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A PROPERTY OF THE COMPLEX SEMIGROUP ALGEBRA OF A FREE MONOID

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Abstract

It is shown that the complex semigroup algebra of a free monoid of rank at least two is $*$-primitive, where $*$ denotes the involution on the algebra induced by word-reversal on the monoid.

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Let A be an algebra over the complex field $\mathbb{C}$ that admits an involution $*$; thus $*$ is a mapping $A \to A$ such that for all $a, b \in A$ and $\lambda \in \mathbb{C}$

$$(a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*, \quad a^{**} = a, \quad (\lambda a)^* = \bar{\lambda} a^*,$$

where $\bar{\lambda}$ denotes the complex conjugate of λ. A right module V for A is termed a $*$-*module* if and only if it admits an inner product $\langle\,|\,\rangle$ such that

$$\langle ua|v\rangle = \langle u|va^*\rangle \quad \text{for all } u, v \in V \text{ and } a \in A.$$

We say that A is $*$-*primitive* if and only if it has a faithful irreducible $*$-module.

The complex semigroup algebra of a semigroup S is denoted by $\mathbb{C}[S]$. For a nonempty set X, the free monoid and the free group on X are denoted, respectively, by M_X and G_X. Note that $\mathbb{C}[M_X]$ is the free complex algebra-with-unity on X. It is well known and easy to see that each of the algebras $\mathbb{C}[M_X]$ and $\mathbb{C}[G_X]$ possesses an involution. Let $*$ denote the involution on $\mathbb{C}[M_X]$ defined by

$$\left(\sum_{i=1}^{n} \alpha_i y_i\right)^* := \sum_{i=1}^{n} \bar{\alpha}_i \overleftarrow{y_i} \quad \text{for } \alpha_i \in \mathbb{C},\ y_i \in M_X,$$

where $\overleftarrow{y_i}$ denotes the reverse of the word y_i, and let † denote the involution on $\mathbb{C}[G_X]$ defined by

$$\left(\sum_{i=1}^{n} \alpha_i g_i\right)^{\dagger} := \sum_{i=1}^{n} \bar{\alpha}_i g_i^{-1} \quad \text{for } \alpha_i \in \mathbb{C},\ g_i \in G_X.$$

Now suppose that X has at least 2 elements. It was shown by Formanek [4] that $\mathbb{C}[G_X]$ is primitive (that is, has a faithful irreducible right module); and his argument can be adapted to also show that $\mathbb{C}[M_X]$ is primitive (see [8, Chapter 9, Ex. 17]). Subsequently, explicit constructions for faithful irreducible right modules for $\mathbb{C}[M_X]$ and $\mathbb{C}[G_X]$ were provided by McGregor ([7] and [6]); and alternative constructions, without cardinality restrictions, appeared in [1] and [2]. As was pointed out by Irving [5], the module constructed for $\mathbb{C}[G_X]$ in [6] is in fact a †-module; thus $\mathbb{C}[G_X]$ is †-primitive. The purpose of the present paper is to show that $\mathbb{C}[M_X]$ is $*$-primitive. This does not appear to follow from the construction in [7]. To obtain the result, we adapt the procedure that establishes the †-primitivity of $\mathbb{C}[G_X]$.

The symbols $\mathbb{N}$ and $\mathbb{Z}$ denote, respectively, the sets of all positive integers and all integers and $|S|$ denotes the cardinal of a set S. Let X be a set with $|X| \geq 2$ and let s, t be distinct elements of X. The identity of G_X (the empty word) is denoted by 1 and the set $\{x^{-1} : x \in X\}$ by X^{-1}. If $g \in G_X \backslash \{1\}$ has reduced form $g = g_1 g_2 \cdots g_n$, where $g_1, g_2, \ldots, g_n \in X \cup X^{-1}$, then we write

$$l(g) := n, \qquad g^{\div} := g_1^{-1} g_2^{-1} \cdots g_n^{-1},$$
$$g^{\Omega} := g_n, \qquad g^{\flat} := g_1 g_2 \cdots g_{n-1} \ (= 1 \text{ if } n = 1).$$

