 + \sum_{r=1}^{\infty} \frac{1}{r!} f^{(r)} \right)(n).$$

Note that for each n the above sums are finite.

It can be proved (cf. [6]) that

$$\log(f \circ g) = \log f + \log g \tag{1}$$

and

$$\log f = g \text{ if, and only if, } f = \exp g. \tag{2}$$

It can also be proved that

$$\log(fg) = g(\log f) \tag{3}$$

for all K-multiplicative functions g. In fact, $(\log(fg))(1) = g(1)(\log f)(1) = 0$ and for $n > 1$

$$\begin{aligned}
\log(fg)(n) &= \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} \sum_{\substack{d_1 \ldots d_r = n \\ d_1, \ldots, d_r \neq 1}} (fg)(d_1) \ldots (fg)(d_r) \\
&\quad \times K(n, d_1) K(d_2 \ldots d_r, d_2) \ldots K(d_{r-1} d_r, d_{r-1}) \\
&= \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} \sum_{\substack{d_1 \ldots d_r = n \\ d_1, \ldots, d_r \neq 1}} f(d_1) \ldots f(d_r) \\
&\quad \times K(n, d_1) K(d_2 \ldots d_r, d_2) \ldots K(d_{r-1} d_r, d_{r-1}) g(d_1 \ldots d_r) \\
&= g(n)(\log f)(n).
\end{aligned}$$

3. **Arithmetical equations.**

THEOREM 1. *Suppose f is an arithmetical function such that $f(1) = 1$. Then $f^{(r)}$ is multiplicative if, and only if, f is multiplicative.*

PROOF. If f is multiplicative, then $f^{(r)}$ is multiplicative by (b). Conversely, suppose $f^{(r)}$ is multiplicative. Then we proceed by induction on mn to prove that $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. If $mn = 1$, the statement holds. Assume it holds for a, b with $a < m$, $b < n$, $(a, b) = 1$. We may omit the trivial case $m = 1$ or $n = 1$.

Thus

$$f^{(r)}(mn) = \sum_{a_1\ldots a_r=m}\sum_{b_1\ldots b_r=n} f(a_1b_1)\ldots f(a_rb_r)K(mn,a_1b_1)K(a_2b_2\ldots a_rb_r,a_2b_2)$$
$$\times\ldots K(a_{r-1}b_{r-1}a_rb_r,a_{r-1}b_{r-1})$$
$$= \sum_{a_1\ldots a_r=m} f(a_1)\ldots f(a_r)K(m,a_1)K(a_2\ldots a_r,a_2)\ldots K(a_{r-1}a_r,a_{r-1})$$
$$\times \sum_{b_1\ldots b_r=n} f(b_1)\ldots f(b_r)K(n,b_1)K(b_2\ldots b_r,b_2)\ldots K(b_{r-1}b_r,b_{r-1})$$
$$- r(f(1))^{2r-2}f(m)f(n) + rf(1)^{r-1}f(mn)$$
$$= f^{(r)}(m)f^{(r)}(n) + r(f(mn) - f(m)f(n)).$$

As $f^{(r)}(mn) = f^{(r)}(m)f^{(r)}(n)$, we have $f(mn) = f(m)f(n)$. Thus f is multiplicative and the proof is complete.

REMARK 1. Using Theorem 1 we can easily see that if f is an arithmetical function such that $f(1) \neq 0$, then $f^{(r)}$ is quasi-multiplicative if, and only if, f is quasi-multiplicative.

It can be shown that if f is semi-multiplicative, then $f^{(r)}$ is semi-multiplicative or identically zero. Conversely, if $f^{(r)}$ is semi-multiplicative, then f is not necessarily semi-multiplicative. Take, for example, $K = U$, $r = 2$, $f(1) = 0$, $f(2) = f(3) = 1$, $f(n) = 0$ for $n \geqq 4$. Then $f^{(2)}(6) = 2$, $f^{(2)}(n) = 0$ for $n \neq 6$. Hence f is not semi-multiplicative but $f^{(r)}$ is semi-multiplicative.

THEOREM 2. *Suppose g is a fixed arithmetical function such that $g(1) \neq 0$. Then the equation $f^{(r)} = g$ has exactly r solutions in f. If f_0 is one solution, then all solutions are given by*

$$f = \omega_i f_0, i = 1, 2, \ldots, r, \tag{4}$$

$\omega_1, \omega_2, \ldots, \omega_r$ being the rth roots of unity. One solution can be found by

$$f_0(n) = g(1)^{1/r}\{\exp[(1/r)\log(g/g(1))]\}(n). \tag{5}$$

The equation has a multiplicative solution if, and only if, g is multiplicative, in which case only one solution is multiplicative.

