Canad. Math. Bull. Vol. **30** (1), 1987

AN AMALGAMATION THEOREM FOR SOLUBLE GROUPS

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ABSTRACT. A theorem of G. Higman about the embeddability of amalgams within the class of all finite p-groups is generalized to classes of soluble groups. We also give best possible bounds for the solubility lengths of the constructed completions. And, as an application, the supersoluble amalgamation bases in the class of all finite soluble π-groups are determined.

If two groups G and H intersect in a common subgroup $U = G \cap H$, then their union (which is in general not a group) is called the *amalgam* $G \cup H|U$ of G and H over U. The following necessary and sufficient condition for amalgams of finite p-groups to be contained in a finite p-group is due to G. Higman.

THEOREM 1. ([5], Theorem) *An amalgam $G \cup H|U$ of finite p-groups is embeddable into a finite p-group if and only if there exist chief series $G = G_0 > G_1 > \ldots > G_n = 1$ in G and $H = H_0 > H_1 > \ldots > H_m = 1$ in H such that $\{U \cap G_i | 0 \leqq i \leqq n\} = \{U \cap H_j | 0 \leqq j \leqq m\}$.*

Notice that chief factors of finite p-groups are always cyclic of order p. Thus, in the situation of Theorem 1, the subgroups $U \cap G_i$ resp. $U \cap H_j$ of U form a chief series of U.

Now, let us consider how Theorem 1 can be generalized to soluble groups. Suppose that an amalgam $G \cup H|U$ is contained in a soluble group W. Clearly, W has a series $W = W_0 \geqq W_1 \geqq \ldots \geqq W_r = 1$ of normal subgroups with abelian factors. Therefore $G = G \cap W_0 \geqq G \cap W_1 \geqq \ldots \geqq G \cap W_r = 1$ and $H = H \cap W_0 \geqq H \cap W_1 \geqq \ldots \geqq H \cap W_r = 1$ are series of normal subgroups in G resp. H with

$$\{U \cap (G \cap W_k) | 0 \leqq k \leqq r\} = \{U \cap W_k | 0 \leqq k \leqq r\} = \{U \cap (H \cap W_k) | 0 \leqq k \leqq r\}.$$

But in general, the converse assumption, that there be series $G = G_0 > G_1 > \ldots > G_n = 1$ and $H = H_0 > H_1 > \ldots > H_m = 1$ of normal subgroups in G resp. H with abelian factors satisfying $\{U \cap G_i | 0 \leqq i \leqq n\} = \{U \cap H_j | 0 \leqq j \leqq m\}$, is not sufficient for the existence of a soluble group W containing the amalgam $G \cup H|U$. This is shown by the following example of B.H. Neumann and J. Wiegold.

Received by the editors November 13, 1984 and, in revised form, August 13, 1985.
AMS Subject Classification (1980): 20F16 20E22.
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https://doi.org/10.4153/CMB-1987-002-7 Published online by Cambridge University Press

EXAMPLE. ([7], pp. 59–60) *Let* $U = \langle u_1, u_2, u_3 \rangle$ *be an elementary abelian group of order eight. Define* $\alpha, \beta \in \mathrm{Aut}(U)$ *via*

$$u_1 \xrightarrow{\alpha} u_2 \xrightarrow{\alpha} u_3 \xrightarrow{\alpha} u_1u_2 \qquad \text{and} \qquad u_1 \xrightarrow{\beta} u_2 \xrightarrow{\beta} u_3 \xrightarrow{\beta} u_1u_3.$$

Then the subgroups $G = U\langle\alpha\rangle$ *and* $H = U\langle\beta\rangle$ *of the holomorph of U form an amalgam* $G \cup H|U$ *of soluble groups. But* α *and* β *jointly generate* $\mathrm{Aut}(U)$. *Hence, in any supergroup W of* $G \cup H|U$, *the factor* $N_W(U)/C_W(U)$ *must be isomorphic to the non-abelian simple group* $\mathrm{PSL}(3,2) \cong \mathrm{GL}(3,2) \cong \mathrm{Aut}(U)$, *and W cannot be soluble.*

It is obvious from the example that a sufficient condition for the embeddability of $G \cup H|U$ into a soluble group must involve some control of the action of G and H on their abelian factors G_i/G_{i+1} resp. H_j/H_{j+1}. This was redundant in G. Higman's theorem, since the chief factors of finite p-groups are central. Our main result is

THEOREM 2. *Let* $\mathfrak{X}$ *be either the class of all soluble groups, or the class of all finite soluble* π*-groups (for some fixed set* π *of primes).*

(A) *An amalgam* $G \cup H|U$ *of* $\mathfrak{X}$*-groups is embeddable into an* $\mathfrak{X}$*-group if and only if there exist series* $G = G_0 \geqq G_1 \geqq \ldots \geqslant G_n = 1$ *and* $H = H_0 \geqq H_1 \geqq \ldots \geqq H_n = 1$ *of normal subgroups in G resp. H satisfying the following three conditions:*

(a) *The factors* G_i/G_{i+1} *and* H_i/H_{i+1} *are abelian for* $0 \leqq i \leqq n-1$.

