3

Historical notes

3.1 Extended charge models (1897–1912)

When in 1897 J. J. Thomson identified the cathode rays as consisting of particles with charge $-e$, not only had he discovered the first elementary particle, but posed the theoretical challenge of computing the energy–momentum relation of this novel object. To put it concisely, we write the equations of motion in approximately uniform $\boldsymbol{E}$ and $\boldsymbol{B}$ fields as

$$m(\boldsymbol{v})\dot{\boldsymbol{v}} = e(\boldsymbol{E} + c^{-1}\boldsymbol{v} \times \boldsymbol{B}) \tag{3.1}$$

with $m(\boldsymbol{v})$ the velocity-dependent mass as a 3×3 matrix. The challenge was to predict the ratio $m(\boldsymbol{v})/e$. For small velocities it was well established that the mass is independent of $\boldsymbol{v}$. But for the electron with its tiny mass and unprecedented range of accessible velocities the case was wide open. In fact, Thomson (1881) himself had pointed out that, in analogy with a ball immersed in a fluid, the coupling to the self-generated electromagnetic field will induce a velocity dependence of the mass.

So which theory could be used to determine $m(\boldsymbol{v})$? In fact, there was little choice. Since the phenomenon under consideration is clearly electromagnetic, the Maxwell–Lorentz equations had to be used, and since the trajectory of a single charge was measured, one had to couple through Newton's equations of motion. Thus the electron was pictured as a tiny sphere charged with electricity. In the inhomogeneous Maxwell equations the current generated by that moving sphere had to be inserted. On the other hand the electromagnetic fields react back on the charge distribution through the Lorentz force. Thereby the so-called extended charge model was introduced. Abraham (1903, 1904) adopted a charge distribution which is rigid in the laboratory frame. The corresponding energy–momentum relation is discussed at length in the second volume of his book on electromagnetism (Abraham 1905), compare with section 4.1. For Abraham's model, Sommerfeld

(1904a, 1905) obtained an exact equation of motion for the electron. As a complicating and unfamiliar feature it contains memory terms through the integration over the retarded fields. Lorentz (1904a, b) proposed a charge distribution which is rigid in its momentary rest frame, and therefore, as seen from the laboratory frame, contracting parallel to its momentary velocity. It was left completely open by which forces this charge distribution would be kept in place. Poincaré (1905, 1906) developed nonelectromagnetic models where additional stresses counteracted the Coulomb repulsion. Bucherer (1904, 1905) and Langevin (1905) introduced a charge distribution Lorentz contracted under the constraint of constant volume.

Up to 1900 electromagnetism was dominated by mechanics, in the sense that physicists felt compelled to introduce mechanical models for electromagnetic fields. Light would propagate through a rather mysterious gas, called the ether, and not simply through vacuum. The great revolution of the young electrodynamicists of the day was to reverse this position and consider inertial mass to be of purely electromagnetic origin. This electromagnetic world picture was nourished by the fact that in all extended charge models the velocity-dependent mass has the additive structure $m(\boldsymbol{v}) = m_\mathrm{b}\mathbb{1} + m_\mathrm{f}(\boldsymbol{v})$, as 3×3 matrices with $\mathbb{1}$ the unit matrix, where m_b is the bare mechanical mass of the particle, in accordance with Newtonian mechanics taken to be velocity independent, and $m_\mathrm{f}(\boldsymbol{v})$ is the mass due to the coupling to the field, which was to be computed from the model charge distribution. In the spirit of the electrodynamic world picture it was natural to set $m_\mathrm{b} = 0$. Then Lorentz predicted the standard relativistic velocity dependence, which only for $|\boldsymbol{v}/c| > 0.3$ differed significantly from the results of Abraham and Bucherer.

While experiments were on the way to decide between the competing theories, the whole enterprise came to a sudden end, since Einstein (1905a, b) forcefully argued that just like electromagnetism in vacuum also the mechanical laws had to be Lorentz invariant. But if Einstein was right, then the energy–momentum relation of the electron had to be the relativistic one, as emphasized independently by Poincaré (1906). Thus the only free parameter was the rest mass of the electron which anyway could not be deduced from theory, since the actual charge distribution was not known. There was simply nothing left to compute. At the latest with the atomic model of Bohr, to say 1913, it became obvious that a theory based on classical electromagnetism could not account for the observed stability of atoms nor for the sharp spectral lines. Classical electron theory, as a tool for explaining properties of atoms, electrons, and nuclei, was abandoned.

