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ISOMETRIC SHIFT OPERATORS ON $C(X)$

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ABSTRACT. Recently A. Gutek, D. Hart, J. Jamison and M. Rajagopalan have obtained many significiant results concerning shift operators on Banach spaces. Using a result of Holsztynski they classify isometric shift operators on $C(X)$ for any compact Hausdorff space X into two (not necessarily disjoint) classes. If there exists an isometric shift operator $T: C(X) \longrightarrow C(X)$ of type II, they show that X is necessarily separable. In case T is of type I, they exhibit a *paticular* infinite countable set $D = \{p, \psi^{-1}(p), \psi^{-2}(p), \psi^{-3}(p), \ldots\}$ of isolated points in X. Under the additional assumption that the linear functional Γ carrying $f \in C(X)$ to $Tf(p) \in \mathbb{C}$ is identically zero, they show that D is dense in X. They raise the question whether D will still be dense in X even when $\Gamma \neq 0$. In this paper we give a negative answer to this question. In fact, given any integer $l \geq 1$, we construct an example of an isometric shift operator $T: C(X) \longrightarrow C(X)$ of type I with $X \setminus \bar{D}$ having exactly l elements, where $\bar{D}$ is the closure of D in X.

1. **Introduction.** R. M. Crownover [1] was the first person to give a basis free definition of a shift on a general Banach space. In [3] J. R. Holub studied isometric shift operators on $C_{\mathbb{R}}(X)$, where $C_{\mathbb{R}}(X)$ is the real Banach space of real valued continuous functions on the compact Hausdorff space X. One of the results proved by him asserts that if X has only finitely many components then $C_{\mathbb{R}}(X)$ does not admit an isometric shift operator. However his techniques do not carry over to the complex Banach space $C_{\mathbb{C}}(X)$. In [2] Gutek *et al* study simultaneously the real as well as the complex case.

We follow the convention that maps between topological spaces are necessarily continuous. In the work of Gutek *et al* [2] a crucial role is played by a result of W. Holsztynski [4] which essentially describes the form of a linear isometry $T: C(X) \longrightarrow C(Y)$ where X and Y are any two compact Hausdorff spaces. Here $C(X)$ denotes the complex Banach space of complex valued continuous functions on X. Using Holsztynski's result they classify isometric shift operators $T: C(X) \longrightarrow C(X)$ into two (but not mutually exclusive) types (Theorem 2.1 in [2]). On p. 100 of [2] three examples are described. Example 2 gives an isometric shift operator of type I which is not of type II whereas Example 3 yields an isometric shift operator which is simultaneously of both types. We denote the range of an operator T by $R(T)$. After proving Theorem 2.1 the authors of [2] correctly remark that the only element $f \in R(T)$ vanishing on X_0 (using the notation in [2]) is 0. On p. 100 of [2] they further assert that when $X_0 \neq X$, the above observation gives the "uniqueness" of p where $X_0 = X \setminus \{p\}$. It is not clear to us what the authors

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have in mind. Some of our results in §4 are devoted to clarifying the situation. Actually Example 3 on p. 100 of [2] turns out to be an isometric shift operator expressible as a shift operator of type I in *two different ways*. Also it turns out that any isometric shift operator $T: C(X) \longrightarrow C(X)$ expressible as an operator of type I in two different ways is automatically of type II. But the converse is not true. We will give a specific example of an isometric shift operator which is simultaneously of types I and II but is expressible as an operator of type I in exactly one way.

Let $T: C(X) \longrightarrow C(X)$ be an isometric shift operator of type I which is not of type II. Then our observation in the earlier paragraph yields a *unique* isolated point p in X, a homeomorphism $\psi: X_0 \longrightarrow X$ where $X_0 = X \setminus \{p\}$ and a map $w: X_0 \longrightarrow S^1$ satisfying

$$Tf(y) = w(y)f\big(\psi(y)\big) \quad \text{for all } y \in X_0, f \in C(X). \tag{1}$$

The statement "the only element $f \in R(T)$ vanishing on X_0 is 0" is equivalent to asserting that the characteristic function χ_p of p is not in $R(T)$. A natural question is whether p is the only isolated point in X with $\chi_p \notin R(T)$. In §4 we will also see that the answer to this question is negative. We will see that Example 2, p. 100 of [2] satisfies the condition that none of χ_1, χ_2 and χ_3 is in $R(T)$.

Let $T: C(X) \longrightarrow C(Y)$ be any linear isometry. In [4] Holsztynski gives a *specific construction* yielding a well determined closed subset Y_0 of Y and well determined maps $\psi: Y_0 \longrightarrow X$, $w: Y_0 \longrightarrow S^1$ with ψ surjective and satisfying

$$Tf(y) = w(y)f\big(\psi(y)\big) \quad \text{for all } y \in Y_0,\ f \in C(X). \tag{2}$$

One of our major results in §2 is a "*universal property*" possessed by Holsztynski's triple $\{Y_0, \psi, w\}$ (Theorem 2.1). This result has some important consequences which will be discussed in §2.