(b) $U \cap G_i = U \cap H_i$ *for* $1 \leqq i \leqq n-1$.

(c) *For every* $i \in \{1, \ldots, n-1\}$ *there exists an abelian* $\mathfrak{X}$*-supergroup* Z_i *of the amalgam*

$$G_i/G_{i+1} \cup H_i/H_{i+1}|(U \cap G_i)G_{i+1}/G_{i+1} = (U \cap H_i)H_{i+1}/H_{i+1}$$

(where $(U \cap G_i)G_{i+1}/G_{i+1}$ *and* $(U \cap H_i)H_{i+1}/H_{i+1}$ *are identified via* $uG_{i+1} = uH_{i+1}$ *for all* $u \in U \cap G_i = U \cap H_i$*), and there exist homomorphisms*

$$\alpha_i : G/G_i \to \mathrm{Aut}(Z_i) \qquad \text{and} \qquad \beta_i : H/H_i \to \mathrm{Aut}(Z_i)$$

such that

(i) $(xG_{i+1})[(gG_i)\alpha_i] = x^gG_{i+1}$ *for all* $x \in G_i$, $g \in G$;

(ii) $(yH_{i+1})[(hH_i)\beta_i] = y^hH_{i+1}$ *for all* $y \in H_i$, $h \in H$;

(iii) $A_i = \langle \mathrm{Im}\,\alpha_i, \mathrm{Im}\,\beta_i \rangle$ *is an* $\mathfrak{X}$*-subgroup of* $\mathrm{Aut}(Z_i)$.

(B) *Let* $G \cup H|U$ *be an amalgam of* $\mathfrak{X}$*-groups satisfying all the conditions of part (A), and assume in addition that*

(iv) $(uG_i)\alpha_i = (uH_i)\beta_i$ *for all* $u \in U$ *and* $1 \leqq i \leqq n-1$.

Put $l_1 = 1$, *and define inductively* $l_{i+1} = 1 + \max\{l_i, m_i\}$ *where* m_i *is the solubility length of* A_i. *Then the amalgam is contained in an* $\mathfrak{X}$*-group of solubility length* $\leqq l_n$.

Notice that condition (b) and the choice $n = m$ in Theorem 2 are not restrictive at all, since we allow $G_i = G_{i+1}$ or $H_i = H_{i+1}$. Moreover, the formulae (i) and (ii) are well-defined, since the factors G_i/G_{i+1} and H_i/H_{i+1} are abelian.

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Observe also that part (B) yields the lowest possible bound l_n for the solubility length which a soluble supergroup of $G \cup H|U$ can have with respect to the solubility lengths of the groups A_i and with respect to the number n. This should make it possible to use Theorem 2 for an investigation of the structure of existentially closed groups in classes of soluble groups with bounded solubility lengths.

PROOF OF THEOREM 2. (A) Suppose firstly that $G \cup H|U$ is contained in an $\mathscr{X}$-group W. Denote by W_i the i-th term of the derived series of W. Put $G_i = G \cap W_i$ and $H_i = H \cap W_i$ and $Z_i = W_i/W_{i+1}$. Define α_i and β_i via

$$(wW_{i+1})[(gG_i)\alpha_i] = w^g W_{i+1} \qquad \text{for all } w \in W_i,\ g \in G, \text{ and}$$

$$(wW_{i+1})[(hH_i)\beta_i] = w^h W_{i+1} \qquad \text{for all } w \in W_i,\ h \in H.$$

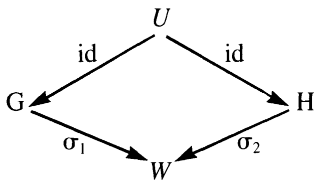
Then the conditions (a), (b) and (c) are satisfied, if we identify G_i/G_{i+1} (via $xG_{i+1} = xW_{i+1}$ for all $x \in G_i$) with the subgroup G_iW_{i+1}/W_{i+1} of Z_i, and similarly H_i/H_{i+1} with $H_iW_{i+1}/W_{i+1} \leqq Z_i$.

For the proof of the converse, suppose that (a), (b) and (c) hold for the amalgam $G \cup H|U$. We will embed the amalgam into an $\mathscr{X}$-group by induction over n. In the case $n = 1$ the groups G and H are abelian, and the amalgam can be embedded canonically into the central product of G and H over U.

