A GENERALISATION OF A RESULT OF ABEL WITH AN APPLICATION TO TREE ENUMERATIONS

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1. Introduction

We prove the following theorem, which was established by Abel (1) for the case $u = 1$.

Theorem. *If u, k are positive integers and $x, \alpha_1, \ldots, \alpha_u, \beta$ are real numbers, then*

$$\left(x+\sum_{j=1}^{u}\alpha_j\right)^k = \sum_{l=0}^{k}\sum_{s_1,\ldots,s_u}^{(l)}\frac{k!}{(k-l)!s_1!\ldots s_u!}(x+l\beta)^{k-l}\prod_{i=1}^{u}\alpha_i(\alpha_i-s_i\beta)^{s_i-1}, \quad \ldots(1)$$

where the sum $\sum\limits_{s_1,\ldots,s_u}^{(l)}$ is taken over all distinct ordered solutions $(s_1, \ldots, s_u)$ in non-negative integers of the equation $\sum\limits_{i=1}^{u} s_i = l$.

It is clear that, when $\beta = 0$, equation (1) reduces to the multinomial expansion. The theorem is applied in Section 3 to obtain a proof by induction of the well-known result of Cayley that the number of rooted trees with n distinct nodes is n^{n-1}.

2. Proof of the Theorem

The theorem is proved by induction on $k+u$. It is trivial to verify that (1) holds for $k = 1$ and all u. In (**1**) Abel showed that (1) is true for $u = 1$ and all k. We assume that (1) holds for $k = m-1$, $u = v$, for $k = m-1$, $u = v-1$ and for $k = m$, $u = v-1$, and we prove that then (1) holds for $k = m$, $u = v$.

By the hypothesis then, we have

$$\left(x+\sum_{j=1}^{v}\alpha_j\right)^{m-1} = \sum_{l=0}^{m-1}\sum_{s_1,\ldots,s_v}^{(l)}\frac{(m-1)!}{(m-l-1)!s_1!\ldots s_v!}(x+l\beta)^{m-l-1} \times\prod_{i=1}^{v}\alpha_i(\alpha_i-s_i\beta)^{s_i-1}. \quad \ldots\ldots(2)$$

Integrating with respect to x, we obtain

$$\left(x+\sum_{j=1}^{v}\alpha_j\right)^{m} = \sum_{l=0}^{m-1}\sum_{s_1,\ldots,s_v}^{(l)}\frac{m!}{(m-l)!s_1!\ldots s_v!}(x+l\beta)^{m-l} \times\prod_{i=1}^{v}\alpha_i(\alpha_i-s_i\beta)^{s_i-1}+C, \quad \ldots\ldots(3)$$

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where C is independent of x. Multiplying (2) by $m\beta$, adding to (3) and then substituting $x = -m\beta$, we obtain

$$C = m\beta\left(\sum_{j=1}^{v}\alpha_j - m\beta\right)^{m-1} + \left(\sum_{j=1}^{v}\alpha_j - m\beta\right)^m$$

$$= m\beta\sum_{l=0}^{m-1}\sum_{s_1,\ldots,s_{v-1}}^{(l)}\frac{(m-1)!}{(m-l-1)!s_1!\ldots s_{v-1}!}\{\alpha_v-(m-l)\beta\}^{m-l-1}\times\prod_{i=1}^{v-1}\alpha_i(\alpha_i-s_i\beta)^{s_i-1}$$

$$+\sum_{l=0}^{m}\sum_{s_1,\ldots,s_{v-1}}^{(l)}\frac{m!}{(m-l)!s_1!\ldots s_{v-1}!}\{\alpha_v-(m-l)\beta\}^{m-l}\prod_{i=1}^{v-1}\alpha_i(\alpha_i-s_i\beta)^{s_i-1}$$

$$= \sum_{l=0}^{m}\sum_{s_1,\ldots,s_{v-1}}^{(l)}\frac{m!}{(m-l)!s_1!\ldots s_{v-1}!}\alpha_v\{\alpha_v-(m-l)\beta\}^{m-l-1}\prod_{i=1}^{v-1}\alpha_i(\alpha_i-s_i\beta)^{s_i-1}$$

$$= \sum_{s_1,\ldots,s_v}^{(m)}\frac{m!}{s_1!\ldots s_v!}\prod_{i=1}^{v}\alpha_i(\alpha_i-s_i\beta)^{s_i-1}.$$

This relation in conjunction with (3) completes the proof by induction.