In §5, given any integer $l \geq 1$ we construct an isometric shift operator $T: C(X) \longrightarrow C(X)$ of type I with $X \setminus \bar{D}$ having exactly l elements. One of the results proved in [2] asserts that if $X = S^n$ the n-sphere or I^n the n-cube then $C(X)$ does not admit an isometric shift operator. Using Theorem 2.6 of [2] this result can easily be generalised. We will show that if M^n is any compact topological manifold with or without boundary then $C(M^n)$ does not admit an isometric shift operator. Actually it turns out that *some* of the results proved in [2] are valid for linear isometries $T: C(X) \longrightarrow C(X)$ with codimension of $R(T)$ in $C(X)$ equal to 1. T need not be a shift operator; namely T need not satisfy the condition $\bigcap_{n\geq 1} R(T^n) = \{0\}$. Our exposition will take this fact into account and clearly point out results which are valid for codimension 1 linear isometries. Actually in §6 we show that $C(M^n)$ does not admit a codimension 1 linear isometry when M^n is a compact manifold.

2. **Universal property of Holsztynski's construction.** For any compact Hausdorff space X let $C(X)$ denote the complex Banach space of complex valued continuous functions on X. Throughout this section X, Y will denote compact Hausdorff spaces and $T: C(X) \longrightarrow C(Y)$ a linear isometry. In [4] Holsztynski describes a *specific construction*

yielding a closed subset Y_0 of Y, well determined maps $\psi\colon Y_0 \longrightarrow X$, $w\colon Y_0 \longrightarrow S^1$ with ψ surjective and satisfying

$$Tf(y) = w(y)f\big(\psi(y)\big) \quad \forall y \in Y_0,\ f \in C(X). \tag{3}$$

We will refer to $\{Y_0, \psi, w\}$ obtained as above as Holsztynski's triple associated to the linear isometry $T\colon C(X) \longrightarrow C(Y)$. We actually need this specific construction. Hence we briefly describe this construction.

For any $x \in X$ let $S_x = \{f \in C(X) \mid \|f\| = 1 = |f(x)|\}$ and $Q_x = \{y \in Y \mid T(S_x) \subset S_y\}$ (where of course $S_y = \{g \in C(Y) \mid \|g\| = 1 = |g(y)|\}$. Holsztynski shows that $Q_x \neq \emptyset$ for any $x \in X$, $Q_x \cap Q_{x'} = \emptyset$ if $x \neq x'$ in X, $Y_0 = \bigcup_{x\in X} Q_x$ is closed in Y and that $\psi\colon Y_0 \longrightarrow X$ defined by $\psi(y) = x$ for any $y \in Q_x$ is continuous. Since $Q_x \neq \emptyset$ for each $x \in X$, it is clear that ψ is surjective. If $w(y) = T1(y)$ where $1 \in C(X)$ is the constant function assigning 1 to each $x \in X$ then it is shown in [4] that Y_0, ψ, w satisfy (3). Also in (3) if we substitute $f = 1 \in C(X)$ we get $w(y) = T1(y)$ for all $y \in Y_0$. This shows that w is unique. The following theorem shows that Holsztynski's triple $\{Y_0, \psi, w\}$ possesses a universal property.

THEOREM 2.1. *Let A be any subspace (not necessarily closed) of Y and $\varphi\colon A \longrightarrow X$, $u\colon A \longrightarrow S^1$ maps satisfying*

$$Tf(a) = u(a)f\big(\varphi(a)\big) \quad \forall a \in A \text{ and } f \in C(X). \tag{4}$$

Then $A \subseteq Y_0$, $\varphi = \psi|A$ and $u = w|A$.

PROOF. Before taking up the proof observe that we do not assume that $\varphi\colon A \longrightarrow X$ is surjective.

We first show that any $a \in A$ satisfies $a \in Q_{\varphi(a)}$. Let $f \in S_{\varphi(a)}$. This means $\|f\| = 1 = \big|f\big(\varphi(a)\big)\big|$. From equation (4) we get $|Tf(a)| = \big|f\big(\varphi(a)\big)\big| = 1$. Since T is an isometry, we get $\|Tf\| = 1$. Thus $\|Tf\| = 1 = |Tf(a)|$, showing that $Tf \in S_a$. Hence $f \in S_{\varphi(a)} \Rightarrow Tf \in S_a$. This yields $a \in Q_{\varphi(a)}$. Since $Y_0 = \bigcup_{x\in X} Q_x$ we see that $A \subseteq Y_0$.

From equation (4) we see that $u(a) = T1(a) = w(a)$ for all $a \in A$, yielding $u = w|A$.

Since $Tf(y) = w(y)f\big(\psi(y)\big)$ for all $y \in Y_0$ and $A \subseteq Y_0$, we get $Tf(a) = w(a)f\big(\psi(a)\big)$. Again equation (4) yields $Tf(a) = u(a)f\big(\varphi(a)\big) = w(a)f\big(\varphi(a)\big)$ since $u = w|A$. From $|w(a)| = 1$, we get $f\big(\psi(a)\big) = f\big(\varphi(a)\big)$. This is valid for all $f \in C(X)$. Since functions in $C(X)$ separate points of X we get $\psi(a) = \varphi(a)$. This shows that $\varphi = \psi|A$. ∎

COROLLARY 2.1. *Let A, B be subspaces of Y, $\varphi\colon A \longrightarrow X$, $\theta\colon B \longrightarrow X$, $u\colon A \longrightarrow S^1$, $v\colon B \longrightarrow S^1$ be maps satisfying equation (4) in Theorem 2.1 and equation (5) below:*

$$Tf(b) = v(b)f\big(\theta(b)\big) \quad \forall b \in B \text{ and } f \in C(X). \tag{5}$$

Then $\varphi|A \cap B = \theta|A \cap B$ and $u|A \cap B = v|A \cap B$. Moreover $\gamma\colon A \cup B \longrightarrow X$, $t\colon A \cup B \longrightarrow S^1$ defined by $\gamma|A = \varphi$, $\gamma|B = \theta$; $t|A = u$, $t|B = v$ are continuous and

$$Tf(x) = t(x)f\big(\gamma(x)\big) \quad \forall x \in A \cup B \text{ and } f \in C(X). \tag{6}$$

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PROOF. From Theorem 2.1, $\varphi = \psi|A$, $\theta = \psi|B$; $u = w|A$ and $v = w|B$. The first part is immediate now. Also we get $\gamma = \psi|A \cup B$, $t = w|A \cup B$ from which we get the second part.