We also take $l(1) = 0$. Next, we write

$$L := \left\{ g \in G_X \;\middle|\; \begin{array}{l} g \text{ has reduced form } s^k g_1 g_2 \cdots g_n \text{ for} \\ k \in \mathbb{Z}\backslash\{0\}, 0 \leq n \leq |k|, g_i \in X \cup X^{-1} \end{array} \right\} \cup \{1\}$$

and $E := \{g \in G_X : g \notin L \text{ and } g^{\flat} \in L\}$. As in [6], we use these sets to define subsets $\mathscr{L}, \mathscr{E}, \mathscr{U}^+, \mathscr{U}^-, \mathscr{U}$, and $\mathscr{B}$ of $G_X \times \mathbb{Z}$ by $\mathscr{L} := L \times \{0\}$, $\mathscr{E} := E \times \{0\}$,

$$\mathscr{U}^+ := \{(w, n) : w \in E,\ w^{\Omega} \in X \text{ and } n \in \mathbb{N}\},$$
$$\mathscr{U}^- := \{(w, -n) : w \in E,\ w^{\Omega} \in X^{-1} \text{ and } n \in \mathbb{N}\},$$

$\mathscr{U} := \mathscr{U}^+ \cup \mathscr{U}^-$, and $\mathscr{B} := \mathscr{L} \cup \mathscr{E} \cup \mathscr{U}$. We also define a subset $\mathscr{U}^*$ of $\mathscr{U}$ by

$$\mathscr{U}^* := \{(t, 3^n) : n \in \mathbb{N} \cup \{0\}\}.$$

In [6], $\mathscr{U}^*$ is taken to be $\{(t, 2^n) : n \in \mathbb{N}\}$, but this change does not affect the validity of the construction.

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It may be verified that $\mathscr{B}$ has cardinal $\max\{|X|, \aleph_0\}$. Let V be the complex vector space consisting of all mappings $\mathscr{B} \to \mathbb{C}$ of finite support, so we may write a typical element of V in the form $\sum_{i=1}^{n} \alpha_i e_i$ for some $n \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$ and $e_i \in \mathscr{B}$. Again, following [6], we define a right action of $\mathbb{C}[G_X]$ on V. First, we define $ex \in V$ for $e \in \mathscr{B}$ and $x \in X$ by the rules below:

$$\text{for all } (w, 0) \in \mathscr{L}, \quad (w, 0)x = (wx, 0) \in \mathscr{L} \cup \mathscr{E},$$

$$\text{for all } (w, 0) \in \mathscr{E}, \quad (w, 0)x = \begin{cases} (w, 1) \in \mathscr{U}^+ & \text{if } w^{\Omega} \in X, \\ (w^{\flat}, 0) \in \mathscr{L} & \text{if } w^{\Omega} = x^{-1}, \\ (w^{\vec{}}, 0) \in \mathscr{E} & \text{otherwise,} \end{cases}$$

$$\text{for all } (w, k) \in \mathscr{U}, \quad (w, k)x = \begin{cases} -(w, k+1) & \text{if } x = s \text{ and } (w, k) \in \mathscr{U}^*, \\ (w, k+1) & \text{otherwise.} \end{cases}$$

It can be shown that for all $x \in X \backslash \{s\}$ the mapping $\mathscr{B} \to \mathscr{B}$, $e \mapsto ex$ is a permutation. Thus we may extend it by linearity to an invertible mapping $V \to V$, $v \mapsto vx$. Although the rule $e \mapsto es$ for $e \in \mathscr{B}$ does not give a permutation of $\mathscr{B}$, it also extends to an invertible mapping $V \to V$, $v \mapsto vs$. For all $v \in V$ we take $v1 := v$. Next, we define $vx^{-1} \in V$ for all $v \in V$ and all $x \in X$ by $vx^{-1} = w$, where $wx = v$. This enables us to define a right action of G_X on V and hence a right action of $\mathbb{C}[G_X]$ on V.

The first lemma states that, with respect to this action, V is a $\dagger$-module.

LEMMA 1 (Irving [5]). *Let $\langle\ |\ \rangle$ be the inner product on V defined by*

$$\text{for all } e, f \in \mathscr{B}, \quad \langle e|f \rangle = \begin{cases} 1 & \text{if } e = f, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle ua|v \rangle = \langle u|va^{\dagger} \rangle$ for all $u, v \in V$ and $a \in \mathbb{C}[G_X]$.