PROOF. Clearly

$$f(1) = g(1)^{1/r},$$

that is,

$$f(1) = \omega_i z \text{ for some } i = 1, 2, \ldots, r,$$

z being an rth root of $g(1)$. Further, the values $f(n)$, $n \geqq 2$, can be found inductively by

$$r(f(1))^{r-1}f(n) + \sum_{\substack{d_1\ldots d_r=n\\ d_1,\ldots,d_r\neq n}} f(d_1)\ldots f(d_r)$$
$$\times K(n,d_1)K(d_2\ldots d_r,d_2)\ldots K(d_{r-1}d_r,d_{r-1}) = g(n).$$

https://doi.org/10.4153/CMB-1989-067-4 Published online by Cambridge University Press

So we deduce (4).

In proving (5) assume firstly that $g(1) = 1$. Then there is a solution f_0 for which $f_0(1) = 1$. By (1), $r(\log f_0) = \log g$. Thus, by (2),

$$f_0 = \exp[(1/r)\log g]. \tag{6}$$

Now, consider the general case $g(1) \neq 0$. Then $(g/g(1))(1) = 1$ and hence applying (6) proves (5).

The results concerning multiplicative functions follow now easily by Theorem 1. This completes the proof.

REMARK 2. Theorem 2 can easily be extended to quasi-multiplicative functions as follows: The equation $f^{(r)} = g$ has a quasi-multiplicative solution if, and only if, g is quasi-multiplicative, in which case all the r solutions are quasi-multiplicative. By Remark 1 this is not valid for semi-multiplicative functions.

THEOREM 3. *Suppose g is a fixed quasi-K-multiplicative function such that $g(n) \neq rg(1)$ for all n. Then the equation $f^{(r)} = fg$ has $r - 1$ solutions f such that $f(1) \neq 0$. The solutions are given by*

$$f = (g(1))^{1/(r-1)}E_0.$$

PROOF. At first, assume $f(1) = g(1) = 1$. Then, by (1) and (3), $r(\log f) = g(\log f)$ or $(\log f)(n)(g(n) - r) = 0$ for all n. Thus $(\log f)(n) = 0$ for all n and consequently, by (2), $f = E_0$.

Now, consider the general case: $f(1), g(1) \neq 0$. Then $(f/f(1))(1) = (g/g(1))(1) = 1$ and hence we have $f/f(1) = E_0$. So we can deduce the result.

THEOREM 4. *Suppose g and h are fixed and $g(1) \neq 0$. Then the equation $f \circ g = h$ has a unique solution given by*

$$f = h \circ g^{(-1)}. \tag{7}$$

If g and h are quasi-multiplicative, then the solution f is quasi-multiplicative. If, in addition, $h(1)/g(1) = 1$, then the solution f is multiplicative.

PROOF. Each arithmetical function g with $g(1) \neq 0$ has a unique inverse with respect to the K-convolution. Hence we have (7). Further, suppose g and h are quasi-multiplicative. We shall prove that f is quasi-multiplicative. As $(f \circ g)(1) = f(1)g(1) = h(1) \neq 0$ and $g(1) \neq 0$, so $f(1) \neq 0$. We are to prove still that

$$f(1)f(mn) = f(m)f(n), \tag{8}$$

whenever $(m, n) = 1$. Suppose (8) holds for $d|m, e|n$ with $de \neq mn$. Then

$$\begin{aligned} h(1)h(mn) &= (f \circ g)(1)(f \circ g)(mn) = f(1)g(1)\sum_{d|m}\sum_{e|n} f(de)g(mn/(de))K(mn, de) \\ &= \sum_{d|m} f(d)g(m/d)K(m,d)\sum_{e|n} f(e)g(n/e)K(n,e) \\ &\quad - f(m)f(n)g(1)^2 + f(1)f(mn)g(1)^2 \\ &= h(m)h(n) - f(m)f(n)g(1)^2 + f(1)f(mn)g(1)^2. \end{aligned}$$

https://doi.org/10.4153/CMB-1989-067-4 Published online by Cambridge University Press

As $h(1)h(mn) = h(m)h(n)$ and $g(1) \neq 0$, we obtain (8) and hence f is quasi-multiplicative. If, in addition, $h(1)/g(1) = 1$, then $f(1) = 1$ and consequently f is multiplicative. This completes the proof.

REMARK 3. If $f \circ g = h$ and g, h are semi-multiplicative, f is not necessarily semi-multiplicative. Take, for example, $K = U$, $f(2) = f(3) = 1$, $f(n) = 0$ for $n \neq 2, 3, g(3) = 1, g(n) = 0$ for $n \neq 3, h(6) = 1, h(n) = 0$ for $n \neq 6$.

REMARK 4. For material relating to arithmetical equations of the types of this paper we refer to [1], [11] and [18] which consider the Dirichlet convolution and the unitary convolution. In [10] the exponential convolution is considered.