The experimental status remained ambiguous for some time. Kaufmann (1901) favored Abraham's model up to 1906. Only through the experiments of Bucherer (1908, 1909) were the predictions of Einstein and Lorentz considered to be reasonably confirmed. Of course, by that time Einstein had already convinced the theoreticians, and any other outcome would have been in serious doubt. A repetition of these historical experiments dryly concludes that "it seems fair to say that the Bucherer–Neumann experiments proved very little, if anything more than the Kaufmann experiments, which indicated a large qualitative increase of mass with velocity", Zahn and Spees (1938).

The effective equation of motion for the electron as given by Eq. (3.1) could not possibly have been the full story. Through the work of Larmor it was already understood that a charge loses energy through radiation at a rate roughly proportional to $\dot{\boldsymbol{v}}^2$. Lorentz observed that in the approximation of small velocities this loss could be accounted for by the friction or radiation reaction force

$$\boldsymbol{F}_{\mathrm{rr}} = \frac{e^2}{6\pi c^3}\,\ddot{\boldsymbol{v}}, \tag{3.2}$$

which had to be added to the Lorentz force in Eq. (3.1). In 1904 Abraham obtained this friction force for arbitrary velocities as

$$\boldsymbol{F}_{\mathrm{rr}} = \frac{e^2}{6\pi c^3}\left[\gamma^4 c^{-2}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} + 3\gamma^6 c^{-4}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\boldsymbol{v} + 3\gamma^4 c^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \gamma^2\ddot{\boldsymbol{v}}\right]. \tag{3.3}$$

He argued that energy and momentum are transported to infinity through the far field. On that scale the charge distribution is like a point charge and the electromagnetic fields can be computed from the Liénard–Wiechert potentials. Using conservation of energy and momentum for the total system he showed that the loss at infinity could be balanced by the friction-like force (3.3). Von Laue (1909) realized that the radiation reaction (3.3) is relativistically covariant and can be written as

$$\mathbf{F}_{\mathrm{rr}} = \frac{e^2}{6\pi c^3}\left[\ddot{\mathbf{u}} - c^{-2}(\dot{\mathbf{u}}\cdot\dot{\mathbf{u}})\mathbf{u}\right], \tag{3.4}$$

with $\boldsymbol{u}$ the four-velocity. It is in this form that the radiation reaction appears in the famous 1921 review article of Pauli on relativity. But apparently there was no incentive to study properties of Newton's equations of motion (3.1) including the full radiation reaction correction (3.3). Using the data from the Kaufmann experiment Abraham estimated the radiation reaction to be down by a factor of 10^{-9} relative to the Hamiltonian motion. Schott (1912) after studying the motion in a uniform electric field concluded: "Hence the effect of the reaction due to radiation is quite inappreciable in this and probably in all practical cases."

The first chapter on the dynamics of classical electrons closes around 1912 with the relativistic version of elasticity theory for deformable bodies by Born (1909) and von Laue (1911a, b). In essence there were two results: (i) a relativistically covariant expression for the radiation reaction and (ii) an energy–momentum relation for the charged particle dependent on the particular model charge distribution.

Of these models only Lorentz's model of a charge distribution properly contracting along its instantaneous velocity is consistent with Einstein's theory of special relativity.

3.2 Nonrelativistic quantum electrodynamics

The time lapse was short: In late 1925 Heisenberg formulated his matrix mechanics and in early 1926 Schrödinger had come to wave mechanics. Through Dirac's transformation theory both approaches were shown to be equivalent. But more importantly in our context, Dirac clearly formulated the rules of canonical quantization, providing the tools for quantizing any Hamiltonian system including those with an infinite number of degrees of freedom. In 1928 Dirac discovered the relativistic generalization of the Schrödinger equation. From then on the theoretician's avant garde strived for creating a relativistic quantum electrodynamics understood as a specific quantum field theory – no small effort – which in a broad sense still continues with us today. The nonrelativistic theory, our concern here, was regarded as being settled. In fact, in its basic theoretical aspects, the research monograph of Heitler (1936) does not differ significantly from modern variants. But obviously, many fascinating phenomena and theoretical developments still lay ahead.

