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Uniformization and Steinness

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Abstract. It is shown that the unit ball in $\mathbb{C}^n$ is the only complex manifold that can universally cover both Stein and non-Stein strictly pseudoconvex domains.

In this note we use methods from [5] to show that the unit ball in $\mathbb{C}^n$ is the only simply connected complex manifold that can cover both Stein and non-Stein strictly pseudoconvex domains.

Here a strictly pseudoconvex domain is a relatively compact domain in a complex manifold such that its boundary admits a C^2-smooth strictly plurisubharmonic defining function.

Theorem *Let Y be the universal cover of a Stein strictly pseudoconvex domain. Suppose that Y is not biholomorphic to the ball. Then any manifold covered by Y does not contain compact complex analytic subsets of positive dimension. In particular, any other strictly pseudoconvex domain covered by Y is Stein.*

Examples of strictly pseudoconvex domains covered by the ball in $\mathbb{C}^2$ that contain compact complex curves (and hence are not Stein) can be found in [2]. It is well known that the ball covers compact complex manifolds as well.

Recall also from [4, 5] that a Stein strictly pseudoconvex domain is covered by the unit ball if and only if its boundary is everywhere locally CR-diffeomorphic to the unit sphere.

The theorem will follow immediately from the two lemmas below.

Lemma 1 *Let $\pi: Y \to D$ be a covering of a complex manifold D admitting a strictly plurisubharmonic function $\varphi: D \to \mathbb{R}$. If $A \subset Y$ is an analytic subset of positive dimension, then its projection $\pi(A)$ cannot lie in a compact subset in D.*

Remark 2 The assumptions of the lemma are satisfied if D is (an unramified domain over) a Stein manifold. However, there exist examples of complex manifolds with strictly plurisubharmonic functions but no non-constant holomorphic functions [3].

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Proof Suppose that $\pi(A)$ is contained in a compact subset of D. Then there exists a sequence of points $x_n = \pi(y_n)$ such that $x_n \to x \in D$, $y_n \in A$, and

$$\sup_{\pi(A)} \varphi = \lim_{n\to\infty} \varphi(x_n) = \varphi(x).$$

Let $U \ni x$ be a small ball in local coordinates centered at x and let $U' \supset \overline{U}$ be a slightly larger ball. Let h be a non-negative smooth function on U' such that $h(x) = 0$, h is positive on $\partial U \cap \{\varphi \le \varphi(x)\}$, and the C^2-norm of h is sufficiently small. Then $\widetilde{\varphi} := \varphi - h$ is a strictly plurisubharmonic function on U' such that

$$\widetilde{\varphi}(x) = \varphi(x) \tag{1}$$

and

$$\widetilde{\varphi} \le \varphi(x) - \varepsilon \text{ on } \partial U \cap \{\varphi \le \varphi(x)\} \text{ for some } \varepsilon > 0. \tag{2}$$

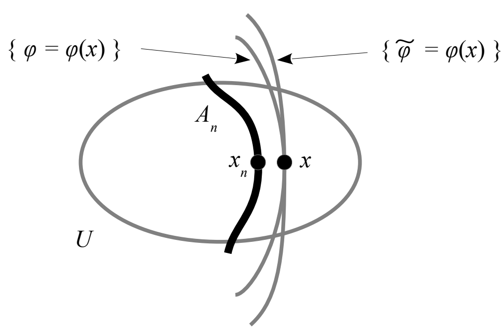

Figure 1: Impossible analytic sets A_n.

Since U' is simply connected, $\pi^{-1}(U') = \bigsqcup V_j$, where V_j are disjoint open sets and $\pi|_{V_j}$ is invertible. We then have $y_n \in V_{\alpha(n)}$ for every $n \gg 1$ and a suitable index $\alpha(n)$. Set $A_n := \pi(A \cap V_{\alpha(n)})$. This is a complex analytic subset of U' containing x_n; see Fig. 1. Since $A_n \subset \{\varphi \le \varphi(x)\}$ by the choice of x, it follows that $\widetilde{\varphi} \le \varphi(x) - \varepsilon$ on $A_n \cap \partial U = \partial(A_n \cap U)$ by property (2) of $\widetilde{\varphi}$. Hence, $\widetilde{\varphi} \le \varphi(x) - \varepsilon$ on $A_n \cap U$ by the maximum principle for plurisubharmonic functions on complex analytic sets (see *e.g.*, [1, §6.3]). However, $\widetilde{\varphi}(x_n) \to \varphi(x)$ as $n \to \infty$ by (1), and we arrive at a contradiction. ■

Lemma 3 *Let $\pi: Y \to D$ be the universal covering of a strictly pseudoconvex domain by a complex manifold Y that is not biholomorphic to the ball. Suppose that $\pi': Y \to M$ is a covering of a complex manifold containing a connected compact complex analytic subset $B \Subset M$ of positive dimension. Then $\pi(\pi'^{-1}(B))$ is contained in a compact subset of D.*

Remark 4 In this lemma, D does not need to be Stein.

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Proof Let $\varphi: \overline{D} \to (-\infty, 0]$ be a plurisubharmonic defining function for D. Following [5, §2.3], consider the function ψ on M defined by

$$\psi(x) := \Big(\sup_{\pi'(y)=x} \varphi \circ \pi(y) \Big)^*,$$

where $*$ denotes the upper semicontinuous regularisation. As shown in [5, §2.3], it follows from [5, Corollary 2.3] that ψ is plurisubharmonic and strictly negative on M. (It is explained in [5, §3.2] how to modify the proof of [5, Corollary 2.3] for non-Stein domains.) By the maximum principle,

$$\psi|_B \equiv \text{const.} < 0.$$

Hence,

$$\varphi \circ \pi(y) \le \text{const.} < 0 \text{ for all } y \in \pi'^{-1}(B),$$

and therefore $\pi\left(\pi'^{-1}(B)\right)$ is relatively compact in D. ■

Remark 5 The key point in the proof of Lemma 3 is the application of [5, Corollary 2.3]. That result is a consequence of [5, Proposition 2.2], which is an extension of the well-known Wong–Rosay theorem [6, 7] to universal coverings of strictly pseudoconvex domains in complex manifolds.

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