

## MODULE HOMOMORPHISMS OF THE DUAL MODULES OF CONVOLUTION BANACH ALGEBRAS

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**ABSTRACT.** Suppose that  $A$  is either the group algebra  $L^1(G)$  of a locally compact group  $G$ , or the Volterra algebra or a weighted convolution algebra with a regulated weight. We characterize: a) Module homomorphisms of  $A^*$ , when  $A^*$  is regarded an  $A^{**}$  left Banach module with the Arens product, b) all the weak\*-weak\* continuous left multipliers of  $A^{**}$ .

**Introduction.** Suppose  $A$  is a Banach algebra. The dual Banach space  $A^*$  of  $A$  can be made into a right Banach  $A$ -module and a left  $A^{**}$ -module by the Arens products (see the definitions below). When  $A$  is the group algebra  $L^1(G)$  of a locally compact group, the bounded  $A$ -module homomorphisms of  $A^*$  were given in [2], see also [11] and [14]. One of the results in this paper is a characterization of bounded  $A^{**}$ -module homomorphisms of  $A^*$  when  $A$  is  $L^1(G)$ , or the Volterra algebra or a weighted convolution algebra with a regulated weight.

Recall that if  $A$  is a Banach algebra then the first (or left) Arens product in  $A^{**}$  is defined as follows. For  $f \in A^*$  and  $a \in A$  define  $fa \in A^*$  by  $\langle fa, b \rangle = \langle f, ab \rangle$ , ( $b \in A$ ). For  $F \in A^{**}$ ,  $f \in A^*$  define  $Ff \in A^*$  by  $\langle Ff, a \rangle = \langle F, fa \rangle$ , ( $a \in A$ ). Finally for  $F$  and  $G$  in  $A^{**}$  define  $FG$  in  $A^{**}$  by  $\langle FG, f \rangle = \langle F, Gf \rangle$ , ( $f \in A^*$ ). Then  $A^{**}$  with this product is a Banach algebra, and the product in  $A^{**}$  extends the product of  $A$  as canonically embedded in  $A$ . Furthermore,  $A^*$  is a left  $A^{**}$ -module by the module product  $Ff$ , ( $F \in A^{**}$ ,  $f \in A^*$ ) and a right  $A$ -module by the module product  $fa$ , ( $f \in A^*$ ,  $a \in A$ ). See [3] for more details. We denote the canonical mapping of  $A$  in  $A^{**}$  by  $\pi$ .

It is well known (and easy to prove) that for every  $m \in A^{**}$  the mapping  $n \mapsto nm$  ( $n \in A^{**}$ ) is weak\*-weak\* continuous. The set of all  $m$  in  $A^{**}$  for which  $n \mapsto mn$  ( $n \in A^{**}$ ) is weak\*-weak\* continuous is called the topological centre (briefly, centre) of  $A^{**}$ . The topological centre of  $A^{**}$  contains  $\pi(A)$ . N. Isik, J. S. Pym and A. Ülger have proved that if  $G$  is a compact topological group, then the centre of  $(L^1(G))^{**}$  is precisely  $\pi(L^1(G))$  [9]. A. T. Lau and V. Losert have extended this result to every locally compact group [13]. Since an element  $m$  belongs to the centre if and only if the left multiplier  $\lambda_m: n \mapsto nm$  ( $n \in A^{**}$ ) is weak\*-weak\* continuous, a question about the description of all weak\*-weak\* continuous left multipliers becomes natural. We answer this question when  $A$  is either the group algebra  $L^1(G)$  or the Volterra algebra  $V$  or a weighted convolution algebra with a regulated weight. Our characterization of weak\*-continuous multipliers

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builds upon the description of the centre of  $L^1(G)^{**}$  obtained in [13] and [9]. However, for abelian groups we even give a new proof to show that the centre of  $L^1(G)^{**}$  is  $\pi(L^1(G))$ .

As usual we will let  $L^1(G)$  be the group algebra of a locally compact group and  $M(G)$  be the measure algebra of  $G$ .