Now let $n \geqq 2$. For convenience write $N, M, Z, A, \alpha, \beta$ instead of $G_{n-1}, H_{n-1}, Z_{n-1}, A_{n-1}, \alpha_{n-1}, \beta_{n-1}$ (resp.). By our induction, we may assume that the amalgam

$$G/N \cup H/M|UN/N = UM/M$$

(where UN/N and UM/M are identified via $uN = uM$ for all $u \in U$) is contained in an $\mathscr{X}$-group V. Denote by R the subgroup $\{(\delta, z)|z \in Z, \delta \in A\}$ of the holomorph of Z (with multiplication $(\delta_1, z_1)(\delta_2, z_2) = (\delta_1\delta_2, z_1\delta_2 \cdot z_2)$). Furthermore, let $W = R\,Wr\,V$ be the unrestricted regular wreath product of R and V, i.e., let $W = \{(v,f)|v \in V, f: V \rightarrow R\}$ with multiplication $(v_1, f_1)(v_2, f_2) = (v_1v_2, f_1^{v_2}f_2)$, where $f_1^{v_2}(v) = f_1(v_2v)$ for all $v \in V$. Clearly, W is an $\mathscr{X}$-group by (iii). We will construct embeddings $\sigma_1: G \rightarrow W$ and $\sigma_2: H \rightarrow W$ such that the diagram



commutes, and such that $G\sigma_1 \cap H\sigma_2 = U\sigma_1$.

Define $\theta: G \cup H \rightarrow V$ via

$$g\theta = gN \in V \qquad \text{for all } g \in G, \text{ and}$$

$$h\theta = hM \in V \qquad \text{for all } h \in H$$

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(notice that $uN = uM$ holds in V for all $u \in U$). As in G. Higman [5], p. 303 we can choose a map $\phi^*: V \to U$ such that

$$(u\theta \cdot v)\phi^*\theta = u\theta \cdot v\phi^*\theta \qquad \text{for all } u \in U,\ v \in V.$$

Now an embedding $\sigma: U \to W$ is given by

$$\begin{aligned} u\sigma &= (u\theta, f_u) && \text{for all } u \in U, \text{ where} \\ f_u(v) &= (1,\ [(u\theta v)\phi^*]^{-1} \cdot u \cdot v\phi^*) \in R && \text{for all } v \in V. \end{aligned}$$

As in the proof of G. Higman [5], Lemma 2.1 it is possible to find a map $\theta_1^*: V \to G$ with the property

$$(g\theta \cdot v)\theta_1^*\theta = g\theta \cdot v\theta_1^*\theta \qquad \text{for all } g \in G,\ v \in V,$$

and such that $\omega_1: V \to G$ given by

$$v\theta_1^* = v\phi^* \cdot v\omega_1 \qquad \text{for all } v \in V$$

is constant on each of the cosets $(U\theta)v$, $v \in V$. Then (i) ensures that $\sigma: U \to W$ can be extended to an embedding $\sigma_1: G \to W$ via

$$\begin{aligned} g\sigma_1 &= (g\theta, f_g) && \text{for all } g \in G, \text{ where} \\ f_g(v) &= ([(g\theta v)\omega_1 \cdot (v\omega_1)^{-1}]\theta\alpha, \\ &\quad ([(g\theta v)\theta_1^*]^{-1} \cdot g \cdot v\theta_1^*)[(v\omega_1)^{-1}]\theta\alpha) && \text{for all } v \in V. \end{aligned}$$

Similarly we can find a map $\theta_2^*: V \to H$ with the property

$$(h\theta \cdot v)\theta_2^*\theta = h\theta \cdot v\theta_2^*\theta \qquad \text{for all } h \in H,\ v \in V,$$

and such that $\omega_2: V \to H$ given by

$$v\theta_2^* = v\phi^* \cdot v\omega_2 \qquad \text{for all } v \in V$$

is constant on each of the cosets $(U\theta)v$, $v \in V$. Then (ii) ensures that $\sigma: U \to W$ can be extended to an embedding $\sigma_2: H \to W$ via

$$\begin{aligned} h\sigma_2 &= (h\theta, f_h) && \text{for all } h \in H, \text{ where} \\ f_h(v) &= ([(h\theta v)\omega_2 \cdot (v\omega_2)^{-1}]\theta\beta, \\ &\quad ([(h\theta v)\theta_2^*]^{-1} \cdot h \cdot v\theta_2^*)[(v\omega_2)^{-1}]\theta\beta) && \text{for all } v \in V. \end{aligned}$$

It remains to show that $G\sigma_1 \cap H\sigma_2 = U\sigma_1$.

