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ON A CLASS OF NEAR-RINGS

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Abstract

It is well known that in a commutative Noetherian ring with identity every ideal has a representation as a finite intersection of primary ideals. The object of the present paper is to generalize this result to a class of near-rings called Q-near-rings which includes rings with dense quasi-centre and consequently all commutative rings.

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1. Preliminaries

We recall that a (right) near-ring $N = (N, +, \cdot)$ is a system where (i) $(N, +)$ is a group which we denote by N^+, (ii) $(N, \cdot)$ is a semigroup, (iii) $a(b + c) = ab + ac$ for all a, b, c in N, and (iv) $0a = 0$ for all a in N where 0 is the identity of N^+. Ideals and right ideals of N are defined in the usual way. An ideal P of a near-ring N is called a prime ideal if for all ideals I and J of N such that $IJ \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$. If I is an ideal of N, call $\mathcal{P}(I) = \bigcap_P P$, where P ranges over all prime ideals of N containing I, the prime radical of I. If $a \in N$ is such that $a \in \mathcal{P}(I)$ then $a^n \in I$ for some $n > 0$ (Pilz (1977), Proposition 2.94). Further $\mathcal{P}(\{0\}) = \mathcal{P}(N)$, the prime radical of N. If A and B are subsets of N, we denote the set $\{n \in N \mid Bn \subseteq A\}$ by $(A : B)$. We denote $(A : \{b\})$ by $(A : b)$.

2. Q-near-rings

In this section a class of near-rings called Q-near-rings is introduced, examples and some properties of such near-rings are presented. We start with the following.

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DEFINITION 2.1. *A near-ring N is called a Q-near-ring if N contains a multiplicatively closed subset Q satisfying the following properties*:

(i) *$a \in Q$ implies aN is a right ideal of N*,

(ii) *$aN = Na$ for all a in Q*,

(iii) *for all ideals A, B of N such that $A \subset B$ (properly) B contains an element of Q which is not in A.*

REMARK. If N is a Q-near-ring and $a \in Q$ then aN is an ideal of N. If N has the identity then $aN = (a)$, the ideal generated by a. Examples of Q-near-rings are:

(1) Any commutative ring.

(2) Any simple ring with 1.

(3) Any division ring.

(4) Any ring with dense quasi-centre (A. P. J. Vander Walt, 1967).

(5) Any near-field.

(6) Any simple near-ring with 1.

(7) Any biregular near-ring (in the sense of Betsch) (Pilz (1977), page 94).

(8) Let G be any additive (not necessarily abelian) group. Define $a \cdot b = 0$ for all a, b in G. Then $(G, +, \cdot)$ is a Q-near-ring.

For Q-near-rings we have the following characterization of prime ideals.

THEOREM 2.2. *An ideal I of a Q-near-ring N with* 1 *is prime if and only if $ab \in I$ with $a, b \in Q$ implies either $a \in I$ or $b \in I$.*

PROOF. Suppose I is a prime ideal of N. Let $a, b \in Q$ and $ab \in I$. Then, $NabN \subseteq I$. Therefore, either $Na \subseteq I$ or $bN \subseteq I$. So $a \in I$ or $b \in I$. Conversely suppose I is an ideal of N such that $A \not\subseteq I$ and $B \not\subseteq I$. Then there exist elements a, b in Q with $a \in A$, $b \in B$, and $a, b \notin I$ (from (iii) of Definition 2.1). Therefore $ab \notin I$. So $AB \not\subseteq I$. Hence I is a prime ideal of N.

COROLLARY 2.3. *Let N be a Q-near-ring with* 1. *An ideal I of N is prime if and only if $Q \cap I'$ is a multiplicatively closed set (where I' is the complement of I in N)*

The proof of this corollary follows directly from Theorem 2.2.

THEOREM 2.4. *If N is a Q-near-ring then every ideal I of N is generated by the elements of Q contained in I.*

PROOF. Suppose I is an ideal of N. Put $S = I \cap Q$. Consider (S), the ideal generated by S in N. Clearly $(S) \subseteq I$. If $(S) \subset I$ (properly), there exists an element $a \in I \cap Q$ such that $a \notin (S)$, that is $a \notin S$. But this is in conflict with the definition of S. Therefore, $(S) = I$.

COROLLARY 2.5. *If N is a Q-near-ring with 1 then $\mathcal{P}(N)$ is the ideal generated by the set of all nilpotent elements of Q.*

The proof is easy and will be omitted.