4. A congruence.

THEOREM 5. *Suppose $f(n), g(n)$ and $K(n,d)$ are integral valued functions and $f(n)$ is multiplicative. Then the congruence*

$$(f \circ g)(n) \equiv 0 \pmod{n} \tag{9}$$

holds for all positive integers n if, and only if,

$$\sum_{i=0}^{a} f(p^i) g(p^{a-i}m) K(p^a, p^i) \equiv 0 \pmod{p^a} \tag{10}$$

for all primes p and positive integers a, m with $(p, m) = 1$.

PROOF. Suppose (10) holds. To prove (9) we can clearly assume $n > 1$. Then denote $n = p^a m$, where $a \geqq 1, (p, m) = 1$. By (b) and the multiplicativity of f we obtain

$$(f \circ g)(n) = \sum_{d|m} f(d) K(m, d) \sum_{i=0}^{a} f(p^i) g(p^{a-i} m/d) K(p^a, p^i).$$

By (10) the inner sum $\equiv 0 \pmod{p^a}$. Thus using a similar argument for each prime divisor of n we have (9).

Conversely, suppose (9) holds. Taking $n = p^a m$, where $(p, m) = 1$, we have by (b) and the multiplicativity of $f^{(-1)}$

$$\begin{aligned} g(p^a m) &= \sum_{d|n} (f \circ g)(d) f^{(-1)}(n/d) K(n, d) \\ &= \sum_{i=0}^{a} f^{(-1)}(p^{a-i}) K(p^a, p^i) \sum_{d|m} (f \circ g)(p^i d) f^{(-1)}(m/d) K(m, d). \end{aligned}$$

Now, applying the above identity, items (c) and (d) and equation (9) we obtain

$$\begin{aligned} \sum_{i=0}^{a} f(p^i) g(p^{a-i} m) K(p^a, p^i) &= \sum_{d|m} (f \circ g)(p^a d) f^{(-1)}(m/d) K(m, d) \\ &\equiv 0 \pmod{p^a}, \end{aligned}$$

https://doi.org/10.4153/CMB-1989-067-4 Published online by Cambridge University Press

that is, we obtain (10). This completes the proof.

REMARK 5. Subbarao [17] has proved Theorem 5 in the case of the Dirichlet convolution. He also briefly recounted the history of the present congruence type. Since Subbarao the present congruence type has been studied by Hanumanthachari [10] and McCarthy [13] in the cases of the exponential convolution and the unitary convolution, respectively.

REFERENCES

1. L. Carlitz and M. V. Subbarao, *Transformation of arithmetic functions*, Duke Math J. **40** (1973), 949–958.

2. E. Cohen, *Arithmetical functions associated with the unitary divisors of an integer,* Math. Z. **74** (1960), 66–80.

3. T. M. K. Davison, *On arithmetic convolutions*, Canad. Math. Bull. **9** (1966), 287–296.

4. M. Ferrero, *On generalized convolution rings of arithmetic functions,* Tsukuba J. Math. **4** (1980), 161–176.

5. I. P. Fotino, *Generalized convolution ring of arithmetic functions*, Pacific J. Math. **61** (1975), 103–116.

6. M. L. Fredman, *Arithmetical convolution products,* Duke Math J. **37** (1970), 231–242.

7. M. D. Gessley, *A generalized arithmetic convolution*, Amer. Math. Monthly **74** (1967), 1216–1217.

8. A. A. Gioia, *The K-product of arithmetic functions,* Canad. J. Math. **17** (1965), 970–976.

9. A. A. Gioia and M. V. Subbarao, *Generalized Dirichlet products of arithmetic functions* (Abstract), Notices Amer. Math. Soc. **9** (1962), 305.

10. J. Hanumanthachari, *On an arithmetic convolution*, Canad. Math. Bull. **20** (1977), 301–305.

11. P. Haukkanen, *Arithmetical equations involving semi-multiplicative functions and the Dirichlet product* (To appear in Rend. Math.).

12. D. B. Lahiri, *Hypo-multiplicative number-theoretic functions*, Aequationes Math. **9** (1973), 184–192.

13. P. J. McCarthy, *Note on some arithmetic sums,* Boll. Un. Mat. Ital. **21** (1966), 239–242.

14. ———, *Introduction to arithmetical functions,* Springer, 1986.

15. W. Narkiewicz, *On a class of arithmetical convolutions*, Colloq. Math. **10** (1963), 81–94.

16. D. Rearick, *Semi-multiplicative functions*, Duke Math. J. **33** (1966), 49–53.

17. M. V. Subbarao, *A congruence for a class of arithmetic functions*, Canad. Math. Bull. **9** (1966), 571–574.

18. ———, *A class of arithmetical equations,* Nieuw Arch. Wisk. **15** (1967), 211–217.

19. K. L. Yocom, *Totally multiplicative functions in regular convolution rings,* Canad. Math. Bull. **16** (1973), 119–129.

Department of Mathematical Sciences
University of Tampere
Tampere, Finland