Let $g \in G$ and $h \in H$ with $(g\theta, f_g) = g\sigma_1 = h\sigma_2 = (h\theta, f_h)$. Then $g\theta = h\theta \in G/N \cap H/M = UN/N = UM/M$. Thus, $g \in UN$ and $h \in UM$. Now straightforward calculations yield that

$$f_g(v) = (1,\ [(g\theta v)\phi^*]^{-1} \cdot g \cdot v\phi^*) \qquad \text{and}$$

$$f_h(v) = (1,\ [(h\theta v)\phi^*]^{-1} \cdot h \cdot v\phi^*) \qquad \text{for all } v \in V.$$

Hence, $g = h \in G \cap H = U$ and $g\sigma_1 = h\sigma_2 \in U\sigma$.

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(B) In the case $n = 1$ the central product of the groups G and H over U is abelian and has solubility length $\leqq 1 = l_1$.

Now, let $n \geqq 2$. We go back into the proof of part (A) and assume by induction that V has solubility length $\leqq l_{n-1}$. We will show that the l_{n-1}-st term $\tilde{Q}$ of the derived series of $Q = \langle G\sigma_1, H\sigma_2 \rangle$ is contained in the normal subgroup

$$W_0 = \{(1,f) | f(V) \subseteq \{(\delta, z) | \delta \in \tilde{A}, z \in Z\}\}$$

of W (where $\tilde{A}$ denotes the l_{n-1}-th term of the derived series of A). This will prove part (B), since W_0 has solubility length $\leqq 1 + \max\{0, m_{n-1} - l_{n-1}\} = l_n - l_{n-1}$.

For convenience, define $\sigma: G \cup H \to W$ via

$$\sigma \restriction G = \sigma_1 \quad \text{and} \quad \sigma \restriction H = \sigma_2,$$

as well as $\gamma: G \cup H \to A$ via

$$\gamma \restriction G = \theta\alpha \quad \text{and } \gamma \restriction H = \theta\beta.$$

Because of (iv) the map γ is well-defined. Put $l = l_{n-1}$. For $1 \leqq i \leqq l$ let $w_i(x_1, \ldots, x_{2^i})$ be the word given recursively by

$$w_1(x_1, x_2) = [x_1, x_2] \qquad \text{and}$$

$$w_1(x_1, \ldots, x_{2^i}) = [w_{i-1}(x_1, \ldots, x_{2^{i-1}}), w_{i-1}(x_{2^{i-1}+1}, \ldots, x_{2^i})] \qquad \text{for } i \geqq 2.$$

Then $\tilde{Q}$ is generated by the set $S = \{w_l(q_1, \ldots, q_{2^l}) | q_j \in Q\}$. Fix some $w_1(q_1, \ldots, q_{2^l}) \in S$. Every q_j is a word $v_j(y_{j1}\sigma, \ldots, y_{jv_j}\sigma)$ for some $y_{jk} \in G \cup H$.

Now observe that every $g \in G$ has the image

$$g\sigma = (g\theta, f_g), \qquad \text{where}$$

$$\begin{aligned} f_g(v) &= ([(g\theta v)\omega_1 \cdot (v\omega_1)^{-1}]\gamma,\ z_g(v)) \\ &= ([(g\theta v)\phi^*]^{-1}\gamma \cdot [(g\theta v)\theta_1^*\theta \cdot v\theta_1^*\theta^{-1}]\alpha \cdot v\phi^*\gamma,\ z_g(v)) \\ &= ([(g\theta v)\phi^*]^{-1}\gamma \cdot g\gamma \cdot v\phi^*\gamma,\ z_g(v)) \qquad \text{for some } z_g(v) \in Z. \end{aligned}$$

Similarly, every $h \in H$ has the image

$$h\sigma = (h\theta, f_h), \qquad \text{where}$$

$$f_h(v) = ([(h\theta v)\phi^*]^{-1}\gamma \cdot h\gamma \cdot v\phi^*\gamma,\ z_h(v)) \qquad \text{for some } z_h(v) \in Z.$$

Thus, for all $x, y \in G \cup H$ we obtain

$$x\sigma \cdot y\sigma = (x\theta \cdot y\theta, f_{x,y}) \qquad \text{where}$$

$$f_{x,y}(v) = ([(x\theta \cdot y\theta \cdot v)\phi^*]^{-1}\gamma \cdot x\gamma \cdot y\gamma \cdot v\phi^*\gamma,\ z_{x,y}(v)) \qquad \text{for some } z_{x,y}(v) \in Z.$$

But then

$$\begin{aligned} w_l(\ldots, q_j, \ldots) &= w_l(\ldots, v_j(\ldots, y_{jk}\sigma, \ldots), \ldots) \\ &= (w_l(\ldots, v_j(\ldots, y_{jk}\theta, \ldots), \ldots), f^*) \end{aligned}$$