Theorem 2.1 can be strengthened as follows:

THEOREM 2.2. *Let A be a subspace of Y, $\varphi: A \longrightarrow X$ and $v: A \longrightarrow \mathbb{C}$ be maps satisfying*

$$Tf(a) = v(a)f\big(\varphi(a)\big) \quad \forall a \in A \text{ and } f \in C(X).$$

Then $A \subseteq Y_0$ if and only if $v(A) \subset S^1$. Moreover when this condition is satisfied we have $\varphi = \psi|A$ and $v = w|A$.

PROOF. In view of Theorem 2.1 we have only to show that $A \subseteq Y_0 \Rightarrow v(A) \subset S^1$. Assume $A \subseteq Y_0$. Then for any $a \in A$, $\exists$ an $x \in A$ with $a \in Q_x$. This means $T(S_x) \subset S_a$. Clearly $1 \in S_x$. Hence $T1 \in S_a$ yielding $1 = |T1(a)| = |v(a)|\,\big|1\big(\varphi(a)\big)\big| = |v(a)|$. Hence $v(A) \subset S^1$. ■

REMARKS 2.1. (a) A bounded linear operator T on a Banach space E is defined to be a shift by Crownover [1] if T is injective, the range $R(T)$ of T has codimension 1 in E and $\bigcap_{n \geq 1} R(T^n) = \{0\}$. We now observe that Theorem 2.1 of [2] is valid for any codimension 1 linear isometry $T: C(X) \longrightarrow C(X)$. Hence we may introduce the concepts of type I and type II codimension 1 linear isometries. We also observe that Lemmas 2.1, 2.2 and Theorem 2.6 of [2] are valid for codimension 1 linear isometries.

(b) From Theorem 2.1 in our present paper it follows that Y_0 is the largest subset of Y admitting maps $\psi: Y_0 \longrightarrow X$, $w: Y_0 \longrightarrow S^1$ satisfying equation (3). It turns out that Y_0 is closed and $\psi: Y_0 \longrightarrow X$ is surjective. It can very well happen that there exists a closed set $Y_1 \subsetneq Y_0$ and $\psi: Y_1 \longrightarrow X$ is surjective. This is what happens in the case of a codimension 1 linear isometry $T: C(X) \longrightarrow C(X)$ which is simultaneously of types I and II.

An immediate consequence of Corollary 2.1 is the following:

PROPOSITION 2.1. *Suppose $T: C(X) \longrightarrow C(X)$ is a codimension 1 linear isometry and there are closed subspaces $X_0 \subsetneq X$, $X_1 \subsetneq X$ with $X_0 \neq X_1$ maps $w: X_0 \longrightarrow S^1$, $w': X_1 \longrightarrow S^1$ and surjective maps $\psi: X_0 \longrightarrow X$, $\psi': X_1 \longrightarrow X$ satisfying*

$$Tf(y) = w(y)f\big(\psi(y)\big) \quad \forall y \in X_0 \text{ and } f \in C(X) \tag{7}$$

as well as

$$Tf(y') = w'(y')f\big(\psi'(y')\big) \quad \forall y' \in X_1 \text{ and } f \in C(X). \tag{8}$$

Then T is simultaneously of types I and II.

PROOF. Theorem 2.1 of [2] yields $|X \setminus X_0| = 1 = |X \setminus X_1|$. Since $X_0 \neq X_1$ we get $X = X_0 \cup X_1$. Now Corollary 2.1 completes the proof of Proposition 2.1. ■

DEFINITION 2.1. When the hypotheses of Proposition 2.1 are satisfied we say that T can be expressed as an operator of type I in two different ways.

As an immediate consequence of Proposition 2.1 we obtain the following:

COROLLARY 2.2. *Let* $T: C(X) \longrightarrow C(X)$ *be a codimension* 1 *linear isometry of type I which is not of type II. Then there exists a unique isolated point* p *in* X*, a unique homeomorphism* $\psi: X_0 \longrightarrow X$ *where* $X_0 = X \setminus \{p\}$*, a unique map* $w: X_0 \longrightarrow S^1$ *satisfying (7).*

Using Corollary 2.1 we can find a necessary and sufficient condition for a given codimension 1 linear isometry $T: C(X) \longrightarrow C(X)$ of type I to be also of type II.

PROPOSITION 2.2. *Let* $T: C(X) \longrightarrow C(X)$ *be a codimension* 1 *linear isometry of type I. Let* p *be an isolated point in* X*,* $\psi: X_0 \longrightarrow X$*,* $w: X_0 \longrightarrow S^1$ *maps with* $X_0 = X \setminus \{p\}$ *and* ψ *homeomorphic satisfying (7).*

Then T *will be of type II if and only if there exist elements* $c \in X$ *and* $\lambda \in S^1$ *satisfying*

$$Tf(p) = \lambda f(c) \quad \text{for all } f \in C(X). \tag{9}$$

PROOF. If T is also of type II, ψ and w admit extensions, also denoted by the same letters $\psi: X \longrightarrow X$, $w: X \longrightarrow S^1$ satisfying (7) for all $y \in X$. Choose $c = \psi(p)$ and $\lambda = w(p)$. Then clearly (9) is satisfied.