Let us briefly recall the major steps. Born, Heisenberg and Jordan (1926) quantized the wave equation by regarding it as corresponding to an infinite set of harmonic oscillators. They studied the energy fluctuations and derived Planck's law. On 2 February 1927 Dirac proudly reported to Bohr that, on the basis of the new quantum theory, he knew how to compute the lifetime and the line shape of an excited state of an atom in the approximation where only a single photon is emitted. A systematic quantum treatment of emission and absorption of radiation is Dirac (1927). Fermi (1930) recognized the importance of the Coulomb gauge and quantized a system with an arbitrary number of charges. His 1932 review article discusses the quantization of the (many-particle) Abraham model as we know it today; compare with chapter 13. With the theoretical foundations laid down, most physical processes of interest could be handled through second-order perturbation. Perturbation theory as applied to an isolated bound state had been well established. However, for radiation one has to deal with resonances, i.e. unperturbed energies embedded in the continuum energy of field modes. On a practical level Fermi's golden rule settled the issue. The reason why and in what sense this was the correct answer triggered a continuing theoretical effort. As the body of radiation phenomena explainable through quantum mechanics accumulated, the trust in the new theory increased. Divergences were of concern, but, according to Heitler, "it seems now that there is a certain limited field within which the present quantum electrodynamics is correct". High frequencies had to be cut off to taste. In this spirit Bethe arrived at his famous prediction for the Lamb shift of the 2S level of the hydrogen atom.

As well as ultraviolet divergence, nonrelativistic quantum electrodynamics is also infrared divergent, as discovered by Bloch and Nordsieck (1937) and more exhaustively studied by Pauli and Fierz (1938). Even today infrared divergence is a somewhat elusive physical phenomenon. It says that an accelerated charge radiates an infinite number of photons. Since their total energy is finite, by necessity these photons must have ever-increasing wavelengths.

3.3 The point charge

In the 1930s and early 1940s it was a fairly widespread belief that one way to overcome the difficulties of quantum electrodynamics is a better understanding of the classical theory of point charges coupled to their radiation field. Of course, this was to be understood only as an intermediate step to the final goal, namely a consistent quantized theory. Our third section deals with a single paper: "Classical theory of radiating electrons" submitted by P. A. M. Dirac on 15 March 1938. Dirac's paper was equally motivated by quantum electrodynamics; however, as such it is concerned only with classical electron theory.

We have to report the findings of Dirac in sufficient detail, since most later activities start from there. The formal argument in the original paper can be well followed and alternative versions can be found in Rohrlich (1990), Teitelbom *et al.* (1980), and Thirring (1997). Thus there is no need for repetition and we can focus on the conclusions. At first reading it is best to disregard all philosophical claims and concentrate on the equations. But before that, let us see how Dirac himself viewed the 1897–1912 period:

> The Lorentz model of the electron as a small sphere charged with electricity, possessing mass account of the energy of the electric field around it, has proved very valuable in accounting for the motion and radiation of electrons in a certain domain of problems, in which the electromagnetic field does not vary too rapidly and the accelerations of the electrons are not too great.

Dirac's goal was to construct quantum electrodynamics. There the electron is regarded as an elementary particle with, almost by definition, no internal structure. Thus Dirac had to dispense with model charges and develop a theory of *point-like* electrons.

