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OPERATORS ON THE FOURIER ALGEBRA WITH WEAKLY COMPACT EXTENSIONS

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Introduction. We let G denote an infinite compact group, and $\hat{G}$ its dual. We use the notation of our book [**3**, Chapters 7 and 8]. Recall that $A(G)$ denotes the Fourier algebra of G (an algebra of continuous functions on G), and $\mathscr{L}^\infty(\hat{G})$ denotes its dual space under the pairing $\langle f, \phi \rangle$, $(f \in A(G), \phi \in \mathscr{L}^\infty(\hat{G}))$. Further, note $\mathscr{L}^\infty(\hat{G})$ is identified with the C^*-algebra of bounded operators on $L^2(G)$ commuting with left translation. The module action of $A(G)$ on $\mathscr{L}^\infty(\hat{G})$ is defined by the following: for $f \in A(G), \phi \in \mathscr{L}^\infty(\hat{G})$, $f \cdot \phi \in \mathscr{L}^\infty(\hat{G})$ by

$$\langle g, f \cdot \phi \rangle = \langle fg, \phi \rangle,$$

$g \in A(G)$. Also $||f \cdot \phi||_\infty \leqq ||f||_A ||\phi||_\infty$. (See [**1**] for the general setting.)

We have previously [**4**] studied the spaces $AP(\hat{G})$, respectively $W(\hat{G})$, consisting of those $\phi \in \mathscr{L}^\infty(\hat{G})$ for which the map $f \mapsto f \cdot \phi$ from $A(G)$ to $\mathscr{L}^\infty(\hat{G})$ is a compact, respectively weakly compact, operator.

In this paper, we renorm $A(G)$ by continuously embedding $A(G)$ into the spaces $C(G)$, $L^p(G)(1 \leqq p < \infty)$, and we characterize the space of those $\phi \in \mathscr{L}^\infty(\hat{G})$ for which the map $f \mapsto f \cdot \phi$ from $A(G)$ to $\mathscr{L}^\infty(\hat{G})$ extends to a weakly compact operator on $C(G)$, $L^p(G)(1 \leqq p < \infty)$.

For G abelian, $\hat{G}$ is a group and $A(G)$ is isomorphic to $l^1(\hat{G})$ by the Fourier transform $\mathscr{F}$. The module action of $A(G)$ on $l^\infty(\hat{G})$ is given by $f \cdot \phi = \mathscr{F}(\tilde{f}) * \phi$, $(\tilde{g}(x) = g(-x), x \in G, g$ a function on $G)$, $(f \in A(G), \phi \in l^\infty(\hat{G}))$. It follows that for G abelian, the spaces $AP(\hat{G})$, respectively $W(\hat{G})$ are the classical spaces of almost periodic, respectively weakly almost periodic, functions on $\hat{G}$. A rewording of a result of Kluvánek [**5**] yields for G abelian that the functions $\phi \in l^\infty(\hat{G})$, for which the map $f \rightarrow f \cdot \phi = \mathscr{F}(\tilde{f}) * \phi$ from $A(G)$ to $l^\infty(\hat{G})$ extends to a weakly compact operator on $C(G)$, are precisely the Fourier-Stieltjes transforms of measures on G. His result states that ϕ is a Fourier-Stieltjes transform on $\hat{H}$ (H locally compact abelian) if and only if the set of functions $\sum_{i=1}^n c_i \phi_{y_i} (c_i \in C,\ y_i \in \hat{H},\ \phi_y$ denoting translation of ϕ by y) where $\sup \{|\sum_{i=1}^n c_i \langle h, y_i \rangle| : h \in H\} \leqq 1$, is relatively weakly compact in the space of continuous, bounded functions on $\hat{H}$. One of our results is the extension of this theorem to the compact nonabelian case (Theorem 1).

The key idea in extending locally compact abelian theorems, which involve translation in the dual group $\hat{G}$, to compact nonabelian groups is to replace

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the translation argument by a convolution argument involving $L^1(\hat{G})$, and then for the compact nonabelian case use the module action of $A(G)$ on $\mathscr{L}^\infty(\hat{G})$ instead of the convolution product.