3. Primary representations

We start with the following:

DEFINITION 3.1. *An ideal I of a near-ring N is called a primary ideal if A, B are ideals of N such that $AB \subseteq I$ then either $A \subseteq \mathcal{P}(I)$ or $B \subseteq I$.*

DEFINITION 3.2. *An ideal I of a near-ring N is called irreducible if $I = A \cap B$ where A and B are ideals of N then either $I = A$ or $I = B$.*

For Q-near-rings we have the following characterization of primary ideals.

LEMMA 3.3. *Let N be a Q-near-ring with 1. An ideal I of N is primary if and only if $ab \in I$ with $a, b \in Q$ implies either $a^n \in I$ for some $n > 0$ or $b \in I$.*

PROOF. Suppose I is an ideal of N satisfying the condition of the lemma. Suppose A, B are ideals of N such that $A \not\subseteq \mathcal{P}(I)$ and $B \not\subseteq I$. Then there exist elements a, b in Q, $a \in A$, $b \in B$ with $a \notin \mathcal{P}(I)$ and $b \notin I$. Suppose, $ab \in I$. Since $b \notin I$, $a^n \in I$ for some $n > 0$. Then $a \in \mathcal{P}(I)$, a contradiction. Hence $ab \notin I$. Therefore, $AB \not\subseteq I$. Hence I is a primary ideal. The converse implication is easy.

The following result shows that to every primary ideal there corresponds a specific prime ideal.

LEMMA 3.4. *Let I be a primary ideal of a Q-near-ring N with 1. Then $\mathcal{P}(I)$ is a prime ideal of N.*

PROOF. Let $ab \in \mathcal{P}(I)$ with $a, b \in Q$. Let n be the least positive integer such that $(ab)^n \in I$. If $n = 1$, $ab \in I$ either $a^k \in I$ for some $k > 0$ or $b \in I$. So we have either $a \in \mathcal{P}(I)$ or $b \in \mathcal{P}(I)$. Suppose $n > 1$. Now, $(ab)^n = a(ba)^{n-1}b \in I$. Hence either $a^m \in I$ for some $m > 0$ or $(ba)^{n-1}b \in I$. If $a^m \in I$ we get $a \in \mathcal{P}(I)$. Suppose $(ba)^{n-1}b \in I$. Now $(ba)^{n-1}b = b(ab)^{n-1} \in I$. Since $(ab)^{n-1} \notin I$, $b^r \in I$ for some $r > 0$ then $b \in \mathcal{P}(I)$. Therefore, $\mathcal{P}(I)$ is a prime ideal of N.

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The following result may be useful for deciding whether a given ideal is actually primary.

THEOREM 3.5. *Let I and J be ideals of a Q-near-ring with* 1 *such that*
(1) $I \subseteq J \subseteq \mathcal{P}(I)$,
(2) $a, b \in Q$ *and* $ab \in I$ *with* $a \notin I$ *then* $b \in J$. *Under these conditions I is a primary ideal of N with* $\mathcal{P}(I) = J$.

The proof of this theorem is easy and will be omitted.

We now state the main theorem of the paper which generalizes the so-called primary Decomposition Theorem of Noether for commutative Noetherian rings.

THEOREM 3.6. *Let N be a Q-near-ring with* 1 *satisfying a.c.c on ideals. Then every ideal of N can be represented as the intersection of a finite number of primary ideals.*

PROOF. It is, of course, sufficient to prove that the condition implies that every irreducible ideal is primary. Suppose I is an irreducible ideal of N and I is not primary. Then there exist elements a, b in Q such that $ab \in I$, $b \in I$ and no power of a belongs to I. Clearly $(I : a)$ is a right ideal of N, since $aN = Na$, $(I : a)$ is an ideal of N. Thus we have an ascending chain of ideals of N: $I \subset (I : a) \subseteq (I : a^2) \subseteq \cdots$. Since N has a.c.c. on ideals there exists an integer n such that $(I : a^n) = (I : a^m)$ for all $m \geqslant n$. Since $a^n \in Q$, a^nN is an ideal of N. Now we claim that $I = (I : a^n) \cap (I + a^nN)$. Let $x \in (I : a^n) \cap (I + a^nN)$. Now, $x = y + a^nt$ for some $y \in I$ and $t \in N$. Then $a^nx = a^ny + a^{2n}t \in I$. Hence $a^{2n}t \in I$. So, $t \in (I : a^{2n}) = (I : a^n)$. Then, $a^nt \in I$. Therefore, $x \in I$. Hence $I = (I : a^n) \cap (I + a^nN)$ where $(I : a^n)$ and $(I + a^nN)$ are ideals of N both of which contain I properly, a contradiction. Therefore every irreducible ideal is primary. This proves the theorem.

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