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RESEARCH ARTICLE

Nonparametric estimation of some dividend problems in the perturbed compound Poisson model

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Abstract

In this paper, we consider some dividend problems in the perturbed compound Poisson model under a constant barrier dividend strategy. We approximate the expected present value of dividend payments before ruin and the expected discounted penalty function based on the COS method, and construct some nonparametric estimators by using a random sample on claim number and individual claim sizes. Under a large sample size setting, we perform an error analysis of the estimators. We also provide some simulation results to verify the effectiveness of this estimation method when the sample size is finite.

1. Introduction

In this paper, the surplus process of an insurance company is described by the perturbed compound Poisson model

$$U_t^\infty = u + ct - S_t + \sigma B_t, \quad t \geq 0, \tag{1.1}$$

where $u \geq 0$ is the initial surplus, and $c > 0$ is the constant premium rate per time. The aggregate claims process $S_t = \sum_{n=1}^{N_t} X_n$ is a compound Poisson process, where $\{N_t\}_{t\geq 0}$ is a Poisson claim number process with constant arrival rate $\lambda > 0$, and the individual claim sizes $\{X_n\}_{n\geq 1}$ form a sequence of positive valued i.i.d. random variables with common probability density function f_X and mean $\mu_X = \int_0^\infty x f_X(x)\,dx$. Finally, $\{B_t\}_{t\geq 0}$, independent of $\{S_t\}_{t\geq 0}$, is a standard Brownian motion starting from zero, and $\sigma > 0$ is a volatility parameter.

The above perturbed compound Poisson model was first proposed by Gerber [11], and it has been studied by many authors; see, for example, Gerber and Landry [12], Wang [26], Tsai [23,24], Tsai and Willmot [25], Chiu and Yin [7] and references therein. In this paper, we consider the model (1.1) modified by a constant barrier dividend strategy. Given a finite barrier of level $b > 0$, we assume that whenever the surplus process reaches level b, dividends are paid off continuously such that the surplus stays at level b until it becomes less than b. Let U_t^b denote the modified surplus process under the above barrier dividend strategy, and let $\tau_b = \inf\{t \geq 0 : U_t^b \leq 0\}$ be the time of ruin. The present value of total dividends paid before the ruin time τ_b is given by

$$D_b = \int_0^{\tau_b} e^{-\delta t}\, dD(t), \quad 0 \leq u \leq b,$$

where $\delta > 0$ is the force of interest for valuation, and $D(t)$ is the aggregate dividends paid by time t. Given the initial surplus level $0 \leq u \leq b$, the expected present value of total dividend payments before ruin is defined by

$$V(u;b) = E[D_b \mid U_0^b = u], \quad 0 \le u \le b. \tag{1.2}$$

Furthermore, we are interested in the expected discounted penalty function associated with model U^b, which is defined by

$$\phi(u;b) = E[e^{-\delta\tau_b} w(|U_{\tau_b}^b|) | U_0^b = u], \quad 0 \le u \le b, \tag{1.3}$$

where $w(\cdot)$ is a nonnegative penalty function of the deficit at ruin. Note that ruin can be caused either by a claim or oscillation due to the Brownian motion, then we have the following decomposition of $\phi(u;b)$,

$$\phi(u;b) = w(0)\phi_d(u;b) + \phi_c(u;b),$$

where

$$\phi_d(u;b) = E[e^{-\delta\tau_b} I(U_{\tau_b}^b = 0) \mid U_0^b = u], \quad 0 \le u \le b,$$
$$\phi_c(u;b) = E[e^{-\delta\tau_b} w(|U_{\tau_b}^b|) I(U_{\tau_b}^b < 0) | U_0^b = u], \quad 0 \le u \le b,$$

are respectively the Laplace transform of the ruin time when ruin is caused by oscillation, and the expected discounted penalty function when ruin is due to a claim.

Dividend problem was first proposed by de Finetti [10] in a binomial model, and since then it has been studied in a number of papers. For example, see Lin *et al.* [17], Dickson and Waters [8], Li and Garrido [16], Albrecher *et al.* [2] and Li [15] to name a few. We know that a common assumption in these papers is that the probability characteristics of the model U^b are known, so that analytic approach such as differential equation, Laplace transform, renewal theory can be used to study the functions $V(u;b)$, $\phi_d(u;b)$ and $\phi_c(u;b)$. However, usually the quantities such as the Poisson intensity λ, the diffusion parameter σ and the claim size density f_X are all unknown. Hence, it is very interesting to consider the statistical estimation of the dividend problems given that the random samples on claim number, individual claim sizes and the surplus levels are available. When the probability characteristics of the surplus process are unknown, some estimation problems have been considered under the model without dividend payments. For example, Zhang and Yang [31,32] proposed some nonparametric estimators of the ruin probability by some Fourier-based methods. The same estimation problem has also been considered by Cai *et al.* [3] and Cai and You [4] by numerical Laplace inversion transform, respectively. The expected discounted penalty functions are estimated by Shimizu [19,20] and Shimizu and Zhang [21] under different risk models.