What then did Dirac really accomplish? Of course, he assumes the validity of the inhomogeneous Maxwell equations. The current is generated by a point charge whose motion is yet to be determined. Mechanically this point charge is relativistic with bare mass m_{b}. There is no explicit reaction of the field back onto the charge,

https://doi.org/10.1017/9781009402286.004 Published online by Cambridge University Press

since at no stage would Dirac invoke the Lorentz force. Instead conservation of energy and momentum should suffice to fix the true trajectory of the point charge. Note that this is very different from the extended charge models where the starting point is a closed system of equations for the particle and the Maxwell field. Dirac studies the flow of energy and momentum through a thin tube of radius R around the world line of the particle. The computation simplifies by writing the retarded fields generated by the motion of the point charge as

$$\mathbf{F}_{\mathrm{ret}} = \frac{1}{2}(\mathbf{F}_{\mathrm{ret}} + \mathbf{F}_{\mathrm{adv}}) + \frac{1}{2}(\mathbf{F}_{\mathrm{ret}} - \mathbf{F}_{\mathrm{adv}}) \tag{3.5}$$

in all of space-time. The difference term turns out to be finite on the world line of the charge and, through a balancing of energy and momentum, yields in the limit $R \to 0$, the relativistic radiation reaction (3.4).

The more delicate term in (3.5) is the sum, which is divergent on the world line of the particle. At the expense of ignoring other divergent terms, cf. Thirring (1997), Eq. (8.4.16), Dirac obtains the expected result, namely

$$-\frac{e^2}{4\pi Rc^2}\dot{\mathbf{u}} = -m_{\mathrm{f}}\dot{\mathbf{u}}\,. \tag{3.6}$$

Adding the radiation reaction (3.4) and equating with the mechanical four-momentum, the final result is an equation of motion which determines the trajectory of the particle,

$$(m_{\mathrm{b}} + m_{\mathrm{f}})\dot{\mathbf{u}} = m_{\mathrm{exp}}\dot{\mathbf{u}} = e\mathbf{F}_{\mathrm{ex}} \cdot \mathbf{u} + \frac{e^2}{6\pi c^3}\left[\ddot{\mathbf{u}} - c^{-2}(\dot{\mathbf{u}} \cdot \dot{\mathbf{u}})\mathbf{u}\right] + \mathcal{O}(R) \tag{3.7}$$

with an error of the size of the tube, where we have added the prescribed electromagnetic field tensor $\mathbf{F}_{\mathrm{ex}}$ of external fields.

To complete his argument, Dirac had to take the limit $R \to 0$. Since $m_{\mathrm{f}} \to \infty$, this amounts to

$$m_{\mathrm{b}} \to -\infty,\ m_{\mathrm{f}} \to \infty,\ m_{\mathrm{exp}} = m_{\mathrm{b}} + m_{\mathrm{f}} \quad \text{fixed}, \tag{3.8}$$

where m_{exp} is adjusted such that it agrees with the experimentally determined mass of the charged particle. The combined limit (3.8) is the classical mass renormalization.

Dirac admits that “such a model is hardly a plausible one according to current physical ideas but this is not an objection to the theory provided we have a reasonable mathematical scheme.”

Equation (3.7), dropping the terms $\mathcal{O}(R)$, is the Lorentz–Dirac equation. Within the framework of Dirac it makes no sense to ask whether the Lorentz–Dirac equation is “exact”, since there is nothing to compare with. The Lorentz–Dirac equation comes as one package, so to speak. One could compare only with real experiments, which is difficult since the radiation reaction is very small, or one could compare with higher-level theories such as quantum electrodynamics. But this has never been seriously attempted, since, to begin with, it would require a well-defined relativistic quantum field theory which is a difficult task.

The Lorentz–Dirac equation is identical to the effective equations of motion obtained from extended charge models, if we ignore for a moment the possibility that the kinetic energy might come out differently depending on which model charge is used. In this sense Dirac has recovered the previous results through a novel approach. However, there is an important distinction. For extended charge models one has a true solution for the position of the charged particle, say $\tilde{\mathbf{q}}(t)$. One can then compare $\tilde{\mathbf{q}}(t)$ with a solution of the Lorentz–Dirac equation and hope for agreement in asymptotic regimes, like slowly varying potentials. In addition, for an extended charge model one can set the bare mass to some negative value and study the consequences.