Conversely, assume that there exist $c \in X$ and $\lambda \in S^1$ satisfying (9). Then $\theta: \{p\} \longrightarrow X$, $v: \{p\} \longrightarrow S^1$ defined by $\theta(p) = c$, $v(p) = \lambda$ are clearly continuous. Taking $A = X_0$, $\varphi = \psi$, $u = w$; $B = \{p\}$ from Corollary 2.1 we immediately conclude that T is of type II. ∎

REMARK 2.2. Suppose $T: C(X) \longrightarrow C(X)$ is a codimension 1 linear isometry of type II. Let $\psi: X \longrightarrow X$, $w: X \longrightarrow S^1$ with ψ surjective satisfy (7) for all $y \in X$. Then there exist $a \neq b$ in X with $\psi(a) = \psi(b)$ and $\psi|X \setminus \{a,b\}: X \setminus \{a,b\} \longrightarrow X \setminus \{c\}$ bijective where $\psi(a) = \psi(b) = c$. It is now straight-forward to see that T is also of type I if and only if one of a, b is an isolated point in X. T is expressible as a type I operator in two different ways if and only if both a and b are isolated points in X. In turn this will be the case if and only if c is an isolated point in X.

In §3 we will discuss methods of constructing codimension 1 linear self isometries of $C(X)$. Using those methods we will construct a codimension 1 linear isometry $T: C(K) \longrightarrow C(K)$ of type II which is not of type I when K is the Cantor set. However our methods do not yield an *isometric shift* operator on $C(K)$. Since K has no isolated points, if there is an isometric shift operator on $C(K)$ it will be of type II which is not of type I.

We end this section by proving the following:

PROPOSITION 2.3. *Let* $T: C(X) \longrightarrow C(X)$ *be a codimension* 1 *linear isometry of type I;* p*,* $X_0 = X \setminus \{p\}$*,* $\psi: X_0 \longrightarrow X$ *and* $w: X_0 \longrightarrow S^1$ *have their usual meanings. Let* $q \in X_0$ *be any isolated point. Then* $\chi_q \in R(T) \Leftrightarrow T\chi_{\psi(q)}(p) = 0$.

PROOF. Suppose $\chi_q \in R(T)$ say $\chi_q = Th$ with $h \in C(X)$. Using the equation $Th(y) = w(y)h\big(\psi(y)\big)$ $\forall y \in X_0$ we immediately see that $h|\big(X - \psi(q)\big) = 0$ and that $h\big(\psi(q)\big) = \frac{1}{w(q)}$. Hence $\chi_q = Th \Rightarrow h = \frac{1}{w(q)}\chi_{\psi(q)}$. From $\chi_q(p) = 0$ we now get $\frac{1}{w(q)}T\chi_{\psi(q)}(p) = 0$ yielding $T\chi_{\psi(q)}(p) = 0$.

Conversely, if $T\chi_{\psi(q)}(p) = 0$, straight-forward checking shows that $\chi_q = Th$ where $h = \frac{1}{w(q)}\chi_{\psi(q)}$. ∎

We will make use of this proposition in §4.

3. **Construction of codimension 1 linear self isometries of $C(X)$.** Throughout this section X will denote a compact Hausdorff space. Let $T: C(X) \longrightarrow C(X)$ be a codimension 1 linear isometry of type I. Then as seen already in §2, there exist an isolated point p in X, a homeomorphism $\psi: X_0 \longrightarrow X$ where $X_0 = X \setminus \{p\}$ and a map $w: X_0 \longrightarrow S^1$ satisfying

$$Tf(y) = w(y)f\big(\psi(y)\big) \quad \text{for all } y \in X_0 \text{ and } f \in C(X). \tag{10}$$

Denoting the continuous linear functional $f \longmapsto Tf(p)$ on $C(X)$ by Γ we see that $|\Gamma f| \leq \|f\|$ for all $f \in C(X)$. We will presently see that the converse to this is true.

PROPOSITION 3.1. *Let p be an isolated point and $\psi: X_0 \longrightarrow X$ a homeomorphism. Let Γ be a continuous linear functional on $C(X)$ satisfying $|\Gamma f| \leq \|f\|$ for all $f \in C(X)$ and $w: X_0 \longrightarrow S^1$ a map. Then $T: C(X) \longrightarrow C(X)$ defined by $Tf(y) = w(y)f\big(\psi(y)\big)$ for all $y \in X_0$ and $Tf(p) = \Gamma f$, for any $f \in C(X)$ is a codimension 1 linear isometry.*