We also need to recall the definition of a Banach algebra closely related to  $L^1(G)^{**}$ . Let  $C(G)$  be the Banach space of all complex valued continuous and bounded functions on  $G$  with the sup-norm. Then  $\text{LUC}(G)$  is defined to be the space of all  $f \in C(G)$  for which the mapping  $x \mapsto \ell_x f$  from  $G$  into  $C(G)$  is continuous, where  $\ell_x f(y) = f(xy)$ , ( $y \in G$ ). An Arens product can be defined in  $\text{LUC}(G)^*$  which makes  $\text{LUC}(G)^*$  a Banach algebra. The product is as follows: for  $n \in \text{LUC}(G)^*$  and  $f \in \text{LUC}(G)$  define  $nf \in \text{LUC}(G)$  by  $(nf)(x) = \langle n, \ell_x f \rangle$ , ( $x \in G$ ). For  $m$  and  $n$  in  $\text{LUC}(G)^*$  define  $mn \in (\text{LUC}(G))^*$  by  $\langle mn, f \rangle = \langle m, nf \rangle$ , ( $f \in \text{LUC}(G)$ ). As with  $A^{**}$ , right multiplication by a fixed element is weak\*-weak\* continuous, and the centre of  $\text{LUC}(G)^*$  is, by definition, the set of elements for which left multiplication is weak\*-weak\* continuous.

Suppose  $(e_i)$  is a bounded approximate identity of  $L^1(G)$  bounded by 1, such that  $\pi(e_i)$  converges weak\* to an element  $E$  of  $L^1(G)^{**}$ . Then  $E$  is a right identity of  $L^1(G)^{**}$  and there exists an isometric embedding  $\Gamma_E: M(G) \rightarrow L^1(G)^{**}$  defined by  $\Gamma_E(\mu) = \text{weak}^*\text{-}\lim \pi(e_i * \mu)$ , ( $\mu \in M(G)$ ). See [7, p. 277] for more details.

**1. Results about  $L^1(G)$ .** For convenience, we state the characterization of the centre of  $L^1(G)^{**}$ , due to Lau and Losert [12] and we also give a new proof for the case of abelian groups, based on the characterization of the centre of  $\text{LUC}(G)^*$  [12, Theorem 1].

**THEOREM 1.1.** *For a locally compact group  $G$ , the (topological) centre of  $L^1(G)^{**}$ , is  $\pi(L^1(G))$ .*

**PROOF, FOR ABELIAN  $G$ .** Let  $(e_i)$  be any bounded approximate identity for  $L^1(G)$ , with  $\|e_i\| \leq 1$  for every  $i$  and such that  $\pi(e_i)$  converges weak\* to a right identity  $E$  in  $L^1(G)^{**}$ . By [6, p. 343], there is an isometric (algebra) isomorphism  $\theta_E$  from  $EL^1(G)^{**}$  onto  $\text{LUC}(G)^*$ , defined by  $\langle \theta_E(Em), f \rangle = \langle m, f \rangle$  (for  $m \in L^1(G)^{**}$  and  $f \in \text{LUC}(G)$ ).

For any  $x$  in  $L^1(G)$ , we have  $E\pi(x) = \pi(x)$ . It follows that if  $x$  and  $y$  are in  $L^1(G)$ , and  $n$  is in  $L^1(G)^{**}$ ,  $\theta_E(\pi(x)n\pi(y))$  is defined. We claim that if  $n$  is in the centre of  $L^1(G)^{**}$ , then  $\theta_E(\pi(x)n\pi(y))$  is in the centre of  $\text{LUC}(G)^*$ , for any  $x, y$  in  $L^1(G)$ . To see that, let  $m_i$  be any net in  $\text{LUC}(G)^*$ , converging weak\* to  $m$ . Let  $m'_i$  (respectively,  $m'$ ) be a Hahn-Banach extension of  $m_i$  (respectively,  $m$ ) to an element of  $L^1(G)^{**}$  (recall that  $\text{LUC}(G)$  is isometrically embedded in  $L^\infty(G)$ ). Then, for  $y$  in  $L^1(G)$ ,  $(\pi(y)m'_i)$  converges weak\* (in  $L^1(G)^{**}$ ) to  $\pi(y)m'$ . To see that, first note that if we put  $\tilde{y}(t) = y(t^{-1})$  for  $y$  in  $L^1(G)$  and  $t$  in  $G$ , then  $\tilde{y}f = \tilde{y} * f$  for  $f$  in  $L^\infty(G)$  and  $y$  in  $L^1(G)$  [16]. Therefore,  $\tilde{y}$  is in  $\text{LUC}(G)$  [8,