We now gather together some further properties of V for ease of reference. These properties are straightforward consequences of the action on V and are mostly stated in [6, Lemma 1].

LEMMA 2. (i) *$et \in \mathscr{U}^+$ for all $e \in \mathscr{U}^+$, $et^{-1} \in \mathscr{U}^-$ for all $e \in \mathscr{U}^-$;*
(ii) *for all $e \in \mathscr{B}$, there exists $n \in \mathbb{N}$ such that $et^n \in \mathscr{U}^+$ and $et^{-n} \in \mathscr{U}^-$;*
(iii) *$(s^r, 0)g \in \mathscr{L}$ for all $r \in \mathbb{N}$ and $g \in G_X$ with $l(g) \leq r$;*
(iv) *for all $r \in \mathbb{N}$ and $g, g' \in G_X$ with $l(g)$, $l(g') \leq r$, $(s^r, 0)g = (s^r, 0)g'$ implies $g = g'$.*

Next, as in the proof of [3, Theorem 1.1], we define a homomorphism $\theta : \mathbb{C}[M_X] \to \mathbb{C}[G_X]$ by $\theta(x) := x + x^{-1}$ for all $x \in X$. Any mapping $X \to \mathbb{C}[G_X]$ extends uniquely to a monoid homomorphism $M_X \to (\mathbb{C}[G_X], \cdot)$ and hence to an algebra homomorphism $\mathbb{C}[M_X] \to \mathbb{C}[G_X]$. The lemma below lists some properties of θ.

LEMMA 3. (i) *θ is an injective homomorphism;*

(ii) *for each $n \in \mathbb{N}$, there exists a polynomial f_n over $\mathbb{Z}$ of degree n such that, for all $x \in X$, $x^n + x^{-n} = f_n(\theta(x))$;*

(iii) *for all $a \in \mathbb{C}[M_X]$, $(\theta(a))^\dagger = \theta(a^*)$.*

PROOF. (i) We may regard M_X as a submonoid of G_X. Let $a \in \mathbb{C}[M_X]\backslash\{0\}$. Consider an element w of $\operatorname{supp}(a)$ with $l(w)$ maximal. Then $w \in \operatorname{supp}(\theta(a))$, which shows that $\theta(a) \neq 0$. Hence θ is injective.

(ii) This can be established by induction. In fact, f_n is closely related to the nth Chebychev polynomial of the first type.

(iii) For all $x \in X$, $(\theta(x))^\dagger = x + x^{-1} = \theta(x)$ and so, for all $y \in M_X$, $(\theta(y))^\dagger = \theta(\overleftarrow{y})$. Hence, for all $a \in \mathbb{C}[M_X]$, $(\theta(a))^\dagger = \theta(a^*)$. □

Denote the element $(t, 1)$ of $\mathscr{B}$ by e_1 and define $W \subseteq V$ by

$$W := \{e_1\theta(a) : a \in \mathbb{C}[M_X]\}.$$

Then W is a nonzero subspace of V. Next, we define $\circ : W \times \mathbb{C}[M_X] \to W$ by $w \circ a = w\theta(a)$ for $w \in W$, $a \in \mathbb{C}[M_X]$. It is straightforward to see that $\circ$ is a right action of $\mathbb{C}[M_X]$ on W. We now show that W is faithful and irreducible under this action.

LEMMA 4. *W is a faithful module for $\mathbb{C}[M_X]$.*

PROOF. Let $a \in \mathbb{C}[M_X]\backslash\{0\}$. Then, by Lemma 3 (i), $\theta(a) \in \mathbb{C}[G_X]\backslash\{0\}$. Thus $\theta(a) = \sum_{i=1}^n \alpha_i g_i$ for some $n \in \mathbb{N}$, some distinct elements $g_i \in G_X$, and some coefficients α_i, not all zero. Take