PROOF. The proof given in [2] for the fact that $\chi_p \notin R(T)$ is valid here also. Still we spell it out. If $\chi_p = Tf$, since $p \notin X_0$, we get $Tf(y) = 0$ for all $y \in X_0$. It follows from the equation $Tf(y) = w(y)f\big(\psi(y)\big)$ that $f = 0$, since $\psi: X_0 \longrightarrow X$ is surjective and $|w(y)| = 1$ for every y. This will mean $\chi_p = 0$, a contradiction. Let $\Delta_1: C(X) \longrightarrow C(X)$ be defined by $\Delta_1 f(x) = f\big(\psi^{-1}(x)\big)/w\big(\psi^{-1}(x)\big)$. A straight-forward verification shows that $f = T\Delta_1 f + \{f(p) - T\Delta_1 f(p)\}\chi_p$. This proves that $C(X)/R(T)$ is of dimension 1, with the class $[\chi_p]$ of χ_p in $C(X)/R(T)$ forming a basis element. Using the facts $\sup_{y\in X_0} |Tf(y)| = \sup_{y\in X_0} \big|f\big(\psi(y)\big)\big| = \sup_{x\in X} |f(x)| = \|f\|$ and $|Tf(p)| = |\Gamma f| \leq \|f\|$ we immediately get $\|Tf\| = \|f\|$. ∎

Suppose $T: C(X) \longrightarrow C(X)$ is a codimension 1 linear isometry of type II. Then we get $\psi: X \longrightarrow X$, $w: X \longrightarrow S^1$ with ψ surjective and satisfying

$$Tf(x) = w(x)f\big(\psi(x)\big) \quad \forall x \in X \text{ and } f \in C(X). \tag{11}$$

Moreover there exist two unique elements $a \neq b$ in X with $\psi(a) = \psi(b)$ and $\psi|X - \{a,b\}: X - \{a,b\} \longrightarrow X - \{c\}$ bijective. Here $\psi(a) = \psi(b) = c$. If W denotes the quotient space obtained from X by identifying a and b, ψ induces a map $\bar{\psi}: W \longrightarrow X$. Then $\bar{\psi}: W \longrightarrow X$ is a homeomorphism (analogue of Theorem 2.6 in [2]). The following proposition yields a converse to this.

PROPOSITION 3.2. *Let $\psi: X \longrightarrow X$, $w: X \longrightarrow S^1$ be given with ψ surjective. Suppose there exist $a \neq b$ in X with $\psi(a) = \psi(b)$ and $\psi|X - \{a,b\}: X - \{a,b\} \longrightarrow X - \{c\}$ bijective, where $c = \psi(a) = \psi(b)$.*

Then $T: C(X) \longrightarrow C(X)$ defined by

$$Tf(x) = w(x)f\big(\psi(x)\big) \quad \forall x \in X \text{ and } f \in C(X)$$

is a codimension 1 linear isometry (of type II).

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PROOF. The proof is somewhat similar to that of Theorem 2.6 of [2]. We omit the details. ∎

EXAMPLE 3.1. Let K denote the Cantor set. Given $a \neq b$ in K it is shown in [2] that there exists a surjection $\psi: K \longrightarrow K$ satisfying $\psi(a) = \psi(b) = a$ with the additional property that $\psi | K \setminus \{a\}: K \setminus \{a\} \longrightarrow K$ is bijective. Proposition 3.2 yields a codimension 1 linear isometry $T: C(K) \longrightarrow C(K)$. Since K has no isolated points, it follows that T can not be of type I.

REMARK 3.1. Given an isolated point p in X, a homeomorphism $\psi: X_0 \longrightarrow X$ (where $X_0 = X \setminus \{p\}$), a map $w: X_0 \longrightarrow S^1$ and a continuous linear functional $\Gamma: C(X) \longrightarrow \mathbb{C}$ satisfying $|\Gamma f| \leq \|f\|$, Proposition 3.1 shows that $T: C(X) \longrightarrow C(X)$ defined by $Tf(y) = w(y)f\big(\psi(y)\big)$ $\forall y \in X_0$ and $Tf(p) = \Gamma f$ is a codimension 1 linear isometry of type I. Let $\Delta_1: C(X) \longrightarrow C(X)$ be defined as earlier, namely $\Delta_1 f(x) = f\big(\psi^{-1}(x)\big) / w\big(\psi^{-1}(x)\big)$ for any $x \in X$. Then Δ_1 is a surjective complex linear map, $\|\Delta_1 f\| = \|f\|$ and $\operatorname{Ker} \Delta_1 = \mathbb{C}\chi_p$. For any integer $n \geq 1$, let $\Delta_n: C(X) \longrightarrow C(X)$ be defined by $\Delta_n = (\Delta_1)^n$; let $\Delta_0 = Id_{C(X)}$. It is easy to see that $f \in R(T^n)$ if and only if $\Delta_j f(p) = \Gamma \Delta_{j+1} f$ for $0 \leq j \leq n-1$. If $\beta_j: C(X) \longrightarrow \mathbb{C}$ denotes the continuous linear functional $\beta_j f = \Delta_j f(p) - \Gamma \Delta_{j+1} f$ then $f \in R(T^n) \Leftrightarrow f \in \bigcap_{j=0}^{n-1} \operatorname{Ker} \beta_j$. Thus T will be an isometric shift $\Leftrightarrow \bigcap_{j \geq 0} \operatorname{Ker} \beta_j = \{0\}$.

We now give an example of a codimension 1 linear isometry $T: C(X) \longrightarrow C(X)$ which is not a shift.

EXAMPLE 3.2. Let $A = \mathbb{N} \cup \{\infty\}$ the one point compactification of $\mathbb{N}$. As usual we identify $C(A)$ with the space of convergent complex sequences $\underline{c} = (c_1, c_2, c_3, \ldots)$.