p. 298]. Therefore

$$\begin{aligned}
 \langle \pi(y)m'_i, f \rangle &= \langle \pi(y), m'_i f \rangle \\
 &= \langle m'_i f, y \rangle \\
 (1.2) \qquad &= \langle m'_i, f y \rangle \\
 &\rightarrow \langle m', f y \rangle \\
 &= \langle \pi(y)m', f \rangle
 \end{aligned}$$

for any  $f$  in  $L^\infty(G)$ . Thus,  $(\pi(y)m'_i)$  converges weak\* to  $\pi(y)m'$ . It is easy to check that  $\pi(x)n$  is in the centre of  $L^1(G)^{**}$  if  $n$  is; so we get

$$(1.3) \qquad \langle \pi(x)n\pi(y)m'_i, f \rangle \rightarrow \langle \pi(x)n\pi(y)m', f \rangle \quad (\text{for } f \text{ in } L^\infty(G)).$$

Since  $\pi(x)n\pi(y)m'_i = E\pi(x)n\pi(y)Em'_i$ , we have

$$\theta_E(\pi(x)n\pi(y)m'_i) = \theta_E(\pi(x)n\pi(y))\theta_E(Em'_i) = \theta_E(\pi(x)n\pi(y))m_i$$

and similarly

$$\theta_E(\pi(x)n\pi(y)m') = \theta_E(\pi(x)n\pi(y))\theta_E(Em') = \theta_E(\pi(x)n\pi(y))m.$$

Combining this with (1.3) and the definition of  $\theta_E$ , we conclude

$$(1.4) \qquad \langle \theta_E(\pi(x)n\pi(y))m_i, f \rangle \rightarrow \langle \theta_E(\pi(x)n\pi(y))m, f \rangle \quad (\text{for } f \text{ in } \text{LUC}(G)).$$

Thus,  $\theta_E(\pi(x)n\pi(y))$  is in the centre of  $\text{LUC}(G)^*$ , as claimed.

Now, by Lau's characterization of the centre of  $\text{LUC}(G)^*$  [12, Theorem 1], there is a measure  $\mu$  in  $M(G)$  such that  $\theta_E(\pi(x)n\pi(y)) = \mu$ . Note that  $(\pi(x)n\pi(y))$  is independent of the right identity  $E$ , so, since  $\theta_E$  is just a restriction mapping, we see that the measure  $\mu = \theta_E(\pi(x)n\pi(y))$  is independent of  $E$ .

Applying [7, Proposition 2.8], we get

$$(1.5) \qquad \pi(x)n\pi(y) = \theta_E^{-1}(\mu) = \Gamma_E(\mu).$$

Since (1.5) holds for every right identity  $E$  of  $L^1(G)^{**}$  by [7, Proposition 2.4 (ii)],

$$(1.6) \qquad \pi(x)n\pi(y) \in \pi(L^1(G))$$

Since  $G$  is abelian,  $\pi(x)n = n\pi(x)$  for any  $x$  in  $L^1(G)$ . By Cohen's factorization theorem if  $z \in L^1(G)$  then  $z = x * y$ , for some  $x$  and  $y$  in  $L^1(G)$ . Thus  $n\pi(z) = n\pi(x)\pi(y) = \pi(x)n\pi(y)$ , and from (1.6) we conclude that  $n\pi(z)$  belongs to  $\pi(L^1(G))$ , for each  $z$  in  $L^1(G)$ . Then it is easy to check that  $z \mapsto \pi^{-1}(n\pi(z))$  is a right multiplier on  $L^1(G)$ . By Wendel's characterization [15], there is a measure  $\mu$  in  $M(G)$  such that  $\pi(z)n = \pi(z * \mu)$  for  $z$  in  $L^1(G)$ . Again let  $(e_i)$  be a bounded approximate identity in  $L^1(G)$  such that  $\pi(e_i)$  converges weak\* to a right identity  $E$  with  $\|E\| = 1$ . Then since  $n$  is in the centre we have  $n = nE = \text{weak}^*\text{-lim } n\pi(e_i) = \text{weak}^*\text{-lim } \pi(e_i)n = \text{weak}^*\text{-lim } \pi(e_i * \mu) = \Gamma_E(\mu)$ .

