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A "CONSTANT OF THE MOTION" FOR THE GEODESIC DEVIATION EQUATION

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Abstract

In this short paper, it is shown that the geodesic deviation equation admits a "constant of the motion" and so can be solved exactly. We also derive an expression for the energy E of relative motion between two freely falling test particles. We can infer that, in general, E will *not* be a linear superposition of kinetic and potential energies.

1. Introduction

It is well known that the geodesic deviation equation in general relativity is a physical equation, because it relates the relative acceleration between two test particles to certain physical components of the Riemann-curvature tensor.

In Section 2 we derive an unfamiliar form of the geodesic deviation equation. A first integral or "constant of the motion" is derived in Section 3. We relate this first integral to the existence of an energy E for the relative motion of the two test particles in Section 4.

2. Synge–Jacobi equation

The standard form of the geodesic deviation equation gives an equation of motion of the space-like part of the deviation vector between two test particles in a gravitational field, namely,

$$\frac{\delta^2}{\delta s^2}(\eta^i)+R^{i}_{\cdot jkl}u^j\eta^k u^l=0, \tag{1}$$

where $\eta_i u^i=0$ and $u^i u_i=-1$, with u^i being the unit time-like tangent vector to

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a geodesic as shown in Fig. 1 and covariant differentiation along the vector field u^i being indicated by $\delta/\delta s$ (see [11, 12, 13, 14]).

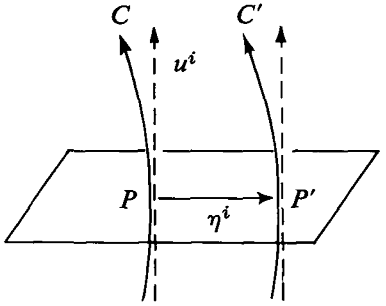

Fig. 1. Deviation vector η^i in the rest space of P.

The form of the equation as given in (1) is extremely difficult to solve exactly except for simple cases. The standard approach is to introduce a tetrad

$$(u^i, e^i_{(\alpha)};\ \alpha = 1, 2, 3),$$

where $e^i_{(\alpha)}$ is parallelly propagated,

$$\frac{\delta}{\delta s}(e^i_{(\alpha)}) = 0,$$

and is space-like orthonormal, that is,

$$\sum_{\alpha=1}^{3} e^i_{(\alpha)} e^j_{(\alpha)} = +\delta^i_j.$$

In this frame equation (1) becomes of the form

$$\left.\begin{aligned} &\frac{d^2\eta_{(\alpha)}}{ds^2} + \sum_\beta K_{\alpha\beta}\eta_{(\beta)} = 0, \\ \eta_{(\alpha)} = e^i_{(\alpha)}\eta_i \quad \text{and} \quad &K_{\alpha\beta} = K_{\beta\alpha} = R_{ijkl}e^i_{(\alpha)}u^j e^k_{(\beta)}u^l. \end{aligned}\right\} \tag{2}$$

However, in general, the matrix $K_{\alpha\beta}$ is not diagonal and so the resulting equations *cannot* be written in the one-dimensional forms

$$\frac{d^2\eta_{(\alpha)}}{ds^2} + L_\alpha\eta_{(\alpha)} = 0, \quad \text{no sum over } \alpha.$$

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We adopt an alternative which is as follows:

(a) The deviation vector is resolved as

$$\eta^i = \eta\mu^i, \tag{3}$$

where $\mu_i\mu^i = +1$. Hence $\mu^i = (0, \mu^\alpha)$ are the direction cosines of the deviation vector in the rest space of P and will depend on the frame of reference chosen.

(b) On substitution of (3) in (1) we obtain

$$\ddot{\eta}\mu^i + 2\dot{\eta}\dot{\mu}^i + \eta\ddot{\mu}^i + \eta R^i_{.jkl} u^j \mu^k u^l = 0, \tag{4}$$

where $\dot{\eta} = d\eta/ds$, $\dot{\mu}^i = \delta\mu^i/\delta s$, $\mu_i\dot{\mu}^i = 0$ and $\mu_i\ddot{\mu}^i + \dot{\mu}_i\dot{\mu}^i_i = 0$. As $\dot{\mu}_i$ is space-like, that is, $\dot{\mu}_i\dot{\mu}^i = \Omega^2 \geqslant 0$, we have $\mu_i\ddot{\mu}^i = -\Omega^2 \leqslant 0$.

(c) On transvecting (4) with μ_i we obtain

$$\ddot{\eta} - \eta\Omega^2 + \eta K = 0, \tag{5}$$

where

$$K = R_{ijkl}\mu^i u^j \mu^k u^l, \tag{6}$$

and the form of the geodesic deviation equation used in this paper,

$$\ddot{\eta} + (K - \Omega^2)\eta = 0, \tag{7}$$

is obtained [11, 12, 13], which we shall call the *Synge–Jacobi equation* as Synge was first to recognize that the n-dimensional geodesic deviation equation can be reduced to the Jacobi equation of two dimensions [11, 12, 13].