Then $T: C(A) \longrightarrow C(A)$ given by $T(c_1, c_2, c_3, \ldots) = (c_1, 0, c_2, c_3, c_4, \ldots)$ is a codimension 1 linear isometry which is not a shift.

4. **Discussion of type (or types) of isometric shift operators.** The object of this section is to remove the vagueness arising from the comment on p. 100, line 1 in [2] concerning the "uniqueness of p".

EXAMPLE 4.1. Consider Example 3, p. 100 of [2]. T in this example is expressible as an isometric shift operator of type I in two different ways. If $p = 1$, $X_0 = X \setminus \{1\}$ and $\psi: X_0 \longrightarrow X$, $w: X_0 \longrightarrow S^1$ are given by $\psi(n+1) = n$ $\forall n \in \mathbb{N}$, $\psi(\infty) = \infty$ and $w(y) = 1$ $\forall y \in X_0$ then we clearly have

$$Tf(y) = w(y)f\big(\psi(y)\big) \quad \forall y \in X_0 \text{ and } f \in C(X). \tag{12}$$

Similarly setting $q = 2$, $X_0' = X \setminus \{2\}$ and defining $\psi' = X_0' \longrightarrow X$, $w': X_0' \longrightarrow S^1$ by $\psi'(1) = 1$, $\psi'(n+1) = n$ for $n \geq 2$, $\psi'(\infty) = \infty$, $w'(1) = -1$ and $w'(x) = 1$ for all $x \in X_0' \setminus \{1\}$ we see that

$$Tf(y') = w(y')f\big(\psi'(y')\big) \quad \forall y' \in X_0' \text{ and } f \in C(X). \tag{13}$$

Thus T is expressible as an isometric shift operator in two different ways.

Propositions 2.1, 2.2 and Corollary 2.2 were proved for codimension 1 linear isometries. In particular they are valid for isometric shift operators.

EXAMPLE 4.2. Consider Example 2 on p. 100 of [2]. In this example T is an isometric shift operator of type I which is not of type II. Thus T is expressible as an isometric shift operator of type I in only one way. $p = 1$, $X_0 = X \setminus \{1\}$; $\psi: X_0 \longrightarrow X$, $w: X_0 \longrightarrow S^1$ with $\psi(n+1) = n \ \forall n \in \mathbb{N}$, $\psi(\infty) = \infty$ and $w(y) = 1 \ \forall g \in X_0$ satisfy $Tf(y) = w(y)f\big(\psi(y)\big)$ $\forall y \in X_0$ and $f \in C(X)$. However, straight-forward checking shows that 2 and 3 are isolated points with χ_2 as wel as χ_3 not in $R(T)$. This means the only function vanishing on either $X \setminus \{1\}$ or $X \setminus \{2\}$ or $X \setminus \{3\}$ and lying in $R(T)$ is the constant function 0.

As an immediate consequence of Proposition 2.3 we get the following:

PROPOSITION 4.1. *Let $T: C(X) \longrightarrow C(X)$ be an isometric shift operator expressible as a shift operator of type I in a unique way. Let p, $X_0 = X \setminus \{p\}$, $\psi: X_0 \longrightarrow X$ and $w: X_0 \longrightarrow S^1$ have their usual meanings. Let q be any isolated point in X with $q \neq p$. Then the following are equivalent:*

(i) $f \in R(T), f|(X - \{q\}) = 0 \Rightarrow f = 0$

(ii) $\chi_q \notin R(T)$

(iii) $T\chi_{\psi(q)}(p) \neq 0$.

The straight-forward proof of this is omitted.

EXAMPLE 4.3. Let $X = \mathbb{N} \cup \{\infty\}$ the one point compactification of $\mathbb{N}$. We identify $C(X)$ with the space of convergent complex sequences $\underline{c} = (c_1, c_2, c_3, \ldots)$ under $f \leftrightarrow \underline{c}$ where $c_n = f(n)$. Under this identification $f(\infty)$ will correspond to $\lim_{n\to\infty} c_n$. We write c_∞ for $\lim_{n\to\infty} c_n$. Consider $T: C(X) \longrightarrow C(X)$ defined by

$$T\underline{c} = (c_\infty, ic_1, -c_2, -ic_3, c_4, c_5, c_6, c_7, \ldots).$$

Let $\psi: X \longrightarrow X$ and $w: X \longrightarrow S^1$ be defined by $\psi(n+1) = n \ \forall n \in \mathbb{N}$, $\psi(1) = \psi(\infty) = \infty$; $w(1) = 1$, $w(2) = i$, $w(3) = -1$, $w(4) = -i$, $w(n) = 1$ for $n \geq 5$ and $w(\infty) = 1$. Clearly $\psi|X - \{1, \infty\}: X - \{1, \infty\} \longrightarrow X - \{\infty\}$ is bijective. T is the codimension 1 linear isometry of type II obtained from ψ and w applying Proposition 3.2. Since 1 is isolated in X we see that T is also of type I (Remark 2.2). Since ∞ is not isolated in X from the same remark we see that T can not be expressed as a type I operator in two different ways.

We will show that T satisfies $\bigcap_{n\geq 1} R(T^n) = \{0\}$. Then it will follow that T is an isometric shift operator simultaneously of types I and II but expressible as a shift operator of type I in exactly one way.

