MAXIMAL GROUPS ON WHICH THE PERMANENT IS MULTIPLICATIVE

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Let Δ_n be the set of all $n \times n$, non-singular matrices of the form PD, where P is a permutation matrix and D is a diagonal matrix with complex entries. In (**1**, conjecture 12), Marcus and Minc asked: Is Δ_n a maximal group on which the permanent function is multiplicative? (that is, per $AB =$ per A per B). The field over which the entries range was not mentioned in the conjecture; however, we assume that the complex number field was intended. Corollary 1 answers this in the affirmative. In fact, Δ_n is the only maximal group (or semigroup) on which the permanent is multiplicative. Let ρ_i be the set of all non-zero entries in the ith row and let λ_j be the set of all non-zero entries in the jth column.

THEOREM. *Δ_n is the maximal semigroup of $n \times n$ matrices with ρ_i and λ_j non-empty for all $i, j = 1, \ldots, n$ on which the permanent function is multiplicative.*

Proof. If K is a maximal semigroup of $n \times n$ matrices with ρ_i and λ_j non-empty for all $i, j = 1, \ldots, n$ on which the permanent is multiplicative, then $\Delta_n \leqq K$, since for any matrix A, per $QA =$ per Q per A, where Q is either a permutation matrix or a diagonal matrix.

Suppose that $\Delta_n < K$, and let $A \in K - \Delta_n$. Then, in A there is at least one row with at least two non-zero entries. We shall show that this implies the existence of a matrix $F \in K$, such that $\text{per}(F^2) \neq (\text{per } F)^2$. Since every permutation matrix is in K, we may assume that the nth row of A has at least two non-zero entries, and that $a_{nn} \neq 0$.

Let

$$\delta_i^A = \begin{cases} \bar{a}_{ni} & \text{if } a_{ni} \neq 0 \\ 1 & \text{if } a_{ni} = 0 \end{cases} \quad \text{for } i = 1, \ldots, n,$$

and let $\delta^A = \text{diag}(\delta_1^A, \ldots, \delta_n^A)$. Now the matrix $B = A\delta^A$ is such that all entries in the nth row are real and positive or zero.

In B, $b_{nn} \neq 0$ and at least one other element of the nth row is non-zero. Let μ be a diagonal matrix such that μ_i is real and strictly positive, for $i = 1, \ldots, n$. If any entry a_{ni} in the nth row of B is zero, then we show that, for suitable μ and some permutation matrix P such that $p_{nn} = 1$, $H = (\mu PB)^2$ is in K and has at least one more non-zero entry in the nth row than did A. Let $G = \mu PB$, so that $H = G^2$. We first show that if $b_{nj} \neq 0$, then $h_{nj} \neq 0$.

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The element $g_{nj} = \mu_n b_{nj} \neq 0$ and

$$h_{nj} = \sum_{k=1}^{n} g_{nk} g_{kj} = g_{nn} g_{nj} + \sum_{k=1}^{n-1} g_{nk} g_{kj} = \mu_n^2 b_{nn} b_{nj} + \sum_{k=1}^{n-1} \mu_n \mu_k b_{n,k} b_{\lambda^{-1}(k),j},$$

where $P(\phi_{ij}) = (\phi_{\lambda(i),j})$. Thus, for μ_{nn} sufficiently larger than μ_i, $i = 1, \ldots, n-1$, $h_{nj} \neq 0$.

Next we show that whereas $b_{ni} = 0$, we can choose μ and P so that $h_{ni} \neq 0$. Some element in the ith column of B, say b_{qi}, is non-zero. Choose P so that premultiplication by P interchanges the qth and jth rows of B. Thus, $g_{ji} \neq 0$, and the element

$$h_{ni} = \sum_{k=1}^{n} g_{nk} g_{ki} = g_{nj} g_{ji} + \sum_{\substack{k=1;\\ k\neq j}}^{n} g_{nk} g_{ki} = \mu_n \mu_j b_{nj} b_{\lambda^{-1}(j),i} + \sum_{\substack{k=1;\\ k\neq j}}^{n} \mu_n \mu_k b_{nk} b_{\lambda^{-1}(k),i}.$$

Thus, for μ_i sufficiently larger than μ_k, $k = 1, \ldots, n-1$, $k \neq i$, we obtain $h_{ni} \neq 0$. Note that here μ_n being large does not affect the result since the last term, $\mu_n^2 b_{nk} b_{\lambda^{-1}(n),i} = \mu_n^2 b_{nk} b_{ni}$, in the sum is zero.