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where

$$\begin{aligned} f^*(v) &= ([(w_l(\ldots, v_j(\ldots, y_{jk}\theta, \ldots), \ldots)\cdot v)\phi^*]^{-1}\gamma\cdot \\ &\quad \cdot(w_l(\ldots, v_j(\ldots, y_{jk}\gamma, \ldots), \ldots)\cdot v\phi^*\gamma,\ z^*(v)) \\ &= ([v\phi^*]^{-1}\gamma\cdot(w_l(\ldots, v_j(\ldots, y_{jk}\gamma, \ldots), \ldots)\cdot v\phi^*\gamma,\ z^*(v)) \\ &\in \{(\delta, z) | \delta \in \tilde{A}, z \in Z\} \qquad \text{for all } v \in V. \end{aligned}$$

Hence $w_l(q, \ldots, q_{2^l}) \in W_0$, and we conclude that $\tilde{Q} = \langle S \rangle \leqq W_0$. □

After writing a first version of this paper the author learned that R.B.J.T. Allenby had used in his Ph.D. thesis the same technique as in the proof of Theorem 2(A) to show the following two results.

THEOREM 3. ([1], Theorem 7.5) *Let $G \cup H | U$ be an amalgam of finite soluble groups in which the amalgamated subgroup U is normal in both constituents G and H. Then $G \cup H | U$ is embeddable into a finite soluble group if and only if G and H have series $G = G_0 \geqq G_1 \geqq \ldots \geqq G_n = 1$ and $H = H_0 \geqq H_1 \geqq \ldots \geqq H_n = 1$ of normal subgroups such that*

(a) *G_i/G_{i+1} and H_i/H_{i+1} are elementary abelian groups,*

(b) *$U \cap G_i = U \cap H_i = U_i$, and*

(c) *G/G_{i+1} and H/H_{i+1} together generate on U_i/U_{i+1} (via conjugation) a soluble group of automorphisms.*

THEOREM 4. ([1], Theorem 7.7) *Let $G \cup H|U$ be an amalgam of finite soluble groups. Let G and H have series of normal subgroups $G = G_0 \geqq G_1 \geqq \ldots \geqq G_n = 1$ and $H = H_0 \geqq H_1 \geqq \ldots \geqq H_n = 1$ in which the factors are elementary abelian, and suppose that $U \cap G_i = U \cap H_i = U_i$ for all i, so that $G_i/G_{i+1} \cap H_i/H_{i+1} = U_i/U_{i+1}$ (identifying U_iG_{i+1}/G_{i+1} and U_iH_{i+1}/H_{i+1} canonically with U_i/U_{i+1}). Let Z_i be the central product of G_i/G_{i+1} and H_i/H_{i+1} over U_i/U_{i+1}. Then, if G_i^* denotes the group of automorphisms generated on Z_i by extending to Z_i the automorphisms induced on G_i/G_{i+1} by G/G_{i+1} via conjugation (this can be done by choosing a basis for Z_i which extends that of G_i/G_{i+1} and forcing G_i^* to act trivially on the basis elements from $Z_i \backslash (G_i/G_{i+1})$), and if H_i^* is defined similarly, it follows that $G \cup H|U$ is embeddable into a finite soluble group, if G_i^* and H_i^* together generate a soluble group of automorphisms on Z_i ($i = 0, 1, \ldots, n-1$).*

Theorem 4 is an immediate consequence of Theorem 2(A). Since the chief factors of finite soluble groups are elementary abelian, the necessity of the conditions in Theorem 3 can be shown as in the proof of Theorem 2(A), while their sufficiency follows from an application of Theorem 2(A): Choose a basis for Z_i which contains bases of U_i/U_{i+1} and G_i/G_{i+1} and H_i/H_{i+1}, and define G_i^* and H_i^* as in Theorem 4; then $\langle G_i^*, H_i^* \rangle$ is a soluble group of automorphisms of Z_i.

Results about amalgams of two soluble groups over a normal common subgroup, which are related to Theorem 3, can also be found in papers of R.J. Gregorac ([2],

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Theorem 6.5), K. Hickin ([4], Theorem 5) and J. Wiegold ([9], Theorem 2.3). The latter two authors also give bounds for the solubility lengths of their completions.

