ENUMERATION OF CERTAIN SUBGROUPS OF ABELIAN p-GROUPS

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The number of distinct types of Abelian group of prime-power order p^n is equal to the number of partitions of n. Let $(\rho) = (\rho_1, \rho_2, \ldots, \rho_r)$ be a partition of n and let $(\mu) = (\mu_1, \mu_2, \ldots, \mu_s)$ be a partition of m, with $\rho_1 \geqq \rho_2 \geqq \ldots \geqq \rho_r$ and $\mu_1 \geqq \mu_2 \geqq \ldots \geqq \mu_s$, $\rho_i \geqq \mu_i$, $r \geqq s$, $n > m$. The number of subgroups of type (μ) in an Abelian p-group of type (ρ) is a function of the two partitions (ρ), (μ) and p, and has been determined as a polynomial in p with integer coefficients by Yeh (**1**), Delsarte (**2**) and Kinosita (**3**). Their results differ in form but are equivalent.

P. Hall (**4**) suggested a refinement of this problem in which we require the number of subgroups of type (μ) in an Abelian p-group of type (ρ) which have a quotient group of type (λ). The result, which is a function $g^{\rho}_{\lambda\mu}(p)$ of the three partitions (ρ), (λ), (μ) and p, is known to be a polynomial in p of degree $\sum_i (i-1)(\rho_i - \lambda_i - \mu_i)$, and the coefficient of its highest power is the coefficient of the Schur function $\{\rho\}$ in the product of the Schur functions $\{\lambda\}\{\mu\}$. The precise form of the polynomial is however not known in general.

In this note, the polynomial is determined when

$$(\rho) = (m_1^{n_1}, m_2^{n_2}, \ldots, m_s^{n_s}),$$

$$(\lambda) = (m_1^{n_1-r_1}, (m_1-k)^{r_1}, m_2^{n_2-r_2}, (m_2-k)^{r_2}, \ldots, m_s^{n_s-r_s}, (m_s-k)^{r_s})$$

and $$(\mu) = (k^{r_1+r_2+\ldots+r_s}),$$

where $r_1 + r_2 + \ldots + r_s = r$. The result is given in the Theorem which is proved by means of the two lemmas which follow.

Lemma 1. *The number of subgroups F of type (k^r) in an Abelian p-group E of type (m^n) such that the quotient group E/F is of type $(m^{n-r}, (m-k)^r)$, where $r \leqq n$, $k \leqq m$, is*

$$p^{kr(n-r)}\phi(n, r;\ 1/p),$$

where

$$\phi(s+t, s;\ u) = \frac{(1-u)(1-u^2)\ldots(1-u^{s+t})}{(1-u)\ldots(1-u^s)(1-u)\ldots(1-u^t)},\quad s, t > 0.$$

Proof. From the work of Yeh, Delsarte and Kinosita, it can be shown that the number of subgroups F of type (k^r) in an Abelian p-group E of type (m^n) is $p^{kr(n-r)}\phi(n, r;\ 1/p)$. It remains to prove that E/F, for every such subgroup F, is of the required type.

Let $E = C_1C_2\ldots C_n$, where C_i is a cyclic group of order p^m, let $F = B_1B_2\ldots B_r$, where B_i is a cyclic group of order p^k, and let $E' = C_1C_2\ldots C_r$ with the quotient group E'/F isomorphic to a group F'. We shall need two results.

(i) The quotient group of a cyclic group with respect to a subgroup is also cyclic, and so C_i/B_i is isomorphic to a cyclic group D_i of order p^{m-k}.

(ii) If X, Y are any two groups such that $X \cap Y = 1$ and, for some group Z, the quotient group Z/X is isomorphic to Y, then Z is the direct product XY of X and Y.

Now $C_i = B_iD_i$, so that $E' = \prod_1^r C_i \simeq \prod_1^r (B_iD_i) = \prod_1^r B_i \prod_1^r D_i = F \prod_1^r D_i$.

Since E' is also equal to FF', we see that $F' = \prod_1^r D_i$, i.e. F' is isomorphic to the direct product of r cyclic groups D_i of orders p^{m-k}. Thus $\dfrac{C_1C_2\ldots C_r}{B_1B_2\ldots B_r}$ is of type $((m-k)^r)$. It follows that E/F, which is $\dfrac{C_1C_2\ldots C_rC_{r+1}\ldots C_n}{B_1\ldots B_r}$, is of type $(m^{n-r}, (m-k)^r)$ and the result follows.