This process may be re-applied until one arrives at a matrix C' such that c'_{ni} is non-zero for all $i = 1, \ldots, n$. Let $C = C'\delta^{C'}$; then $C \in K$, c_{ni} is real and $c_{ni} > 0$ for all $i = 1, \ldots, n$.

In a similar manner we can obtain a matrix $E \in K$, such that e_{in} is real and $e_{in} > 0$ for all $i = 1, \ldots, n$. Now, for matrices α and β, where $\alpha = \operatorname{diag}(1, \ldots, 1, \alpha_n)$ and $\beta = \operatorname{diag}(1, \ldots, 1, \beta_n)$ and α_n and β_n are sufficiently large positive real numbers, $F = (E\alpha)(\beta C)$ is in K, $f_{ij} \neq 0$ for all $i, j = 1, \ldots, n$, and $\operatorname{Re}(f_{ij})$ is positive and so much greater than $|\operatorname{Im}(f_{ij})|$ that

$$\operatorname{Re}\left(\prod_{i=1}^{n} f_{i\tau(i)} f_{\tau(i)\sigma(i)} \right) > 0$$

for every $\sigma \in S_n$, the symmetric group on n letters, and every $\tau \in C_n$, the set of all mappings $\tau\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$.

Now,

$$\operatorname{per} AB = \sum_{\sigma \in S_n} \prod_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{k\sigma(i)}$$

$$= \sum_{\sigma \in S_n} \sum_{\tau \in C_n} \prod_{i=1}^{n} a_{i\tau(i)} b_{\tau(i)\sigma(i)}$$

and

$$\operatorname{per} A \operatorname{per} B = \left(\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \right) \left(\sum_{\sigma \in S_n} \prod_{i=1}^{n} b_{i\sigma(i)} \right)$$

$$= \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \prod_{i=1}^{n} a_{i\tau(i)} b_{\tau(i)\sigma(i)}.$$

Hence, for $A, B \in K$ we must have that

$$0 = \operatorname{per} AB - \operatorname{per} A \operatorname{per} B = \sum_{\sigma \in S_n} \sum_{\tau \in C_n - S_n} \prod_{i=1}^{n} a_{i\tau(i)} b_{\tau(i)\sigma(i)}.$$

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In particular, when $A = B = F$, this sum must be zero. However, $\operatorname{Re}(\prod_{i=1}^{n} f_{i\tau(i)} f_{\tau(i)\sigma(i)}) > 0$; hence

$$\operatorname{Re}\left(\sum_{\sigma \in S_n} \sum_{\tau \in C_n - S_n} \prod_{i=1}^{n} f_{i\tau(i)} f_{\tau(i)\sigma(i)} \right) > 0,$$

which contradicts the fact that $F \in K$. Therefore, $K = \Delta_n$. Since Δ_n is contained in any maximal semigroup, it is the only one.

The following corollaries are immediate consequences of the theorem.

COROLLARY 1. *Δ_n is the maximal group of $n \times n$, non-singular matrices on which the permanent is multiplicative.*

In the above we considered matrices with complex entries. Let Δ_n^R be the set of all $n \times n$, non-singular matrices of the form PD, where P is a permutation matrix and D is a diagonal matrix with real entries. Then as a special case of the theorem we have the following corollary.

COROLLARY 2. *Δ_n^R is the maximal semigroup of $n \times n$, non-singular matrices with real entries on which the permanent is multiplicative.*

Remark. In the semigroup of all $n \times n$ matrices, a maximal semigroup on which the permanent is multiplicative is the set of all $n \times n$ matrices with at least one row [one column] of zeros together with the set Δ_n.

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REFERENCE

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