Since this is independent of  $E$ , we again apply [7, Proposition 2.4 (ii)], and conclude that  $n$  belongs to  $\pi(L^1(G))$ . That completes the proof.

Now we use the above result of Lau and Losert to obtain a representation of all the weak\*-weak\* continuous left multipliers of  $L^1(G)^{**}$ , and to describe all the bounded module homomorphisms of  $L^\infty(G)$ , when  $L^\infty(G)$  is an  $L^1(G)^{**}$ -module, by the Arens product. For convenience, we summarize in a lemma some facts about the interaction of a continuous multiplier  $T$  on a Banach algebra  $A$ , and its adjoints, with the module actions of  $A^{**}$  on  $A^*$ , and with the Arens product on  $A^{**}$ .

LEMMA 1.7. *Let  $T$  be a bounded left multiplier on a Banach algebra  $A$ . Then*

- (i) *for  $a$  in  $A$  and  $f$  in  $A^*$ ,  $fT(a) = T^*(f)a$ ;*
- (ii) *for  $f$  in  $A^*$  and  $n$  in  $A^{**}$ ,  $T^*(nf) = nT^*(f)$ ; so  $T^*$  is an  $A^{**}$ -module homomorphism of  $A^*$ ;*
- (iii) *an operator  $U$  on  $A^{**}$  is a weak\*-continuous left multiplier if and only if  $U = S^*$  for some bounded  $A^{**}$ -module homomorphism  $S$  on  $A^*$ . In particular  $T^{**}$  is a weak\*-continuous left multiplier on  $A^{**}$ .*

PROOF. (i) follows directly from

$$\langle fT(a), b \rangle = \langle f, T(ab) \rangle = \langle f, T(ab) \rangle = \langle T^*f, ab \rangle = \langle (T^*f)a, b \rangle \quad (f \in A^*, a, b \in A).$$

To prove (ii) use (i). Finally to prove (iii) use (ii) and the fact that an operator on the dual space  $X^*$  of a Banach space  $X$  is weak\*-weak\* continuous if and only if it is the adjoint of a bounded operator on  $X$ .

For the next result, we introduce the following notation. If  $\mu$  is a (bounded Borel) measure on the locally compact group  $G$ , we write  $\lambda_\mu$  for the left multiplier on  $L^1(G)$  defined by  $\lambda_\mu(f) = \mu * f$ .

THEOREM 1.8. (a) *A bounded operator  $T$  on  $L^1(G)^{**}$  is a weak\*-weak\* continuous left multiplier if and only if  $T = (\lambda_\mu)^{**}$  for some  $\mu$  in  $M(G)$ .*

(b) *A bounded operator  $S$  on  $L^\infty(G)$  is an  $L^1(G)^{**}$ -module homomorphism if and only if  $S = (\lambda_\mu)^*$  for some  $\mu$  in  $M(G)$ .*

PROOF. Recall that the left multipliers on  $L^1(G)$  are exactly the operators  $\lambda_\mu$ , for  $\mu$  in  $M(G)$  [15]. Lemma 1.7 now implies the ‘‘if’’ part of both (a) and (b). It also follows from Lemma 1.7 (iii) that (a) and (b) are equivalent. Thus, it is enough to prove: if  $T$  is a weak\*-continuous left multiplier on  $A^{**}$ , then  $T = (\lambda_\mu)^{**}$  for some  $\mu$  in  $M(G)$ . We first show  $T(\pi(L^1(G))) \subseteq \pi(L^1(G))$ . By Theorem (1.1), it is enough to prove that  $T(\pi(x))$  is in the centre of  $L^1(G)^{**}$ , for each  $x$  in  $L^1(G)$ . Let  $n_i$  be a net in  $L^1(G)^{**}$ , converging weak\* to an element  $n$ . Since  $\pi(x)$  is in the centre of  $L^1(G)^{**}$ , we have  $\pi(x)n_i \rightarrow \pi(x)n$  (weak\*). Since  $T$  is weak\*-continuous and a left multiplier, we have  $T(\pi(x))n_i = T(\pi(x)n_i) \rightarrow T(\pi(x)n) = T(\pi(x))n$ . Thus  $T(\pi(x))$  is in the centre of  $L^1(G)^{**}$ . From this it easily follows that  $\pi^{-1}T\pi$  is a left multiplier on  $L^1(G)$ , so by the characterization in [15], there is a measure  $\mu$  in  $M(G)$  such that  $\pi^{-1}T\pi = \lambda_\mu$ . Thus  $T\pi = \pi\lambda_\mu$  on  $L^1(G)$ ; that implies  $T = (\lambda_\mu)^{**}$  on  $\pi(L^1(G))$ ; since  $\pi(L^1(G))$  is weak\*-dense in  $L^1(G)^{**}$  [4, p. 425], and both operators are weak\*-continuous, we have  $T = (\lambda_\mu)^{**}$ , as required.