$$r := \max\{l(g_i) : i = 1, \dots, n\} + 5$$

and write

$$w := e_1(t^2 + t^{-2})(s^r + s^{-r}).$$

Since $(t^2 + t^{-2})(s^r + s^{-r}) = \theta(f_2(t)f_r(s))$, by Lemma 3 (ii), we have that $w \in W$. The action of t and of s on certain elements of $\mathscr{B}$ can be represented diagrammatically as

$$t : \cdots \to (t^{-1}, -1) \to (t^{-1}, 0) \to (1, 0) \to (t, 0) \to (t, 1) \to (t, 2) \to \cdots,$$
$$s : \cdots \to (t^{-1}, -1) \to (t^{-1}, 0) \to (t, 0) \to (t, 1) \to -(t, 2) \to -(t, 3) \to \cdots.$$

Hence we have that

$$(1) \qquad w = [(t,3)+(1,0)](s^r+s^{-r}) \\ = \pm(t,r+3)-(t^{-1},-r+4)+(s^r,0)+(s^{-r},0).$$

From the choice of r,

$$(2) \qquad \pm(t,r+3)g_i \in \mathscr{U}^+, \quad (t^{-1},-r+4)g_i \in \mathscr{U}^- \quad \text{for } i=1,2,\dots,n;$$

and

$$(3) \qquad \{(s^{-r},0)g_i : i=1,2,\dots,n\} \cap \{(s^r,0)g_i : i=1,2,\dots,n\} = \emptyset.$$

Now, by Lemma 2 (iv), since the g_i are distinct so are the elements $(s^r,0)g_i$ for $i=1,\dots,n$. However, by Lemma 2 (iii), these lie in $\mathscr{L}$. Hence, from (1)–(3), $w\theta(a) \neq 0$, that is, $w \circ a \neq 0$. Thus W is faithful. □

LEMMA 5. *W is an irreducible module for $\mathbb{C}[M_X]$.*

PROOF. Take $\langle\,|\,\rangle$ to be the inner product on V defined as in Lemma 1. Let $w \in W\backslash\{0\}$. Then $w = e_1\theta(a)$ for some $a \in \mathbb{C}[M_X]$ and so $\langle e_1\theta(a)|e_1\theta(a)\rangle \neq 0$. However, by Lemma 1 and Lemma 3 (iii),

$$\langle e_1\theta(a)|e_1\theta(a)\rangle = \langle e_1|e_1\theta(a)(\theta(a))^{\dagger}\rangle = \langle e_1|w\theta(a^*)\rangle = \langle e_1|w \circ a^*\rangle$$

and so the coefficient of e_1 in $w \circ a^*$ is nonzero. Thus we may write $w \circ a^* = \sum_{i=1}^n \alpha_i e_i$ for some $n \in \mathbb{N}$, some distinct $e_i \in \mathscr{B}$ with $e_1 = (t,1)$, and some nonzero coefficients α_i for $i=1,\dots,n$.

By Lemma 2 (i) and (ii), there exists $p \in \mathbb{N}$ such that

$$e_it^p \in \mathscr{U}^+, \quad e_it^{-p} \in \mathscr{U}^- \quad \text{for } i=1,\dots,n.$$

These $2n$ elements are distinct. Write $(g_i,k_i) := e_it^p$ for $i=1,\dots,n$. In particular, $(g_1,k_1) = (t,1)t^p = (t,p+1)$. Let $l \in \mathbb{N}$ be defined by

$$l := \max\{k_i : 1 \le i \le n \text{ and } g_i = t\}.$$

Choose $m \in \mathbb{N}$ such that $3^{m-1} > l$ and take $q := 3^m - l$. Then

$$(4) \qquad e_it^{p+q} = (g_i, k_i+q) = (g_i, 3^m - l + k_i) \quad \text{for } i=1,\dots,n.$$

Let $j \in \{1,\dots,n\}$ be such that $g_j = t$ and $k_j = l$. Then, by (4), $e_jt^{p+q} = (t,3^m)$ and so

$$(5) \qquad e_jt^{p+q}(t-s) = 2(t,3^m+1), \quad e_jt^{p+q}(t^{-1}-s^{-1}) = 0.$$

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We next show that

(6) $$e_i t^{p+q}(t-s) = 0, \quad e_i t^{p+q}(t^{-1}-s^{-1}) = 0 \quad (i \neq j).$$