Recently, Xie and Zhang [27] study the statistical estimation of dividend problems in the classical compound Poisson model. In their paper, the Fourier-cosine (COS) method was first used to estimate the expected present value of dividend payments before ruin, then the expected discounted penalty function was estimated by using the dividends-penalty identity. We remark that the COS method was first proposed by Fang and Oosterlee [9] to price European options, and since then, it has been widely used in the field of financial mathematics. In insurance risk theory, Chan *et al.* [5,6] applied the COS method to approximate the ruin probability and the expected discounted penalty function, respectively; Zhang used the COS method to approximate the density function of the time to ruin; Yang *et al.* [28] applied a two-dimensional COS method to estimate the discounted density function of the deficit at ruin; Lee *et al.* [14] studied the finite time ruin probabilities by the COS method. On the estimation of the expected discounted penalty function with dividend payments, we also would like to point out the work Strini and Thonhauser [22]. In their paper, ruin problems in a renewal risk model with a general surplus dependent premium rate are studied. Note that both barrier and band type strategies can be expressed by some special settings of their premium rate function. Different from Xie and Zhang [27], they first study the expected discounted penalty function by numerical solution of a partial-integro-differential equation, and then consider the statistical estimation through the estimated model parameters.

In this paper, we shall follow the approach in Xie and Zhang [27] to estimate the functions $V(u;b)$, $\phi_d(u;b)$ and $\phi_c(u;b)$ under the model U^b. Note that different from Xie and Zhang [27], in this paper, we add an independent Brownian motion to describe the uncertainty of the surplus flow of the insurance company. This additional noise puts an obstacle for constructing the estimators of the interesting functions from the random samples on the individual claim sizes and claim number. As is shown in Section 4, the information on dividend payments has to be used to construct the estimators. When studying the consistency properties of our estimators, we need to study the uniform convergence properties of some functions related to the Laplace exponent of the process U_t^∞, and this is more difficult to deal with than the compound Poisson model in Xie and Zhang [27]. The remainder of this paper is organized as follows. In Section 2, we present some preliminaries on $V(u;b)$, $\phi_c(u;b)$ and $\phi_d(u;b)$. In Section 3, we show how to approximate $V(u;b)$, $\phi_c(u;b)$ and $\phi_d(u;b)$ by the COS method. In Section 4, we show how to estimate those functions. The consistency properties for the estimators are studied in Section 5. Finally, in Section 6, we present some numerical results to illustrate the performance of our method.

2. Some preliminaries

In this section, we recall some known results on the expected present value of dividend payments before ruin and the expected discounted penalty functions. Throughout this paper, for any function f defined on the positive real line, we denote its Fourier transform and Laplace transform by

$$\mathcal{F}f(s) = \int_0^\infty e^{isx} f(x)\,dx, \quad \mathcal{L}f(s) = \int_0^\infty e^{-sx} f(x)\,dx,$$

where s is such that the above integrals are well defined.

Note that U_t^∞ is a spectrally negative Lévy process, and its Laplace exponent is defined by

$$\psi_U(s) = \frac{1}{t}\ln E[e^{sU_t^\infty}] = \frac{\sigma^2}{2}s^2 + cs - \lambda(1 - \mathcal{L}f_X(s)). \tag{2.1}$$

For each $q \geq 0$, let

$$\rho_q = \sup\{s \geq 0 : \psi_U(s) = q\}$$

be the right inverse of ψ_U. For the special case $q = \delta$, we shall put $\rho = \rho_\delta$ for simplicity.

For each $q, x \geq 0$, let $W_q(x)$ denote the q-scale function associated with the process U_t^∞, which is a strictly increasing and continuous function with Laplace transform given by

$$\int_0^\infty e^{-sx} W_q(x)\,dx = \frac{1}{\psi_U(s) - q}, \quad \text{for } s > \rho_q. \tag{2.2}$$

We can extend W_q to the whole real line by setting $W_q(x) = 0$ for $x < 0$. The q-scale function and its various extensions have many applications in the fields related to the spectrally negative Lévy processes. For a thorough introduction to the q-scale function and some of its applications, we refer the interested readers to the survey Kuznetsov *et al.* [13].