It remains open whether the sufficient conditions of Theorem 4 are necessary too, and in how far they are influenced by the choices of bases for the Z_i. However, in applications, it should be easier to continue the automorphisms induced on G_i/G_{i+1} and H_i/H_{i+1} by conjugation with elements from G/G_{i+1} resp. H/H_{i+1} to automorphisms of *some* elementary abelian supergroup of the amalgam $G_i/G_{i+1} \cup H_i/H_{i+1} | U_i/U_{i+1}$ rather than to automorphisms of the central product of G_i/G_{i+1} and H_i/H_{i+1} over U_i/U_{i+1}. We will use this idea in the proof of

THEOREM 5. *Let $G \cup H | U$ be an amalgam of two finite soluble π-groups G and H over a supersoluble group U. If there exist chief series $G = G_0 \geqq G_1 \geqq \ldots \geqq G_n = 1$ and $H = H_0 \geqq H_1 \geqq \ldots \geqq H_n = 1$ in G resp. H such that $U \cap G_i = U \cap H_i = U_i$ for $0 \leqq i \leqq n$ and such that $U = U_0 \geqq U_1 \geqq \ldots \geqq U_n = 1$ is a chief series in U, then the amalgam can be embedded into a finite soluble π-group.*

PROOF. In order to apply Theorem 2(A) we only need to show that condition (c) is satisfied. Fix $i \in \{1, \ldots, n-1\}$, and denote epimorphic images modulo U_{i+1}, G_{i+1} or H_{i+1} by bars.

If $U_i = U_{i+1}$, then we choose $Z_i = \bar{G}_i \times \bar{H}_i$ and define $\sigma_i : G/G_i \rightarrow \mathrm{Aut}(Z_i)$ and $\beta_i : H/H_i \rightarrow \mathrm{Aut}(Z_i)$ via

$$(\overline{xy})[(gG_i)\alpha_i] = \bar{x}^{\bar{g}} \cdot \bar{y} \qquad \text{for all } x \in G_i,\ y \in H_i,\ g \in G, \text{ and}$$

$$(\overline{xy})[(hH_i)\beta_i] = \bar{x} \cdot \bar{y}^{\bar{h}} \qquad \text{for all } x \in G_i,\ y \in H_i,\ h \in H.$$

Clearly α_i and β_i are homomorphisms satisfying the conditions (i) and (ii) of Theorem 2(A), and $A_i = \mathrm{Im}\,\alpha_i \times \mathrm{Im}\,\beta_i$ is a finite soluble π-group.

Now let $U_i \neq U_{i+1}$. As a chief factor of a finite super soluble π-group, $\bar{U}_i$ is cyclic of prime order $p \in \pi$. Therefore, $\bar{G}_i$ and $\bar{H}_i$ are elementary abelian p-groups. Put

$$\bar{G}_i = \langle \bar{x}_1 \rangle \oplus \langle \bar{x}_2 \rangle \oplus \ldots \oplus \langle \bar{x}_r \rangle \qquad \text{and}$$

$$\bar{H}_i = \langle \bar{y}_1 \rangle \oplus \langle \bar{y}_2 \rangle \oplus \ldots \oplus \langle \bar{y}_s \rangle \qquad \text{where}$$

$$\bar{x}_1 = \bar{y}_1 \in \bar{U}_i.$$

Let Z_i be an elementary abelian p-group with basis $B = \{z_{kl} | 1 \leqq k \leqq r,\ 1 \leqq l \leqq s\}$. Identify

$$z_{k1} = \bar{x}_k \quad \text{for} \quad 1 \leqq k \leqq r, \qquad \text{and} \qquad z_{1l} = \bar{y}_l \quad \text{for} \quad 1 \leqq l \leqq s.$$

For $g \in G$ and $h \in H$, define endomorphisms $(gG_i)\alpha_i$ resp. $(hH_i)\beta_i$ of Z_i on the basis B as follows:

—If $\bar{x}_{k_o}^{\bar{g}} = \sum_{k=1}^{r} \eta_k \bar{x}_k$, then $(z_{k_o l})[(gG_i)\alpha_i] = \sum_{k=1}^{r} \eta_k z_{kl}$ for $1 \leqq l \leqq s$.

—If $\bar{y}_{l_o}^{\bar{h}} = \sum_{l=1}^{s} \zeta_l \bar{y}_l$, then $(z_{kl_o})[(hH_i)\beta_i] = \sum_{l=1}^{s} \zeta_l z_{kl}$ for $1 \leqq k \leqq r$.

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Then $(gG_i)\alpha_i$ acts on each of the subspaces

$$\langle z_{1l}\rangle \oplus \ldots \oplus \langle z_{rl}\rangle \qquad (1 \leqq l \leqq s)$$

of Z_i as $\bar{g}$ acts via conjugation on $\langle \bar{x}_1\rangle \oplus \ldots \oplus \langle \bar{x}_r\rangle$. Similarly, $(hH_i)\beta_i$ acts on each of the subspaces

$$\langle z_{k1}\rangle \oplus \ldots \oplus \langle z_{ks}\rangle \qquad (1 \leqq k \leqq r)$$

as h acts on $\langle \bar{y}_1\rangle \oplus \ldots \oplus \langle \bar{y}_s\rangle$ via conjugation. Therefore it is obvious that $(gG_i)\alpha_i$ and $(hH_i)\beta_i$ are automorphisms of Z_i satisfying the conditions (i) and (ii) of Theorem 2(A), and that the maps $\alpha_i: G/G_i \to \mathrm{Aut}(Z_i)$ and $\beta_i: H/H_i \to \mathrm{Aut}(Z_i)$ are homomorphisms.