Dirac continues with a remark which shattered the naive trust in classical electron theory. He observes that even for zero external fields Eq. (3.7) has solutions where $|\boldsymbol{v}(t)/c| \to 1$ as $t \to \infty$ and $|\dot{\boldsymbol{v}}(t)|$ increases beyond any bound. Such unphysical solutions he called runaway solutions. It is somewhat surprising that runaways apparently went completely unnoticed before, which only indicates that no attempt was made to apply the Lorentz–Dirac equation to a concrete physical problem. If one inserts numbers, then runaways grow very fast. For instance, for an electron $\dot{\boldsymbol{v}}(t) = \dot{\boldsymbol{v}}(0)e^{t/\tau}$ with $\tau = 10^{-23}$ s. Thus if the Lorentz–Dirac equation (3.7) is a valid approximation in an extended charge model, which after all was the general understanding of the 1897–1912 period, then this model must also have runaway solutions – a conclusion in obvious conflict with empirical evidence.

Dirac proposed to eliminate the runaway solutions by requiring the asymptotic condition

$$\lim_{t\to\infty} \dot{\mathbf{u}}(t) = 0 \,. \tag{3.9}$$

As a bonus the problem of the missing initial condition is resolved: since in (3.7) the third derivative appears, one has to know $\mathbf{q}(0), \dot{\mathbf{q}}(0)$, as in any mechanical problem, and in addition $\ddot{\mathbf{q}}(0)$. If one accepts (3.9), the initial condition for $\dot{\mathbf{u}}$ is replaced by the asymptotic condition (3.9). Dirac checked that for zero external forces and for a spatially constant but time-dependent force the asymptotic condition singles out physically meaningful solutions. By the end of 1938 the classical electron theory was in an awkward shape, in fact in a much worse shape than by the end of 1912. Formal, but even by strict standards careful, derivations yielded an equation with unphysical solutions. How did they come into existence? While Dirac's asymptotic condition seemed to be physically sensible, it was very much ad hoc and imposed *post festum* to get rid of unwanted guests. Even physicists willing to accept the asymptotic condition as a new principle, like Haag (1955), could not be too happy. Solutions satisfying the asymptotic condition are acausal in the sense that the charge starts moving even before any force is acting. To be sure, the causality violation is on the time scale of $\tau = 10^{-23}$ s for an electron, and even shorter for a proton, and thus has no observable consequences. But acausality remains as a dark spot in relativistic theory. The clear recognition of runaway solutions generated a sort of consensus that the coupled Maxwell–Newton equations have internal difficulties.

In the preface of his book Rohrlich writes:

> Most applications treat electrons as point particles. At the same time, there was the widespread belief that the theory of point particles is beset with various difficulties such as infinite electrostatic self-energy, a rather doubtful equation of motion which admits physically meaningless solutions, violation of causality, and others. It is not surprising, therefore, that the very existence of a consistent classical theory of charged particles is often questioned.

In Chapter 28 of the Feynman Lectures we read:

> Classical mechanics is a mathematically consistent theory; it just doesn't agree with experience. It is interesting, though, that the classical theory of electromagnetism is an unsatisfactory theory all by itself. The electromagnetic theory predicts the existence of an electromagnetic mass, but it also falls on its face in doing so, because it does not produce a consistent theory.

And finally to quote from the textbook on mathematical physics by Thirring:

> Not all solutions to (3.7) are crazy. Attempts have been made to separate sense from nonsense by imposing special initial conditions. It is to be hoped that some day the real solution of the problem of the charge–field interaction will look differently, and the equations describing nature will not be so highly unstable that the balancing act can only succeed by having the system correctly prepared ahead of time by a convenient coincidence.

To be sure, these issues were of concern only to theoretical physicists in search of a secure foundation. Synchrotron radiation sources were built anyhow. The loss in energy of an electron during one revolution can be accounted for by Larmor's formula. This is then the amount of energy which has to be supplied in order to maintain a stationary electron current. The radiation emitted from the synchrotron source is computed from the inhomogeneous Lorentz–Maxwell equations with a point charge source, i.e. from the Liénard–Wiechert potentials. No problem.

3.4 Wheeler–Feynman electrodynamics

To avoid the infinities of self-interaction Wheeler and Feynman (1945, 1949) designed a radical solution, at least on the classical level, since the quantized version of their theory was never accomplished.