Let $M(G)$ denote the measure algebra on G. For $\mu \in M(G)$, the Fourier-Stieltjes transform of μ, $\hat{\mu}$ or $\mathscr{F}\mu$, is a matrix-valued function in $\mathscr{L}^\infty(\hat{G})$ defined for $\alpha \in \hat{G}$ by

$$\alpha \mapsto \hat{\mu}_\alpha = \int_G T_\alpha(x^{-1})d\mu(x), \quad (T_\alpha \in \alpha).$$

For $\hat{\mu} \in \mathscr{L}^\infty(\hat{G})$ and $f \in A(G)$,

$$\langle f, \hat{\mu}\rangle = \int_G \tilde{f} d\mu.$$

To see this consider:

$$\begin{aligned}\langle f, \hat{\mu}\rangle &= \sum_{\alpha\in\hat{G}} n_\alpha \mathrm{Tr}(\hat{f}_\alpha \hat{\mu}_\alpha)\\ &= \sum_{\alpha\in\hat{G}} n_\alpha \sum_{i,j=1}^{n_\alpha} \hat{f}_{\alpha ij} \int_G T_{\alpha ji}(x^{-1})d\mu(x)\\ &= \sum_{\alpha\in\hat{G}} n_\alpha \int_G \mathrm{Tr}(\hat{f}_\alpha T_\alpha(x^{-1}))d\mu(x)\\ &= \int_G \sum_{\alpha\in\hat{G}} n_\alpha \mathrm{Tr}(T_\alpha(x^{-1})\hat{f}_\alpha)d\mu(x)\\ &= \int_G \tilde{f}d\mu.\end{aligned}$$

Thus for $f \in A(G)$ and $\hat{\mu} \in \mathscr{L}^\infty(\hat{G})$ the map $f \mapsto f \cdot \hat{\mu}$ has the explicit form $f \mapsto (\tilde{f}\mathrm{d}\mu)\hat{}$.

1. Embedding $A(G)$ into $C(G)$.

THEOREM 1. *Let $\phi \in \mathscr{L}^\infty(\hat{G})$. The map $f \mapsto f \cdot \phi$ from $A(G)$ to $\mathscr{L}^\infty(\hat{G})$ extends to a weakly compact operator on $C(G)$ if and only if $\phi \in M(G)\hat{}$.*

Proof. Let $\phi \in \mathscr{L}^\infty(\hat{G})$ be such that ϕ extends (uniquely, since $A(G)$ is uniformly dense in $C(G)$) to a weakly compact operator on $C(G)$. In particular, ϕ defines a bounded operator on $C(G)$ so there exists $M < \infty$ such that $||f \cdot \phi||_\infty \leqq M||f||_\infty$. The linear functional $f \mapsto \langle \tilde{f}, \phi\rangle$ on $A(G)$ is bounded in $C(G)$-norm since

$$|\langle \tilde{f}, \phi\rangle| = |\langle 1, \tilde{f}\cdot\phi\rangle| \leqq ||1||_A ||\tilde{f}\cdot\phi||_\infty \leqq M||f||_\infty.$$

Thus it extends by the Hahn-Banach theorem to all of $C(G)$, and by the Riesz representation theorem there exists $\mu \in M(G)$, $||\mu|| \leqq M$, such that

$$\langle \tilde{f}, \phi\rangle = \int_G f d\mu = \langle \tilde{f}, \hat{\mu}\rangle, \quad (f \in A(G)).$$

Hence $\phi = \hat{\mu}$.

Conversely, let $\mu \in M(G)$. We may assume $\mu \geqq 0$. Choose p with $1 < p < \infty$. The map $f \mapsto f\,d\mu \mapsto (f\,d\mu)^\wedge$ from $L^p(\mu) \to M(G) \to \mathscr{L}^\infty(\hat{G})$ is weakly compact since $L^p(\mu)$ is reflexive [**2**, p. 483]. Thus the map

$$f \mapsto \tilde{f} \mapsto \tilde{f}\,d\mu \mapsto (\tilde{f}\,d\mu)^\wedge = f \cdot \hat{\mu}$$

from $C(G) \to L^p(\mu) \to M(G) \to \mathscr{L}^\infty(\hat{G})$ is also weakly compact.

2. Embedding $A(G)$ into $L^p(G)$ $(1 < p < \infty)$.

THEOREM 2. *Let $\phi \in \mathscr{L}^\infty(\hat{G})$ and $1 < p < \infty$. The map $f \mapsto f \cdot \phi$ from $A(G)$ to $\mathscr{L}^\infty(\hat{G})$ extends to a weakly compact* (*equivalently, bounded* [**2**, p. 483]) *operator on $L^p(G)$ if and only if $\phi \in \mathscr{F}\, L^q(G)$* $(1/p + 1/q = 1)$.