In our applications, it is useful to introduce the following auxiliary function,

$$h(x) := \frac{\delta}{\psi_U'(\rho)} e^{\rho x} - \delta W_\delta(x), \quad x \in \mathbb{R}. \tag{2.3}$$

It follows from Zhang and Cui [30] that $h(x)$ is the probability density function of the random variable $U_0^\infty - U_{\mathbf{e}_\delta}^\infty$, where $\mathbf{e}_\delta$, independent of U^∞, denotes an exponential random variable with rate $\delta > 0$.

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For the expected present value of total dividend payments before ruin, it follows from Albrecher and Gerber [1] that

$$V(u;b) = \frac{W_\delta(u)}{W'_\delta(b)}, \quad 0 \le u \le b. \tag{2.4}$$

Then, by formula (2.9), in Xie and Zhang [27], we obtain

$$V(u;b) = \frac{\delta/\psi'_U(\rho) - e^{-\rho u}h_+(u)}{\delta\rho e^{\rho(b-u)}/\psi'_U(\rho) - e^{-\rho u}g_+(b)}, \quad 0 \le u \le b, \tag{2.5}$$

where

$$h_+(x) = \begin{cases} h(x), & x \ge 0, \\ 0, & x < 0, \end{cases} \qquad g_+(x) = \begin{cases} h'_+(x), & x \ge 0, \\ 0, & x < 0. \end{cases}$$

Let $\tau_\infty = \inf\{t \ge 0 : U_t^\infty \le 0\}$ denote the ruin time of the model U^∞, and accordingly define

$$\phi_d(u;\infty) = E[e^{-\delta\tau_\infty} I(U_{\tau_\infty}^\infty = 0, \tau_\infty < \infty) | U_0^\infty = u], \quad u \ge 0,$$
$$\phi_c(u;\infty) = E[e^{-\delta\tau_\infty} w(|U_{\tau_\infty}^\infty|) I(U_{\tau_\infty}^\infty < 0, \tau_\infty < \infty) | U_0^\infty = u], \quad u \ge 0,$$

to be the Laplace transform of ruin time when ruin is caused by the Brownian motion, and the expected discounted penalty function when ruin is due to a claim. For the expected discounted penalty functions $\phi_d(u;b)$ and $\phi_c(u;b)$, they can be expressed via the dividends-penalty identities as follows,

$$\phi_d(u;b) = \phi_d(u;\infty) - V(u;b)\phi'_d(b;\infty), \quad 0 \le u \le b, \tag{2.6}$$

$$\phi_c(u;b) = \phi_c(u;\infty) - V(u;b)\phi'_c(b;\infty), \quad 0 \le u \le b. \tag{2.7}$$

For example, see Lin *et al.* [17].

3. The COS approximation

In this section, we shall use the COS method to approximate $V(u;b)$, $\phi_d(u;b)$ and $\phi_c(u;b)$. Recall that for an integrable function f with a finite support $[a_1, a_2]$, it has the following cosine series expansion,

$$f(x) = {\sum_{k=0}^{\infty}}' A_{f,k} \cos\left(k\pi \frac{x - a_1}{a_2 - a_1}\right), \tag{3.1}$$

where $\sum'$ denotes a summation with its first term weighted by a half, and the cosine coefficients are given by

$$A_{f,k} = \frac{2}{a_2 - a_1}\mathrm{Re}\left\{\int_{a_1}^{a_2} f(x) \exp\left(ik\pi \frac{x - a_1}{a_2 - a_1}\right) dx\right\}, \quad k = 0, 1, 2, \ldots, \tag{3.2}$$

where $\mathrm{Re}(\cdot)$ means taking real part and $i = \sqrt{-1}$ is the imaginary unit.

If f is an integrable function supported on the positive real line $[0, \infty)$, we can expand f on a closed interval $[0, a]$, and the COS method suggests that, for large enough a, the COS coefficients can be approximated as follows,

$$A_{f,k} \approx B_{f,k} := \frac{2}{a}\mathrm{Re}\left\{\mathcal{F}f\left(\frac{k\pi}{a}\right)\right\}, \quad k = 0, 1, 2, \ldots. \tag{3.3}$$

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Hence, we have

$$f(x) \approx \tilde{f}(x) := \sum_{k=0}^{K}{}' B_{f,k} \cos\left(k\pi \frac{x-a_1}{a_2-a_1}\right), \quad a_1 \le x \le a_2, \tag{3.4}$$

where K is a large integer applied to truncate the infinite series.

The following lemma is proved in Xie and Zhang [27], which gives the approximation error for the COS method.