Their basic tenet is to have as dynamical degrees of freedom only the trajectories of the particles. As such there are no electromagnetic fields, even though one still uses them as a familiar and convenient notational device. As Wheeler (1998) puts it later on, the 1940s were his period of "all particles – no fields" and he wanted to understand how far this point of view could be pushed.

Wheeler–Feynman electrodynamics starts from an action which was first written down by Fokker (1929). Let us consider N particles, where the i-th particle has mass m_i, charge e_i, and a motion given by the world line $\mathbf{q}_i(\tau_i)$, $i = 1, \dots, N$. The world line is parametrized by its eigentime τ_i and the dot '˙' denotes differentiation with respect to this eigentime. The action functional has the form

$$\mathsf{S} = -\sum_{i=1}^{N} m_i c^2 \int \mathrm{d}\tau_i + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} e_i e_j \iint \delta((\mathbf{q}_i - \mathbf{q}_j)^2)(\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_j) \mathrm{d}\tau_i \mathrm{d}\tau_j \,. \tag{3.10}$$

A formal variation of S leads to the equations of motion

$$m_i \ddot{\mathbf{q}}_i = \frac{e_i}{c} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{2} \big(\mathbf{F}_{\mathrm{ret}(j)}(\mathbf{q}_i) + \mathbf{F}_{\mathrm{adv}(j)}(\mathbf{q}_i)\big) \cdot \dot{\mathbf{q}}_i \,. \tag{3.11}$$

Here $\mathbf{F}_{\mathrm{ret}(j)}(\boldsymbol{q}_i)$ and $\mathbf{F}_{\mathrm{adv}(j)}(\boldsymbol{q}_i)$ are the retarded and advanced Liénard–Wiechert fields generated by the charge at $\boldsymbol{q}_j$ and evaluated at $\boldsymbol{q}_i$. They are derived from the retarded and advanced potentials

$$\mathbf{A}_{\mathrm{ret}(j)}(\mathbf{x}) = e_j \dot{\mathbf{q}}_j(\tau_{j\mathrm{ret}})[(\mathbf{x} - \mathbf{q}_j(\tau_{j\mathrm{ret}})) \cdot \dot{\mathbf{q}}_j(\tau_{j\mathrm{ret}})]^{-1} \,, \tag{3.12}$$

$$\mathbf{A}_{\mathrm{adv}(j)}(\mathbf{x}) = e_j \dot{\mathbf{q}}_j(\tau_{j\mathrm{adv}})[(\mathbf{x} - \mathbf{q}_j(\tau_{j\mathrm{adv}})) \cdot \dot{\mathbf{q}}_j(\tau_{j\mathrm{adv}})]^{-1} \tag{3.13}$$

with $\tau_{j\mathrm{ret}}$, respectively $\tau_{j\mathrm{adv}}$, the eigentime when the trajectory $\mathbf{q}_j$ crosses the backward, respectively the forward, light cone with apex at $\mathbf{x}$. Notationally (3.11) looks like a set of ordinary differential equations. In fact, the locations of the other particles have to be known both at the advanced and retarded times, a situation which is not covered by any of the standard techniques. Even if the existence of solutions is taken for granted, it is widely open which data would single out a specific one.

To transform (3.11) into a familiar form, we use the decomposition (3.5) and Dirac's observation that $(\mathbf{F}_{\mathrm{ret}} - \mathbf{F}_{\mathrm{adv}})/2$ at the trajectory of the particle yields the radiation reaction. Then

$$m_i \ddot{\mathbf{q}}_i = \frac{e_i}{c} \sum_{\substack{j=1 \\ j \neq i}}^{N} \mathbf{F}_{\mathrm{ret}(j)}(\mathbf{q}_i) \cdot \dot{\mathbf{q}}_i + \frac{e_i^2}{6\pi c^3} \left(\dddot{\mathbf{q}}_i - c^{-2} \left(\ddot{\mathbf{q}}_i \cdot \ddot{\mathbf{q}}_i \right) \dot{\mathbf{q}}_i \right)$$
$$+ \frac{e_i}{c} \sum_{j=1}^{N} \frac{1}{2} \big(\mathbf{F}_{\mathrm{adv}(j)}(\mathbf{q}_i) - \mathbf{F}_{\mathrm{ret}(j)}(\mathbf{q}_i) \big) \cdot \dot{\mathbf{q}}_i \,. \qquad (3.14)$$

Of course, being symmetric in time, we could have equally transformed to the advanced fields for the force and a radiation reaction with reversed sign.