It remains to show that $A_i = \langle \mathrm{Im}\,\alpha_i, \mathrm{Im}\,\beta_i\rangle$ is a finite soluble π-group. We will prove that A_i is a central product of $\mathrm{Im}\,\alpha_i$ and $\mathrm{Im}\,\beta_i$. (Actually it can also be seen easily that $\mathrm{Im}\,\alpha_i \cap \mathrm{Im}\,\beta_i$ contains only power automorphisms of Z_i.)

Fix $z_{k_o l_o} \in B$. Suppose that $g \in G$ and $h \in H$ satisfy

$$\bar{x}_{k_o}^{\bar{g}} = \sum_{k=1}^{r} \eta_k \bar{x}_k \qquad \text{and} \qquad \bar{y}_{l_o}^{\bar{h}} = \sum_{l=1}^{s} \zeta_l \bar{y}_l.$$

Then

$$\begin{aligned}(z_{k_o l_o})[(gG_i)\alpha_i \cdot (hH_i)\beta_i] &= \left(\sum_{k=1}^{r} \eta_k z_{k l_o}\right)[(hH_i)\beta_i] \\ &= \sum_{l=1}^{s}\sum_{k=1}^{r} \eta_k \zeta_l z_{kl} = \left(\sum_{l=1}^{s} \zeta_l z_{k_o l}\right)[(gG_i)\alpha_i] \\ &= (z_{k_o l_o})[(hH_i)\beta_i \cdot (gG_i)\alpha_i].\end{aligned}$$

Hence $\mathrm{Im}\,\alpha_i$ and $\mathrm{Im}\,\beta_i$ commute. □

Let $\mathscr{X}$ be a class of groups. An $\mathscr{X}$-group U is called an *amalgamation basis* in $\mathscr{X}$, if every amalgam $G \cup H | U$ of two $\mathscr{X}$-groups G and H over U can be embedded into an $\mathscr{X}$-group. Theorem 5 can be used to determine the supersoluble amalgamation bases in the class $\mathscr{F}_\pi \cap \mathscr{S}$ of all finite soluble π-groups.

THEOREM 6. (a) *If U is an amalgamation basis in $\mathscr{F}_\pi \cap \mathscr{S}$, then U has a unique chief series.*

(b) *A supersoluble group U is an amalgamation basis in $\mathscr{F}_\pi \cap \mathscr{S}$ if and only if U is either a cyclic p-group for some $p \in \pi$ or the split extension of a cyclic p-group P by a cyclic q-group Q with $\mathbf{C}_Q(P) = 1$ for some $p, q \in \pi$ with $q | p - 1$.*

PROOF. (a) Suppose that U does not have a unique chief series. Then there exist normal subgroups K, L and M in U such that K/M and L/M are different chief factors in U. Choose $g \in K \backslash M$ and $h \in L \backslash M$. Because of $K \cap L = M$ we have $g \in K \backslash L$ and $h \in L \backslash K$.

From P. Hall [3], Lemma 7 there exists a finite nilpotent π-group F such that $h \in F''$. Since chief factors of $\mathscr{F}_\pi \cap \mathscr{S}$-groups are elementary abelian, the intersections

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of $\langle h \rangle$ with the terms of a chief series in any $\mathcal{F}_\pi \cap \mathcal{S}$-supergroup of $\langle h \rangle$ must form the unique chief series in $\langle h \rangle$. Therefore Theorem 5 ensures that the amalgam $U \cup F | \langle h \rangle$ is contained in an $\mathcal{F}_\pi \cap \mathcal{S}$-group V. Clearly, $h \in F'' \leqq V''$.

Let $\sigma : U \to G = V\, Wr\, U/L$ be a Krasner-Kaloujnine-embedding, i.e., for some fixed transversal $T = \{t_{uL} | u \in U\}$ of L in U let

$$x\sigma = (xL, f_x) \qquad \text{for all } x \in U$$

where $f_x : U/L \to V$ is given by

$$f_x(uL) = t_{xuL}^{-1} \cdot x \cdot t_{uL} \qquad \text{for all } u \in U.$$

Then $h\sigma = (1, f_h)$ where $f_h(uL) = h^{t_{uL}} \in V''$ for all $u \in U$, and $g\sigma = (gL, f_g)$ does not lie in the base group $\{(1,f) | f : U/L \to V\}$ of G. Therefore, a combination of P.M. Neumann [8], Lemma 5.1 and F. Leinen [6], Lemma 4.3.(b) yields that $h\sigma \in \langle g\sigma^G \rangle'$. Identify U via σ with $U\sigma \leqq G$. Then G is an $\mathcal{F}_\pi \cap \mathcal{S}$-supergroup of U with $h \in \langle g^G \rangle'$.