It is noteworthy that (7) is similar to the equation of a time-dependent harmonic oscillator

$$\ddot{\eta} + \omega^2(s)\eta = 0 \tag{8}$$

for $K - \Omega^2 \geqslant 0$, and similar to equation (8a) for $K - \Omega^2 \leqslant 0$:

$$\ddot{\eta} - \omega^2(s)\eta = 0. \tag{8a}$$

The coefficient $K - \Omega^2$ in (7) can be expected to change its sign in finite intervals of proper time. We need study only (8) in Section 3 because our results hold also for (8a).

3. The Lewis invariant

In this section we state certain mathematical properties of equation (8):

(i) it possesses a constant of motion which is called the *Lewis invariant*, L, and

(ii) it has an associated differential equation

$$\ddot{\rho} + \omega^2(s)\rho = 1/(\rho^3), \tag{9}$$

which is known as *Pinney's differential equation.* It can be shown that properties (i) and (ii) are equivalent [6, 7, 8].

If ρ is some particular integral of (9), then we can define the *Lewis invariant* as follows:

$$L=\tfrac{1}{2}[(\eta/\rho)^2+(\eta\dot{\rho}-\rho\dot{\eta})^2]. \tag{10}$$

It is easy to verify that L is indeed a constant of motion of (8). Also L is unaffected by the changes of sign of $\pm\omega^2$ (or, in the terms of equation (7), L remains unaffected by changes in the sign of K). Further, we can now formally solve (8).

If $a=\sqrt{(2L)}$ or $L=\tfrac{1}{2}a^2$, and if also $z=\eta/\rho$ and $\Phi=\int(ds/\rho^2)$, then equation (10) can be transformed to

$$a^2=z^2+\rho^4\dot{z}^2=z^2+\left(\frac{dz}{d\Phi}\right)^2,$$

which has for solution $z=a\cos(\Phi+\varepsilon)$ or

$$\eta=a\rho\cos\left[\int^s\frac{ds'}{\rho^2(s')}+\varepsilon\right]. \tag{11}$$

Hence we see that $A(s)=a\rho$ is the *amplitude*, and $\Phi(s)=\int^s(ds'/\rho^2(s'))$ is the *phase* [2, 3].

Now we can invert (11) to obtain

$$\rho(s)=\frac{\eta(s)}{a}\left\{1+\left(\int^s\frac{ds'}{\eta^2(s')}\right)^2\right\}^{\frac{1}{2}}. \tag{12}$$

Thus we know the phase Φ in terms of ρ and so in terms of η [15], that is, in principle, the observable quantity η determines the phase Φ and the amplitude

$$a\rho=\rho\sqrt{(2L)}. \tag{13}$$

For equation (8a) the solution (11) will have the circular function replaced by the hyperbolic. There is a corresponding adjustment to equation (12).

We shall, *in general,* call $W=1/(\rho^2)$ *the analogue of frequency.*

4. Energy received by test particles

Since we have established the concept of an amplitude and a phase for the magnitude of the space-like part of the deviation vector between two test particles in a gravitational field, we introduce the concept of the "energy" E of the relative motion.

We shall adapt a discussion of (11) by Lorentz and Einstein at the 1911 Solvay

Conference: to Lorentz' question as to how the amplitude of a simple pendulum would vary if its period were slowly altered by shortening its string, Einstein replied that the Action $= E/\nu$, where E is the energy and ν the frequency, would remain constant if $\dot{\nu}/\nu$ were small enough (adiabatic invariance). Lewis showed that the hypothesis of adiabatic invariance is unnecessaıy [1, 4, 5, 7].

Assuming L has the dimensions of

$$Action = Energy/Frequency \tag{14}$$

we can use this to *define* the energy E. The analogue of frequency in this context is W. Hence

$$L = E/W = E\rho^2 \tag{15}$$

or

$$E = L/\rho^2 = \frac{1}{2}\left[\left(\frac{\eta}{\rho^2}\right)^2 + \left(\dot{\eta} - \frac{\dot{\rho}}{\rho}\eta\right)^2\right] \tag{16}$$

$$= \frac{1}{2}\left[W^2\eta^2 + \left(\dot{\eta} + \frac{2\dot{W}\eta}{W}\right)^2\right]. \tag{17}$$

We note that E is *not* a linear superposition of kinetic and potential energies.

In de Sitter space–time $K = \omega_0^2$, a constant [14], and it is possible to choose $\dot{\mu}^i = 0$, thus giving

$$E = \tfrac{1}{2}[\omega_0^2\eta^2 + \dot{\eta}^2]. \tag{18}$$

As $K \to 0$ we get Minkowski space–time: $E \to \frac{1}{2}\dot{\eta}^2$ in the absence of gravitation.

Note added in proof. The energy expression E will be shown to remain positive definite in the case of equation (8a) in a later paper.

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