As a specific example let us consider the scattering of two charges with all other charges far apart. In the framework of the Lorentz model one would start with two charges and their comoving Coulomb field, sufficiently far apart and with incoming velocities. If radiation reaction is neglected, the bare mass is renormalized, and the force on one particle is due to the other particle at the retarded time. In the Wheeler–Feynman theory for *two* particles, the mass is just the bare mass, the forces are the average of retarded and advanced, and there is no radiation reaction. The Wheeler–Feynman theory seems to be at variance with empirical observations.

The crucial new element of their theory is that even in the case of two-particle scattering, the motion of all other charges cannot be ignored. Thus in (3.14), we take only $i = 1, 2$, but sum over large N. Wheeler and Feynman spend a considerable amount of effort to argue that when averaged over the random-like motion of all other charges, the last term in (3.14) vanishes and they call this the condition of a perfect absorber. The exact cancellation is hard to check and one has to be satisfied with qualitative arguments. The perfect absorber granted, in the first sum of (3.14) only the terms $j = 1, 2$ contribute by assumption and one has achieved the reduction to a two-particle problem with retarded forces. In its 18-dimensional phase space there is a 12-dimensional submanifold of physical solutions; all others run away. Wheeler and Feynman discuss an energy-like quantity for the system of N charges which seems to ensure that all solutions to (3.11) are well behaved. As a consequence, only the physical solutions to (3.14) with perfect absorber are a valid approximation to the motion of N charges as governed by (3.11) and agreement with the conventional theory is accomplished.

Notes and references

Section 3.1

An authorative, highly recommended source on the history of the classical electron theory is Miller (1997), which should be augmented by Pais (1972, 1982), by Rohrlich (1973), and by the introductory chapters of Rohrlich (1990). For a discussion of the Kaufmann experiments I refer to Cushing (1981) and Miller (1997). The monograph by Schott (1912) is the most complete technical account. It contains lots of material which has become an integral part of our present-day textbooks on electrodynamics and discusses in detail properties of various electron models. Reviews of classical electron theory are Hönl (1952), Caldirola (1956), Erber (1961), Barut (1980), Teitelbom *et al.* (1980), Coleman (1982), and Pearle (1982). The interconnection with quantum electrodynamics before the 1947 Shelter Island conference is vividly described in Schweber (1994).

Section 3.2

There are excellent studies of the historical development of quantum electrodynamics as culminating in the work of Dyson, Feynman, Schwinger, and Tomonaga, in which as one part also the nonrelativistic theory is discussed. The most complete coverage is Schweber (1994), where the mentioned letter by Dirac is reproduced. Miller (1994) covers the history up to 1938 and includes reprints of the most important papers. A somewhat different selection is Schwinger (1958) with a recommended introduction. A further source is the monumental work of Mehra and Rechenberg (2000) on *The Historical Development of Quantum Theory*. The relevant volume is no. 6, part 1. Modern textbooks and research monographs on nonrelativistic quantum electrodynamics are Heitler (1936, 1958), Power (1964), Louisell (1973), Healy (1982), Craig and Thirunamachandran (1984), Cohen-Tannoudji, Dupont-Roc and Grynberg (1989, 1992), Milonni (1994) among others. They all have a common core, but emphasize rather diverse aspects once it comes to applications.

Section 3.3

Kramers' (1948) investigations on the mass renormalization in the classical theory were instrumental for a correct computation of the Lamb shift. We refer to Dresden (1987) and Schweber (1994).

Section 3.4

The two-body problem in Wheeler–Feynman electrodynamics is discussed by Schild (1963). The existence and classification of solutions is studied by Bauer (1997). A few explicit solutions are listed in Stephas (1992).

The opposite extreme "no particles – all fields" is briefly mentioned in the Notes to section 2.5.