Proof. Let $\phi \in \mathscr{L}^\infty(\hat{G})$ be such that $f \mapsto f \cdot \phi$ extends (uniquely since $A(G)$ is $\|\cdot\|_p$-dense in $L^p(G)$) to a weakly compact (bounded) operator on $L^p(G)$. Since $L^p(G)^* = L^q(G)$, it follows as above that $\phi \in \mathscr{F}\, L^q(G)$.

Conversely, let $h \in L^q(G)$. The map $f \mapsto f \cdot \hat{h}$ from $A(G)$ to $\mathscr{L}^\infty(\hat{G})$ is $L^p(G)$-bounded since

$$\begin{aligned}\|f \cdot \hat{h}\|_\infty &= \sup\{|\langle g, f \cdot \hat{h}\rangle| : g \in A(G), \|g\|_A \leqslant 1\}\\ &= \sup\left\{\left|\int_G fgh\,dm_G\right| : g \in A(G), \|g\|_A \leqslant 1\right\}\\ &\leqslant \sup\{\|fg\|_p \|h\|_q : g \in A(G), \|g\|_A \leqslant 1\}\\ &\leqslant \|f\|_p \|h\|_q.\end{aligned}$$

3. Embedding $A(G)$ into $L^1(G)$.

THEOREM 3. *Let G be an infinite compact group, and let $\phi \in \mathscr{L}^\infty(\hat{G})$ with $\phi \neq 0$. The map $f \mapsto f \cdot \phi$ from $A(G)$ to $\mathscr{L}^\infty(\hat{G})$ cannot be extended to a weakly compact operator on $L^1(G)$.*

Proof. By way of contradiction, suppose $\phi \neq 0$ is such that $f \mapsto f \cdot \phi$ extends to a weakly compact operator on $L^1(G)$. Analogously to Theorem 2, we see that $\phi \in \mathscr{F}\, L^\infty(G)$. Let $\phi = \hat{h}$, $h \in L^\infty(G)$. Since $\|h\|_\infty \neq 0$, there exists a point $x \in G$ with the property that given a neighborhood V of x, there exists a positive measurable subset E of V with $|h(x)| \geqq \|h\|_\infty/2$ for all $x \in E$. Let $\{V_\lambda\}$ be a neighborhood basis for the point x. Define $g_\lambda = (1/(m_G(E)h))\chi_E$ where $E \subset V_\lambda$ is as above, and χ_A denotes the characteristic function of the set A. Note that $\|g_\lambda\|_1 \leqq 2/\|h\|_\infty$. Thus by the weak compactness of the map $f \mapsto f \cdot \phi$, the set $\{\tilde{g}_\lambda \cdot \phi\}$ has a cluster point $\Upsilon \in \mathscr{L}^\infty(\hat{G})$. Let $\{W_\alpha\}$ be a weak neighborhood basis of Υ in $\mathscr{L}^\infty(\hat{G})$. For each $(V_\lambda, W_\alpha) \in \{V_\lambda\} \times \{W_\alpha\}$, there exists $g_{\lambda,\alpha} \in \{g_\lambda\}$ such that $\tilde{g}_{\lambda,\alpha} \cdot \hat{h} \in W_\alpha$ and support $g_{\lambda,\alpha} \subset V_\lambda$. Thus

$$\tilde{g}_{\lambda,\alpha} \cdot \hat{h} \xrightarrow{(\lambda,\alpha)} \Upsilon$$

weakly, and hence also weak-*.

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Now $\{\tilde{g}_{\lambda,\alpha} \cdot \hat{h}\}_{(\lambda,\alpha)}$ converges weak-* to $\hat{\delta}_{x^{-1}}$: for $k \in A(G)$,

$$\langle k, \tilde{g}_{\lambda,\alpha} \cdot \hat{h} \rangle = \langle k\tilde{g}_{\lambda,\alpha}, \hat{h} \rangle = \int_G \tilde{k} g_{\lambda,\alpha} h dm_G \xrightarrow{(\lambda, \alpha)} \tilde{k}(x) = k(x^{-1}).$$

Thus $\Upsilon = \hat{\delta}_{x^{-1}}$.