Let $i \in \{1, \dots, n\}$ with $i \neq j$. First, suppose that $g_i = t$. Then $k_i < l$ and so $3^m - l + k_i < 3^m$. Further, since $3^{m-l} > l$, we have that $3^m - l > 3^{m-l} + l$, and so $3^m - l + k_i > 3^{m-1} + 1$. Since $e_i t^{p+q} = (t, 3^m - l + k_i)$, by (4), it follows that (6) holds. Now suppose that $g_i \neq t$. Then, from (4), we see that (6) holds in this case also. Thus we have established (6). Since $e_i t^{-p} \in \mathscr{U}^-$,

(7) $$e_i t^{-p-q}(t-s) = 0, \quad e_i t^{-p-q}(t^{-1}-s^{-1}) = 0 \quad \text{for } i = 1, \dots, n.$$

Write $u := t + t^{-1} - s - s^{-1}$. Then, by (5)–(7),

$$\begin{aligned}(w \circ a^*)(t^{p+q} + t^{-p-q})u &= (w \circ a^*)t^{p+q}(t-s) + (w \circ a^*)t^{p+q}(t^{-1}-s^{-1}) \\ &\quad + (w \circ a^*)t^{-p-q}(t-s) + (w \circ a^*)t^{-p-q}(t^{-1}-s^{-1}) \\ &= 2\alpha_j(t, 3^m + 1).\end{aligned}$$

Now write $r := 3^m - 1$. Then $(t, 3^m + 1)(t^r + t^{-r}) = (t, 2.3^m) + (t, 2)$ and so

$$\begin{aligned}(t, 3^m + 1)(t^r + t^{-r})u &= (t, 2.3^m)(t-s) + (t, 2.3^m)(t^{-1}-s^{-1}) \\ &\quad + (t, 2)(t-s) + (t, 2)(t^{-1}-s^{-1}) \\ &= 2(t, 1) = 2e_1.\end{aligned}$$

Hence

(8) $$(w \circ a^*)(t^{p+q} + t^{-p-q})u(t^r + t^{-r})u = 4\alpha_j e_1.$$

Let $b \in \mathbb{C}[M_X]$ be defined by $b := f_{p+q}(t)(t-s)f_r(t)(t-s)$, where f_{p+q} and f_r are the polynomials defined in Lemma 3 (ii). Then $\theta(b) = (t^{p+q} + t^{-p-q})u(t^r + t^{-r})u$ and so, from (8), $w \circ (a^*b) = (w \circ a^*)\theta(b) = 4\alpha_j e_1$. Since $\alpha_j \neq 0$, it follows that $w \circ \mathbb{C}[M_X] = W$. Thus W is irreducible. □

The main result now follows.

THEOREM 6. *Let M_X denote the free monoid on a set X with at least two elements and let $*$ denote the involution on $\mathbb{C}[M_X]$ induced by word-reversal. Then $\mathbb{C}[M_X]$ is $*$-primitive.*

PROOF. By Lemmas 4 and 5, W is a faithful irreducible module for $\mathbb{C}[M_X]$. Now, by Lemma 1, there exists an inner product $\langle \mid \rangle$ on V such that, for all $u, v \in V$ and all $b \in \mathbb{C}[G_X]$, $\langle ub|v\rangle = \langle u|vb^\dagger\rangle$. Consider the restriction of this inner product to W. Then, for all $w_1, w_2 \in W$ and all $a \in \mathbb{C}[M_X]$,

$$\begin{aligned}\langle w_1 \circ a|w_2\rangle &= \langle w_1\theta(a)|w_2\rangle = \langle w_1|w_2(\theta(a))^\dagger\rangle \\ &= \langle w_1|w_2\theta(a^*)\rangle, \quad \text{by Lemma 3 (iii)}, \\ &= \langle w_1|w_2 \circ a^*\rangle.\end{aligned}$$

Hence W is a $*$-module and so W is $*$-primitive. □

REMARK. The construction in [7] also shows that the Banach algebra $l^1(M_X)$ is primitive for the case $|X| \geq 2$. The question of whether $l^1(M_X)$ is $*$-primitive in this case remains open.

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