Lemma 2. *The number of subgroups F of type $(k^{r_1+r_2})$, where $r_1+r_2 = r$, in an Abelian p-group E of type $(m_1^{n_1}, m_2^{n_2})$, $m_1 > m_2$, such that E/F is of type $(m_1^{n_1-r_1}, (m_1-k)^{r_1}, m_2^{n_2-r_2}, (m_2-k)^{r_2})$, where $r_1 \leqq n_1$, $r_2 \leqq n_2$, $k \leqq m_2$,* is

$$p^{k[r_1(N_1-R_1)+r_2(N_2-R_2)]}\phi(n_1, r_1;\ 1/p)\phi(n_2, r_2;\ 1/p),$$

where $N_t = \sum_1^t n_i$, $R_t = \sum_1^t r_i$.

Proof. Let E be generated by n_1 elements x_i, each of order p^{m_1}, and n_2 elements y_j, each of order p^{m_2}. Let $a_i = x_i^{p^{m_1-k}}$, $i = 1, 2, \ldots, n_1$, and $b_j = y_j^{p^{m_2-k}}$, $j = 1, 2, \ldots, n_2$. Then $a_i^{p^k} = b_j^{p^k} = 1$. Let the cyclic groups generated by x_i and y_j be C_{1i} and C_{2j} respectively. Every C_{1i} has one and only one subgroup of order p^k, namely that generated by a_i, and every C_{2j} has one and only one subgroup of order p^k, namely that generated by b_j. We denote these by $[a_i]$ and $[b_j]$.

The number of subgroups generated by r_1 of the a_i's is, as in Lemma 1, $p^{kr_1(n_1-r_1)}\phi(n_1, r_1;\ 1/p)$ and the number of subgroups generated by r_2 of the b_j's is $p^{kr_2(n_2-r_2)}\phi(n_2, r_2;\ 1/p)$. Consider a particular subgroup generated by r_2 of the b_j's, say the one generated by $b_1, b_2, \ldots, b_{r_2}$. If any of these b_j's is replaced by $b_j \times a_{r_1+1}^{\alpha_1}a_{r_1+2}^{\alpha_2}\ldots a_{n_1}^{\alpha_{n_1-r_1}}$, where $\alpha_1, \ldots, \alpha_{n_1-r_1}$ have any prescribed values in the range 0, 1, 2, ..., p^k-1, then the group generated by this "augmented" generator is also a cyclic group of order p^k. The number of these monomials $a_{r_1+1}^{\alpha_1}a_{r_1+2}^{\alpha_2}\ldots a_{n_1}^{\alpha_{n_1-r_1}}$ is $p^{k(n_1-r_1)}$, since every index α can range from 0 to p^k-1 and the number of a_i's involved is n_1-r_1. Further, any of these monomials may be used to "augment" any of the r_2 b_j's and so, in this way, we can construct $p^{kr_2(n_2-r_2)}\phi(n_2, r_2;\ 1/p) \times p^{kr_2(n_1-r_1)}$ subgroups of type (k^{r_2}) and consequently

$$p^{k[r_1(N_1-R_1)+r_2(N_2-R_2)]}\phi(n_1, r_1;\ 1/p)\phi(n_2, r_2;\ 1/p)$$

subgroups of type $(k^{r_1+r_2})$.

It remains to prove that the quotient groups of these subgroups with respect to E are of the type $(m_1^{n_1-r_1}, (m_1-k)^{r_1}, m_2^{n_2-r_2}, (m_2-k)^{r_2})$ and, further, that there are no other subgroups of E of type $(k^{r_1+r_2})$ having this type of quotient group.

Let F be one of the subgroups of type $(k^{r_1+r_2})$. Without loss of generality, we may take it to be

$$[a_1][a_2]\dots[a_{r_1}][M_1b_1][M_2b_2]\dots[M_{r_2}b_{r_2}],$$

where M_j is any of the monomials $a_{r_1+1}^{\alpha_1}a_{r_1+2}^{\alpha_2}\dots a_{n_1}^{\alpha_{n_1-r_1}}$, $(\alpha = 0, 1, \dots, p^k-1)$, and M_j's in different brackets might possibly be the same. (Note, however, that $[M_jb_j]$ and $[M_jb_r]$, $r \neq j$, have no elements in common except the identity.)

Then the quotient groups $\dfrac{C_{1i}}{[a_i]}$, $i = 1, 2, \dots, r_1$, are cyclic of order p^{m_1-k}, and so, as in Lemma 1, $\dfrac{C_{11}C_{12}\dots C_{1r_1}}{[a_1][a_2]\dots[a_{r_1}]}$ is of type $((m_1-k)^{r_1})$.