Recall that $\mathscr{C}_0(\hat{G})$ denotes the subspace of $\mathscr{L}^\infty(\hat{G})$ consisting of those ϕ for which the set $\{\alpha \in \hat{G} : ||\phi_\alpha||_\infty \geqq \epsilon\}$ is finite for $\epsilon > 0$. However, $\hat{\delta}_{x^{-1}}$ is unitary, so $\hat{\delta}_{x^{-1}} \notin \mathscr{C}_0(\hat{G})$. Now $\tilde{g}_{\lambda,\alpha} \cdot \hat{h} \in L^1(G)\hat{} \subset \mathscr{C}_0(\hat{G})$, which being strongly closed is also weakly closed. But then $\hat{\delta}_{x^{-1}}$ cannot be a weak cluster point of $\{\tilde{g}_{\lambda,\alpha} \cdot \hat{h}\}_{(\lambda,\alpha)}$, the required contradiction.

4. Abelian results for weakly compact extensions. Here G is a locally compact abelian (LCA) group. Theorem 1 has an LCA setting.

THEOREM 4. *Let $\phi \in L^\infty(\hat{G})$. The map $f \mapsto \hat{f} * \phi$ from $A(G)$ to $L^\infty(\hat{G})$ extends to a weakly compact operator on $C_0(G)$ if and only if $\phi \in M(G)\hat{}$.*

Proof. The proof is similar to Theorem 1.

5. Abelian results for compact extensions. In this section G is an LCA group.

THEOREM 5. *Let $\phi \in L^\infty(\hat{G})$. The map $f \mapsto \hat{f} * \phi$ from $A(G)$ to $L^\infty(\hat{G})$ extends to a compact operator on $C_0(G)$ if and only if $\phi \in M_d(G)\hat{}$.*

Proof. Let μ be a discrete measure on G, and let μ_n be finitely supported measures with

$$||\mu_n - \mu|| \xrightarrow{n} 0.$$

Note that $f \mapsto \hat{f} * \hat{\mu}_n = (fd\mu_n)\hat{}$ is a finite rank operator from $C_0(G)$ to $L^\infty(\hat{G})$. Thus $f \mapsto (fd\mu)\hat{}$ from $C_0(G)$ to $L^\infty(\hat{G})$ being a limit of compact operators is compact: note that

$$||(fd\mu_n)\hat{} - (fd\mu)\hat{}||_\infty \leqq ||f||_\infty ||\mu_n - \mu|| \quad (f \in C_0(G)).$$

Conversely, suppose $\phi \in L^\infty(\hat{G})$ is such that $f \mapsto \hat{f} * \phi$ extends to a compact (in particular, weakly compact) operator on $C_0(G)$. Thus $\phi \in M(G)\hat{}$ by Section 4. Also we note that $f \mapsto \hat{f} * \phi$ is a compact operator on $A(G)$ since $||f||_\infty \leqq ||f||_A$, $(f \in A(G))$. Thus ϕ is an almost periodic function on $\hat{G}$ [**4**]; and thus $\hat{\mu} \in M_d(G)\hat{}$.

6. Compact extensions for compact groups. Here G is again an infinite compact group.

THEOREM 6. *Let G be an infinite compact nonabelian group of bounded representation type. The map $f \mapsto f \cdot \phi$ from $A(G)$ to $\mathscr{L}^\infty(\hat{G})$ extends to a compact operator on $C(G)$ if and only if $\phi \in M_d(G)\hat{}$.*

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Proof. For $\phi \in M_d(G)^\wedge$, the proof is similar to the first part of Theorem 5.

Conversely, let $f \mapsto f \cdot \phi$ from $A(G)$ to $\mathscr{L}^\infty(\hat{G})$ extend to a compact operator on $C(G)$. By Theorem 1, $\phi \in M(G)^\wedge$. Write $\phi = \hat{\mu}$. We must show that μ is a discrete measure.

For compact groups of bounded representation type, there exists an open abelian subgroup H of finite index in G [**6**]. Since discrete measures do induce compact operators on $C(G)$, we may suppose μ is purely continuous.

By translation, if necessary, we may assume $|\mu|H \neq 0$. Observe that there is a natural imbedding of $A(H) \rightarrow A(G)$, (by extending $f \in A(H)$ to be 0 off H). Thus we have a continuous map $\mathscr{L}^\infty(\hat{G}) \rightarrow l^\infty(\hat{H})$. Hence the map $C(H) \rightarrow L^\infty(\hat{G}) \rightarrow l^\infty(\hat{H})$ defined by

$$g \mapsto \mathscr{F}(gd\mu) \rightarrow \mathscr{F}_H(gd(\mu|H))$$

(where $\mathscr{F}_H$ denotes the Fourier-Stieltjes transform on H) is a compact operator which is a contradiction by Theorem 5.

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