Lemma 1. *For real-valued integrable function f supported on $[0,\infty)$, suppose that $|f'(0+)| < \infty$, $|f'(a)| < \infty$, and $\int_0^\infty |f''(y)|\,dy < \infty$. Then, for some positive constants c_1 and c_2, we have*

$$\sup_{x\in[0,a]} |f(x) - \tilde{f}(x)| \le c_1 a K^{-1} + c_2 K a^{-1} \int_a^\infty |f(y)|\,dy. \tag{3.5}$$

Remark 1. Lemma 1 shows that the uniform approximation error of the COS method in the interval $[0,a]$ depends on the parameters a and K. Note that the parameter a is in fact an integration domain truncation parameter in the COS method, which is used in the approximation of the COS coefficients by the Fourier transform $\mathcal{F}f$. The parameter K is used to truncate the COS series expansion formula. Usually, large K will result in good approximation. The upper bound in (3.5) consists of two terms. The first error term aK^{-1} means that larger K can yields better approximation, but large a may slow down the convergence rate. Usually, we should choose an appropriate truncation parameter a to effectively capture the support of the objective function f. In the literature, it is usually selected according to some cumulant-based methods (see [9]). The second error term $Ka^{-1}\int_a^\infty |f(y)|\,dy$ means that larger a can yield better approximation, since larger a usually can result in more accurate approximation of the COS coefficients. However, this error term is increasing w.r.t. K, since larger K means more COS coefficients have to be approximated. Usually, the term $a^{-1}\int_a^\infty |f(y)|\,dy$ can decay very fast w.r.t. a, especially when the tail of the function f converges to zero at the exponential rate. Overall, the first error can dominate the second error. Hence, for a fixed parameter a, the approximation error in (3.5) can achieve order $O(K^{-1})$.

Remark 2. Usually, the decay rate of the COS coefficients depends heavily on the smoothness of the objective function f. If f is infinitely times differentiable, the series $\{A_{f,k}\}$ will show exponential convergence; otherwise, it will yield algebraic convergence. See, for example, Fang and Oosterlee [9]. In our paper, note that the function f that we approximate usually has support $[0,\infty)$, then using integration by parts we can find that

$$A_{f,k} = \frac{2a}{k^2\pi^2}\left\{f'(a)\cos(k\pi) - f'(0+) - \int_0^a f''(x)\cos\left(k\pi\frac{x}{a}\right)dx\right\},$$

which yields that $A_{f,k} = O(k^{-2})$ for fixed a. Furthermore, if f is $2n$ times differentiable and $f^{(2n)}$ is integrable, and $f^{(j)}(0+) = 0$ for $j = 1, 3, \ldots, 2n-1$, then repeatedly using integration by parts we can prove that

$$A_{f,k} = O\left(a^{-1}\sum_{j=1}^{n}\left(\frac{a}{k}\right)^{2j} f^{(2j-1)}(a) + \frac{a^{2n-1}}{k^{2n}}\right).$$

Hence, if the derivatives $f^{(1)}, f^{(3)}, \ldots, f^{(2n-1)}$ all have exponential decay rate, we can first choose large a to ignore the term $a^{-1}\sum_{j=1}^{n}(a/k)^{2j} f^{(2j-1)}(a)$, then for such a, $A_{f,k} = O(k^{-2n})$, which can further lead to faster convergence for the COS approximation.

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Remark 3. Although faster convergence can be achieved under the additional conditions in Remark 2, these conditions are usually not satisfied by the functions in this paper. For example, the objective functions to be approximated in this paper may take the form of exponential functions or their finite mixture, and we can only obtain algebraic convergence. However, in this case, spectral filters can be applied to accelerate the convergence rate according to Ruijter *et al.* [18]. Recall that filtering is carried out in Fourier space and the idea is to pre-multiply the expansion coefficients by a decreasing function. It should be noted that filtering does not add any significant computational costs, and usually can improve the algebraic convergence. In Section 6, we shall illustrate more details on filtering.