Since $M_j^{p^k} = 1$, every M_jy_j generates a cyclic group C'_{2j} of order p^{m_2}. Thus $\dfrac{C'_{2j}}{[M_jb_j]}$ is cyclic of order p^{m_2-k} and it follows that

$$\frac{C_{11}C_{12}\dots C_{1r_1}C'_{21}C'_{22}\dots C'_{2r_2}}{F}$$

is of type $((m_1-k)^{r_1}, (m_2-k)^{r_2})$. But since $C_{1,\,r_1+1}C_{1,\,r_1+2}\dots C_{1n_1}C_{21}\dots C_{2r_2}$ is the same group as $C_{1,\,r_1+1}\dots C_{1n_1}C'_{21}\dots C'_{2r_2}$, we can write E in either of the forms

$$C_{11}\dots C_{1r_1}\dots C_{1n_1}C_{21}\dots C_{2r_2}C_{2,\,r_2+1}\dots C_{2n_2}$$

$$C_{11}\dots C_{1r_1}\dots C_{1n_1}C'_{21}\dots C'_{2r_2}C_{2,\,r_2+1}\dots C_{2n_2}$$

and so E/F is of type $(\lambda) = (m_1^{n_1-r_1}, (m_1-k)^{r_1}, m_2^{n_2-r_2}, (m_2-k)^{r_2})$.

To show that there are no other subgroups of E of type $(k^{r_1+r_2})$ having a quotient group of type (λ), we note that we are obliged to use r_1 of the n_1 a_i's to give the $(m_1-k)^{r_1}$ part of (λ). We must then choose r_2 elements of order p^k so as to give the $(m_2-k)^{r_2}$ part of (λ) without affecting the $m_1^{n_1-r_1}$ and $m_2^{n_2-r_2}$ parts. These r_2 elements must contain a non-vanishing monomial in the b_j's and may contain also a monomial in the a_i's. A monomial in $a_1, a_2, \dots, a_{r_1}$, say N_j, will give a group $[N_jb_j]$ of order p^k, but

$$[a_1]\dots[a_{r_1}][N_1b_1]\dots[N_{r_2}b_{r_2}]$$

is nothing more than $[a_1]\dots[a_{r_1}][b_1]\dots[b_{r_2}]$. Hence the only monomials in the a_i's which give distinct subgroups F of the required type are the M_j as defined above, which proves the lemma.

Using these lemmas, we can now prove the main result.

Theorem. *If* $(\rho) = (m_1^{n_1}, m_2^{n_2}, \ldots, m_s^{n_s})$, $m_1 > m_2 > \ldots > m_s$, *and*

$$(\lambda) = (m_1^{n_1-r_1}, (m_1-k)^{r_1}, m_2^{n_2-r_2}, (m_2-k)^{r_2}, \ldots, m_s^{n_s-r_s}, (m_s-k)^{r_s}),$$

where $r_1+r_2+\ldots+r_s = r$, $r_i \leqq n_i$ $(i = 1, \ldots, s)$ *and* $k \leqq m_s$, *then the number of subgroups F of an Abelian p-group E of type* (ρ) *which are of type* $(k^{r_1+r_2+\ldots+r_s})$ *and for which E/F is of type* (λ) *is*

$$g^{\rho}_{\lambda,\, k^r}(p) = p^{k\sum_1^s r_i(N_i-R_i)} \prod_{i=1}^{s} \phi(n_i, r_i;\ 1/p).$$

Proof. We assume the result is true for a group E' of type $(\rho') = (m_1^{n_1}, \ldots, m_t^{n_t})$, $t < s$. We now have to form the direct product of E' with n_{t+1} cyclic groups of orders $p^{m_{t+1}}$. Let these be generated by z_d where $d = 1, 2, \ldots, n_{t+1}$. Let $w_d = z_d^{p^{m_{t+1}-k}}$ so that $w_d^{p^k} = 1$. The number of subgroups of type $(k^{r_{t+1}})$ generated by the w_d's is, as in Lemma 1, equal to

$$p^{kr_{t+1}(n_{t+1}-r_{t+1})}\phi(n_{t+1}, r_{t+1};\ 1/p).$$

But any w_d can be " augmented ", as in Lemma 2, by a monomial

$$a_{r_1+1}^{\alpha_1} a_{r_1+2}^{\alpha_2} \ldots a_{n_1}^{\alpha_{n_1-r_1}} b_{r_2+1}^{\beta_1} b_{r_2+2}^{\beta_2} \ldots b_{n_2}^{\beta_{n_2-r_2}} \ldots$$

containing $(n_1-r_1)+(n_2-r_2)+\ldots+(n_t-r_t)$ distinct symbols a_i, b_j, ... with every index α_i, β_j, ... capable of any of the values from 0 to p^k-1. Hence the number of subgroups of type $(k^{r_{t+1}})$ which we can construct from the " augmented " w_d's is

$$p^{kr_{t+1}(N_{t+1}-R_{t+1})}\phi(n_{t+1}, r_{t+1};\ 1/p).$$

As in Lemma 2, the quotient group is of type $(m_1^{n_1-r_1}, (m_1-k)^{r_1}, \ldots, m_{t+1}^{n_{t+1}-r_{t+1}}, (m_{t+1}-k)^{r_{t+1}})$ and there are no further subgroups possible under the conditions for F prescribed in Lemma 2.

The theorem now follows by induction.

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