Similarly, we can construct an $\mathcal{F}_\pi \cap \mathcal{S}$-supergroup H of U with $g \in \langle h^H \rangle'$.

Since U is an amalgamation basis in $\mathcal{F}_\pi \cap \mathcal{S}$, the amalgam $G \cup H | U$ must be contained in an $\mathcal{F}_\pi \cap \mathcal{S}$-group W. Let $W = W_0 > W_1 > \ldots > W_r = 1$ be a series of normal subgroups in W with abelian factors. Choose $k \in \{0, \ldots, r-1\}$ such that $g \in W_k \backslash W_{k+1}$. Then $\langle g^G \rangle \leqq \langle g^W \rangle \leqq W_k$. Since W_k/W_{k+1} is abelian, we obtain $h \in \langle g^G \rangle' \leqq W_k' \leqq W_{k+1}$. But now $g \in \langle h^H \rangle \leqq \langle h^W \rangle \leqq W_{k+1}$, in contradiction to the choice of k.

(b) Let $U = PQ$ as described in the theorem. Then, for any $h \in Q \backslash 1$, the p'-group $\langle h \rangle$ acts on the abelian p-group P. So $P = [P, \langle h \rangle] \times C_P(\langle h \rangle)$. But P is a cyclic p-group. Thus $P = [P, \langle h \rangle]$ or $P = C_P(\langle h \rangle)$. Since $C_Q(P) = 1$, we know that $P \neq C_P(\langle h \rangle)$. Hence $P = [P, \langle h \rangle] \leqq \langle h^U \rangle$ for every $h \in Q \backslash 1$.

Therefore, if $U = PQ$ as described in the theorem, or if U is a cyclic p-group, then U has a unique chief series, and every elementary abelian factor of U is cyclic of prime order. Thus, the intersections of U with the terms of a chief series in any $\mathcal{F}_\pi \cap \mathcal{S}$-supergroup of U must form the chief series in U. Hence Theorem 5 ensures that U is an amalgamation basis in $\mathcal{F}_\pi \cap \mathcal{S}$.

For the converse, let U be a supersoluble amalgamation basis in $\mathcal{F}_\pi \cap \mathcal{S}$. If U is a p-group for some $p \in \pi$, then $U/\Phi(U)$ (where $\Phi(U)$ denotes the Frattini subgroup of U) is elementary abelian. But U has a unique chief series by (a). Hence $U/\Phi(U)$ is cyclic, and this forces U to be cyclic.

Now assume, that U is not a p-group. Let $p \in \pi$ be the maximal prime dividing $|U|$. As a finite supersoluble group, U has a normal Sylow-p-subgroup P. So $P/\Phi(P)$ is an elementary abelian factor of U. Since U has a unique chief series, it is clear that $C_{U/\Phi(P)}(P/\Phi(P)) = P/\Phi(P)$. In particular, the p'-group U/P acts faithfully via conjugation on the abelian p-group $P/\Phi(P)$. Moreover, because chief factors of finite supersoluble groups are cyclic, $P/\Phi(P)$ has a cyclic U/P-invariant subgroup. Now Maschke's theorem and the uniqueness of the chief series in U imply that $P/\Phi(P)$ is

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cyclic. Thus, P is cyclic. But then U/P is isomorphic to a subgroup of $\mathrm{Aut}(P/\Phi(P))$, which is in turn cyclic of order $p - 1$. Again the uniqueness of the chief series in U yields that U/P is even a cyclic q-group for some $q \in \pi$ dividing $p - 1$. Finally, $C_U(P)/\Phi(P) \leqq C_{U/\Phi(P)}(P/\Phi(P)) = P/\Phi(P)$, and so $C_U(P) = P$. □

It would of course be interesting to find all the amalgamation bases in $\mathcal{F}_\pi \cap \mathcal{S}$. The first thing to settle might be the

QUESTION. *Is the alternating group* A_4 *an amalgamation basis in the class of all finite soluble* $\{2,3\}$*-groups?*

Finally it should be noted that all theorems of this paper can be extended to amalgams of finitely many groups G_k, $1 \leqq k \leqq \nu$, over a common subgroup $U = G_k \cap G_l$ (for all k, l).

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https://doi.org/10.4153/CMB-1987-002-7 Published online by Cambridge University Press