For any $\underline{a} = (a_1, a_2, a_3, \ldots) \in C(X)$ let us denote the conventional shift $\underline{a} \mapsto (0, a_1, a_2, a_3, \ldots)$ by S. Given $\underline{c} = (c_1, c_2, c_3, c_4, \ldots) \in C(X)$ let us denote the element $(-c_1, ic_2, -ic_3, c_4, c_5, c_6, \ldots)$ by $\gamma(\underline{c})$. An easy calculation shows that

$$T^3\underline{c} = (c_\infty, ic_\infty, -ic_\infty, 0, 0, 0, 0, \ldots) + S^3\gamma(\underline{c})$$
$$T^6\underline{c} = (c_\infty, ic_\infty, -ic_\infty, -c_\infty, -c_\infty, -c_\infty, 0, 0, 0, 0, \ldots) + S^6\gamma(\underline{c}).$$

Denote the element $(c_\infty, ic_\infty, -ic_\infty, \overbrace{-c_\infty, -c_\infty, -c_\infty, \dots, -c_\infty}^{3l\text{ terms}}, 0, 0, 0, 0, \dots)$ of $C(X)$ by $\mu_l(c_\infty)$. Then by induction on l we show that

$$T^{3(l+1)}\underline{c} = \mu_l(c_\infty) + S^{3(l+1)}\gamma(\underline{c}) \quad \text{for } l \geq 1 \tag{14}$$

Suppose $\underline{a} = (a_1, a_2, a_3, \dots)$ is in $\bigcap_{l\geq 1} R(T^{3(l+1)})$. Then from (14) we see that there should exist an element $c_\infty \in \mathbb{C}$ with $a_1 = c_\infty$, $a_2 = ic_\infty$, $a_3 = -ic_\infty$ and $a_k = -c_\infty$ for all $k \geq 4$. Writing λ for c_∞ we should have

$$\underline{a} = (\lambda, i\lambda, -i\lambda, -\lambda, -\lambda, -\lambda, -\lambda, \dots).$$

Also $\underline{a} = T\underline{b}$ for some $\underline{b} = (b_1, b_2, b_3, \dots) \in C(X)$. This means $\underline{a} = (b_\infty, ib_1, -b_2, -ib_3, b_4, b_5, \dots)$ yielding $\lambda = b_\infty$ and $b_n = -\lambda$ for $n \geq 4$. But $\underline{b}$ being a convergent sequence, we should have $b_\infty = \lim_{n\to\infty} b_n = -\lambda$. Thus we get $\lambda = -\lambda$ or $\lambda = 0$. This yields $\underline{a} = 0 \in C(X)$ thereby showing that $\bigcap_{n\geq 1} R(T^n) = \{0\}$. This completes the proof that T is an isometric shift operator.

5. Isometric shift operators of type I with $\bar{D} \neq X$. In this section, given any integer $l \geq 1$ we construct an isometric shift operator of type I with $X \setminus \bar{D}$ having exactly l elements. Let $A = \mathbb{N} \cup \{\infty\}$ the one point compactification of $\mathbb{N}$. As usual $C(A)$ will be identified with the space of convergent complex sequences $\underline{c} = (c_1, c_2, c_3, \dots)$. Let $\{a_1, a_2, \dots, a_l\}$ be a discrete space with l elements and $X = A \cup \{a_1, a_2, \dots a_l\}$ (disjoint union). Any element of $C(X)$ can be uniquely written as $\underline{c} \oplus \sum_{j=1}^{l} \lambda_j \chi_{a_j}$ with $\underline{c} \in C(A)$. Let $T\colon C(A) \longrightarrow C(A)$ be the usual lateral shift, namely $T\underline{c} = (0, c_1, c_2, c_3, \dots)$. Let $S\colon C(X) \longrightarrow C(X)$ be defined by

$$S\Big(\underline{c} \oplus \sum_{j=1}^{l} \lambda_j \chi_{a_j}\Big) = (\lambda_1, c_1, c_2, c_3, \dots) \oplus (\lambda_2 \chi_{a_1} + \cdots + \lambda_l \chi_{a_{l-1}} - \lambda_1 \chi_{a_l}) \tag{15}$$

We could rewrite the formula for S as

$$S\Big(\underline{c} \oplus \sum_{j=1}^{l} \lambda_j \chi_{a_j}\Big) = (\lambda_1, 0, 0, 0, \dots) + T\underline{c} \oplus (\lambda_2 \chi_{a_1} + \cdots + \lambda_l \chi_{a_{l-1}} - \lambda_1 \chi_{a_l}). \tag{16}$$

Let $X_0 = X \setminus \{1\} = (A \setminus \{1\}) \cup \{a_1, \dots, a_l\}$. Define $\psi\colon X_0 \longrightarrow X$, $w\colon X_0 \longrightarrow S^1$ by

$$\begin{gathered} \psi(n+1) = n \quad \forall n \in \mathbb{N},\ \psi(\infty) = \infty,\ \psi(a_1) = a_2, \\ \psi(a_2) = a_3, \dots, \psi(a_{l-1}) = a_l \text{ and } \psi(a_l) = a_1 \end{gathered} \tag{17}$$

$$w(a_l) = -1 \text{ and } w(y) = 1 \quad \text{for all } y \in X_0 \setminus \{a_l\}. \tag{18}$$

Then it is clear that

$$Sf(y) = w(y)S\big(\psi(y)\big) \quad \forall y \in X_0 \text{ and } f \in C(X). \tag{19}$$