2. **Topological centres of  $L^1(R^+, w)^{**}$  and  $L^1(0, 1)^{**}$ .** Now we examine the centre of  $A^{**}$  when  $A$  is either a weighted convolution algebra  $L^1(R^+, w)$  with a regulated weight or the Volterra algebra  $V = L^1(0, 1)$ . Here  $w$  is a continuous positive function on  $R^+$  with  $w(s+t) \leq w(s)w(t)$  and  $w(0) = 1$ . The algebra  $L^1(R^+, w)$  consists of all (equivalence classes) of functions  $f$  satisfying

$$\|f\|_w = \int_{R^+} |f(x)|w(x) dx < \infty,$$

where the product  $*$  is given by convolution

$$(2.1) \quad (f * g)(x) = \int_0^x f(x-y)g(y) dy \quad (f, g \in L^1(R^+, w), a. e. x \in R^+).$$

The product in the Volterra algebra is also given by convolution as in formula (2.1).

A weight  $w$  is said to be regulated at 0 if  $\lim_{x \rightarrow \infty} \frac{w(x+y)}{w(x)} = 0$  for all  $y > 0$  [1]. If  $w$  is regulated at 0, then for every  $f \in L^1(R^+, w)$ , the operator  $\rho_f: g \mapsto g * f$  ( $g \in L^1(R^+, w)$ ) is compact [1, p. 90]. It is also well known that the action of the elements of the Volterra algebra is compact. Both  $L^1(R^+, w)$  and  $V$  have bounded approximate identities bounded by 1. For example the sequence  $e_n = n\chi_{[0, 1/n]}$ ,  $n = 1, 2, \dots$ , is a bounded approximate identity for  $V$  and  $L^1(R^+, w)$ .

**THEOREM 2.2.** *Suppose that  $A$  is either a weighted convolution algebra with weight regulated at zero, or the Volterra algebra  $V$ . Then,*

- a) *the centre of  $A^{**}$  is the image of  $A$  under the canonical mapping,*
- b) *a left multiplier  $T$  on  $A^{**}$  is weak\*-weak\* continuous if and only if  $T = \rho_\mu^{**}$ ,*
- c) *a linear mapping  $S$  is a module homomorphism  $A^*$  (when  $A$  is regarded an  $A^{**}$  module) if and only if  $S = \rho_\mu^*$ .*

**PROOF.** It suffices to prove part (a). The proof of parts (b) and (c) are similar to the proofs of parts (a) and (b) of Theorem 1.8.

Since the elements of  $A$  act compactly by multiplication, the image of  $A$  under the canonical mapping of  $A$  in  $A^{**}$  is a two-sided ideal in  $A^{**}$  [3]. Now suppose  $n$  is in the centre of  $A^{**}$ . Since for every  $x \in A$ ,  $n\pi(x) \in \pi(A)$ , by identification of the multiplier algebra of  $A$  with a measure algebra [10, Remark 10] and [5, Theorem 1.4] there exists a measure  $\mu$  in the multiplier algebra of  $A$  such that  $n\pi(x) = \pi(\mu * x)$ , ( $x \in A$ ).

Let  $(e_i)$  be a bounded approximate identity of  $A$  such that  $(\pi(e_i))$  converges weak\* to a right identity  $E$  of  $A^{**}$ . Then  $n = nE = \text{weak}^*\text{-lim } n\pi(e_i) = \text{weak}^*\text{-lim } \pi(\mu * e_i)$ . Since this holds for every such bounded approximate identity  $(e_i)$ , a similar argument to the one in the proof of [7, Proposition 2.4] shows that  $\mu$  is in  $A$ . Hence  $n \in \pi(A)$ , and the proof is complete.

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