3. Enumeration of Rooted Trees

We now prove by induction that the number of rooted trees with n distinct nodes is n^{n-1}. As there is only one rooted tree with one node, the formula holds for $n = 1$. We assume that the number of rooted trees with i nodes is i^{i-1} for all $i \leqq n$. Now, rooted trees with $n+1$ nodes are formed by first choosing any one of the $n+1$ nodes as root and joining it in any one of $\binom{n}{r}$ ways to r of the other nodes. The remaining $n-r$ nodes are divided into r ordered sets (some of which may be empty), each of which forms a tree with one of the previous r nodes as root. Therefore, the number of rooted trees with $n+1$ nodes is equal to

$$(n+1)\sum_{r=1}^{n}\binom{n}{r}\sum_{s_1,\ldots,s_r}^{(n-r)}\frac{(n-r)!}{s_1!\ldots s_r!}(s_1+1)^{s_1-1}\ldots(s_r+1)^{s_r-1}$$

and this is equal to $(n+1)^n$ whenever, for $1\leqq r\leqq n$,

$$\binom{n}{r}\sum_{s_1,\ldots,s_r}^{(n-r)}\frac{(n-r)!}{s_1!\ldots s_r!}(s_1+1)^{s_1-1}\ldots(s_r+1)^{s_r-1} = \binom{n-1}{r-1}n^{n-r},$$

i.e. whenever

$$\sum_{s_1,\ldots,s_r}^{(n-r)}\frac{(n-r)!}{s_1!\ldots s_r!}(s_1+1)^{s_1-1}\ldots(s_r+1)^{s_r-1} = rn^{n-r-1}. \qquad \ldots\ldots\ldots\ldots(4)$$

Now,

$$\frac{(n-r)!}{s_1!\ldots s_r!} = \frac{(n-r-1)!}{s_1!\ldots s_r!}\sum_{l=1}^{r}s_l = \sum_{l=1}^{r}\frac{(n-r-1)!}{s_1!\ldots s_{l-1}!(s_l-1)!s_{l+1}!\ldots s_r!}, \qquad \ldots\ldots(5)$$

where every $s_l > 0$. When some s_l are zero the corresponding terms are omitted in the final summation of (5). Therefore the left side of (4) is equal to

$$\sum_{l=1}^{r} \sum_{s_1, \ldots, s_r}^{(n-r-1)} \frac{(n-r-1)!}{s_1! \ldots s_r!} (s_1+1)^{s_1-1} \ldots (s_{l-1}+1)^{s_{l-1}-1}(s_l+2)^{s_l}(s_{l+1}+1)^{s_{l+1}-1} \ldots (s_r+1)^{s_r-1}$$

$$= r \sum_{s_1, \ldots, s_r}^{(n-r-1)} \frac{(n-r-1)!}{s_1! \ldots s_r!} (s_1+1)^{s_1-1} \ldots (s_{r-1}+1)^{s_{r-1}-1}(s_r+2)^{s_r}$$

and so, from (4), the formula is verified if

$$\sum_{s_1, \ldots, s_r}^{(n-r-1)} \frac{(n-r-1)!}{s_1! \ldots s_r!} (s_1+1)^{s_1-1} \ldots (s_{r-1}+1)^{s_{r-1}-1}(s_r+2)^{s_r} = n^{n-r-1}.$$

This relation follows from (1) by putting $u = r-1$, $k = n-r-1$, $x = n-r+1$, $\alpha_1 = \ldots = \alpha_n = 1$, $\beta = -1$, and so the proof is complete.

REFERENCE

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