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We will check that S is an isometric shift operator. One can check that

$$S^l\Big(\underline{c}\oplus\sum_{j=1}^{l}\lambda_j\chi_{a_j}\Big)=(\lambda_l,\lambda_{l-1},\ldots,\lambda_1,0,0,0,\ldots)+T^l\underline{c}\oplus\Big(-\sum_{j=1}^{l}\lambda_j\chi_{a_j}\Big) \tag{20}$$

$$S^{2l}\Big(\underline{c}\oplus\sum_{j=1}^{l}\lambda_j\chi_{a_j}\Big)$$
$$=(-\lambda_l,-\lambda_{l-1},\ldots,-\lambda_1,\lambda_l,\ldots,\lambda_1,0,0,0,\ldots)+T^{2l}\underline{c}\oplus\sum_{j=1}^{l}\lambda_j\chi_{a_j}. \tag{21}$$

Let us denote $(\lambda_l,\lambda_{l-1},\ldots,\lambda_1,0,0,0,\ldots)$ by $\underline{u}$. Then we have

$$S^{(2n+1)l}\Big(\underline{c}\oplus\sum_{j=1}^{l}\lambda_j\chi_{a_j}\Big)=\sum_{r=0}^{n}T^{2rl}\underline{u}-\sum_{r=0}^{n-1}T^{(2r+1)l}\underline{u}+T^{(2n+1)l}\underline{c}\oplus\Big(-\sum_{j=1}^{l}\lambda_j\chi_{a_j}\Big) \tag{22}$$

$$S^{2nl}\Big(\underline{c}\oplus\sum_{j=1}^{l}\lambda_j\chi_{a_j}\Big)=-\sum_{r=0}^{n-1}T^{2rl}\underline{u}+\sum_{r=0}^{n-1}T^{(2r+1)l}\underline{u}+T^{2nl}\underline{c}\oplus\sum_{j=1}^{l}\lambda_j\chi_{a_j} \tag{23}$$

From (22) and (23) we see that if

$$\underline{x}\oplus\sum_{j=1}^{l}\mu_j\chi_{a_j}\text{ is in }R(S^n)\text{ for all }n\geq 1,$$

then $\underline{x}$ will not be convergent unless $\underline{x}=0$ in $C(A)$ and $\mu_1=\cdots=\mu_l=0$. This proves that $\bigcap_{n\geq 1}R(S^n)=\{0\}$. Thus S is an isometric shift operator of type I. In this example $D=\mathbb{N}$, $\bar{D}=A$ and $X\setminus\bar{D}=\{a_1,\ldots,a_l\}$. ∎

In this example it is easily seen that $I_D\cap R(S^l)=\{0\}$. Also $I_D\cap R(S^{l-1})\neq\{0\}$ because if $I_D\cap R(S^{l-1})=\{0\}$, we would have $|X\setminus\bar{D}|\leq l-1$, which is not the case here.

6. **Non existence of codimension 1 linear isometries on $C(M^n)$.** The main result proved in this section is:

THEOREM 6.1. *Let M be any compact manifold with or without boundary. Then $C(M)$ does not admit a codimension 1 linear isometry. In particular $C(M)$ does not admit an isometric shift operator.*

PROOF. Any compact manifold M has only finitely many connected components. Hence M can not admit an infinite number of isolated points. Thus to prove Theorem 6.1 we have only to show that $C(M)$ does not admit a codimension 1 linear isometry of type II. As remarked earlier in §3, if there existed a codimension 1 linear isometry $T\colon C(M)\longrightarrow C(M)$, M would be homeomorphic to a quotient of M obtained by identifying exactly two points. Let $a\neq b$ be any two points of M. If M were of dimension 0, M would be a finite discrete space. Hence $C(M)$ can not admit any injective linear map which is not surjective. Thus we may assume that $\dim M=n\geq 1$.

Suppose $\delta M = \emptyset$. Let X be the quotient space obtained from M by identifying a and b. Let $c \in X$ be the point represented by a or b. Let $B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ and $B^n \vee B^n$ the wedge where $0 \in B^n$ is chosen as the base point. The element $c \in X$ will have a fundamental system of neighbourhoods homeomorphic to $B^n \vee B^n$ with c corresponding to the base point in $B^n \vee B^n$. But $B^n \vee B^n$ is not locally Euclidean around the base point. Hence X can not be homeomorphic to M^n.

Suppose $\delta M \neq \emptyset$. If a and b are both in $\operatorname{Int} M^n$, $c \in X$ will have a fundamental system of neighbourhoods homeomorphic to $B^n \vee B^n$ with c corresonding to the base point of $B^n \vee B^n$. Let $B^n_+ = \{x \in \mathbb{R}^n \mid x_1 \geq 0, \|x\| < 1\}$. If one of a, b is in $\operatorname{Int} M^n$ and the other is in δM then c will admit a fundamental system of neighbourhoods homeomorphic to $B^n \vee B^n_+$ with c corresponding to the base point. If both a and b are in δM, c will admit a fundamental system of neighbourhoods homeomorphic to $B^n_+ \vee B^n_+$. For $B^n \vee B^n$ and $B^n \vee B^n_+$ the manifold condition fails at the base point. Also when $n \geq 2$, the manifold condition fails at the base point for $B^n_+ \vee B^n_+$.

When $n = 1$, M will be a disjoint union of k copies of S^1 and l copies of $[0, 1]$ for some integers $k \geq 0$, $l \geq 0$ and $k + l \geq 1$. If two boundary points in M are identified, the quotient X will have strictly less than l copies of $[0, 1]$, hence cannot be homeomorphic to M. ∎

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