OPERATORS WITH POWERS CLOSE TO A FIXED OPERATOR

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It is intuitively obvious that if z is a complex number such that $|1-z^p| \leqq b < 1$ for all positive integers p and some real number b, then $z = 1$. The purpose of this note is to exhibit a proof of the following generalisation of this observation:

THEOREM. *Let A be a continuous linear operator on a reflexive Banach space B. If there exists a continuous linear operator T on B, a real number b, and a positive integer p' such that*

$$||T-A^p|| \leqq b \text{ for } p \geqq p', \quad p \text{ an integer} \tag{1}$$

and

$$b < \inf \{||Tx|| : x \in B, ||x|| = 1\}, \tag{2}$$

then $A = I$. Moreover, in this case $||I-T|| \leqq b$.

Both in the statement of the theorem and the following discussion I denotes the identity operator on the Banach space B under consideration.

The following lemma is a restatement of the Mean-Ergodic Theorem found in [1, pp. 54—56].

LEMMA. *Let A be a uniformly bounded ($||A^n|| \leqq M, n = 0, 1, \cdots$) continuous linear operator on a reflexive Banach space. Let $V(n)$ be the operator defined by*

$$V(n) = \frac{I+A+\cdots+A^{n-1}}{n}.$$

Then the sequence $\{V(n)\}_{n=1}^{\infty}$ converges strongly to a projection V such that $Vx = x$ if and only if $Ax = x$.

PROOF OF THEOREM. The last observation in the statement of the theorem follows immediately from the hypothesis that $||T-A^p|| \leqq b$ for $p \geqq p'$.

We note first that for $n \geqq 0$

$$||A^n|| \leqq \max \{1, ||A||, \cdots, ||A^{p'-1}||, ||T||+b\}.$$

Thus A satisfies the hypotheses of the lemma. Let

$$V = \lim_{n\to\infty} \frac{I+A+\cdots+A^{n-1}}{n}.$$

For each $r > 0$ and each x in B there exists an integer n' such that

$$\left\| Vx - \frac{I+A+\cdots+A^{n-1}}{n} x \right\| < r \text{ for } n \geqq n'. \tag{3}$$

It follows from (1) and (3) that for $n \geqq n'$ and $n > p'$

$$||(T-V)x|| < r + \frac{1}{n}\left(\sum_{k=0}^{p'-1} ||T-A^k||\,||x||\right) + \frac{(n-p')}{n} b||x||.$$

Taking the limit as n approaches infinity, we have $||(T-V)x|| \leqq r+b||x||$ for each $r > 0$ and each x in B. Consequently,

$$||T-V|| \leqq b. \tag{4}$$

Assume now that (2) holds. If $V \neq I$, there exists an x in B, $||x|| = 1$, such that $Vx = 0$. It follows from inequality (4) that $||Tx|| \leqq b$. However, this contradicts (2). Therefore, $V = I$; using the lemma, we see that $A = I$.

COROLLARY. *Let A be a continuous linear operator on a reflexive Banach space B. If $||I-A^p|| \leqq b < 1$ for some real number b and all positive integers p, then $A = I$.*

The corollary follows immediately from the theorem by taking $T = I$ and $b < 1$. The original observation in this paper is a special case of this corollary.

In concluding this paper I should like to note that hypothesis (2) of the theorem is indeed necessary to force $A = I$, for $A = 0$ satisfies (1) with all T such that $||T|| \leqq b$.

Bibliography

[1] E. R. Lorch, *Spectral Theory* (Oxford University Press, New York, 1962).

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