In order to use the COS method to approximate $V(u;b)$, we should first approximate h_+ and g_+. It follows from Xie and Zhang [27] that

$$\mathcal{L}h_+(s) = \frac{\delta}{\psi_U'(\rho)} \cdot \frac{1}{s-\rho} - \frac{\delta}{\psi_U(s)-\delta}, \quad s \neq \rho, \tag{3.6}$$

and

$$\mathcal{F}h_+(s) = \mathcal{L}h_+(-is) = \frac{\delta}{\psi_U'(\rho)} \cdot \frac{1}{-is-\rho} - \frac{\delta}{\psi_U(-is)-\delta}, \quad s \in \mathbb{R}. \tag{3.7}$$

Since $W_\delta(0) = 0$ as $\sigma > 0$, we have from (2.3) that $h_+(0) = \delta/\psi_U'(\rho)$, and this yields

$$\mathcal{L}g_+(s) = s\mathcal{L}h_+(s) - h_+(0) = \frac{\delta}{\psi_U'(\rho)} \cdot \frac{\rho}{s-\rho} - \frac{\delta s}{\psi_U(s)-\delta}, \quad s \neq \rho, \tag{3.8}$$

and

$$\mathcal{F}g_+(s) = \mathcal{L}g_+(-is) = \frac{\delta}{\psi_U'(\rho)} \cdot \frac{\rho}{-is-\rho} + \frac{i\delta s}{\psi_U(-is)-\delta}, \quad s \in \mathbb{R}. \tag{3.9}$$

Now using the closed-form Fourier transforms $\mathcal{F}h_+(s)$ and $\mathcal{F}g_+(s)$, the functions h_+ and g_+ can be approximated by the COS method as follows,

$$h_+(x) \approx \tilde{h}_+(x) := {\sum_{k=0}^{K}}' B_{h_+,k} \cos\left(k\pi\frac{x}{a}\right), \quad 0 \leq x \leq a, \tag{3.10}$$

and

$$g_+(x) \approx \tilde{g}_+(x) := {\sum_{k=0}^{K}}' B_{g_+,k} \cos\left(k\pi\frac{x}{a}\right), \quad 0 \leq x \leq a, \tag{3.11}$$

where the COS coefficients are given by

$$B_{h_+,k} := \frac{2}{a}\mathrm{Re}\left\{\mathcal{F}h_+\left(\frac{k\pi}{a}\right)\right\}, \quad B_{g_+,k} := \frac{2}{a}\mathrm{Re}\left\{\mathcal{F}g_+\left(\frac{k\pi}{a}\right)\right\}, \quad k = 0, 1, \ldots. \tag{3.12}$$

As for the expected present value of dividend payments before ruin, using formula (2.5) we can approximate it as follows,

$$V(u;b) \approx \tilde{V}(u;b) := \frac{\delta/\psi_U'(\rho) - e^{-\rho u}\tilde{h}_+(u)}{\delta\rho e^{\rho(b-u)}/\psi_U'(\rho) - e^{-\rho u}\tilde{g}_+(b)}, \quad 0 \leq u \leq b, \tag{3.13}$$

where we take $a > b$.

Remark 4. Suppose that $|h'(0+)| < \infty$, $|h''(0+)| < \infty$, $|h'(a)| < \infty$, $|h''(a)| < \infty$ and

$$\int_0^\infty |h''(y)|\, dy < \infty, \quad \int_0^\infty |h'''(y)|\, dy < \infty.$$

Furthermore, let C be a positive generic constant that may vary at different steps. Then by Lemma 1 we can obtain

$$|\tilde{V}(u;b) - V(u;b)| \le C \cdot \mathcal{E}_h(a,K), \tag{3.14}$$

where

$$\mathcal{E}_h(a,K) = aK^{-1} + Ka^{-1}\int_a^\infty h(y)\, dy + Ka^{-1}\int_a^\infty |h'(y)|\, dy. \tag{3.15}$$

Next, we consider how to use the COS method to approximate the expected discounted penalty functions $\phi_d(u;b)$ and $\phi_c(u;b)$. To this end, we use the dividends-penalty identities given in (2.6) and (2.7). It is easily seen that we should use the COS method to approximate $\phi_d(u;\infty)$, $\phi_c(u;\infty)$ and their first-order derivatives.

For $\phi_d(u;\infty)$, by Zhang [29], we know that its Laplace transform and Fourier transform are given by

$$\mathcal{L}\phi_d(s;\infty) = \frac{(\sigma^2/2)(s-\rho)}{\psi_U(s) - \delta}, \quad \mathcal{F}\phi_d(s;\infty) = \frac{(\sigma^2/2)(-is-\rho)}{\psi_U(-is) - \delta}. \tag{3.16}$$

Then, the Laplace transform of the derivative $\phi'_d(u;\infty)$ is given by

$$\mathcal{L}\phi'_d(s;\infty) = s\mathcal{L}\phi_d(s;\infty) - \phi_d(0;\infty) = \frac{\delta - (c+\sigma^2/2\rho)s + \lambda[1 - \mathcal{L}f_X(s)]}{\psi_U(s) - \delta}, \tag{3.17}$$

from which we obtain

$$\mathcal{F}\phi'_d(s;\infty) = \mathcal{L}\phi'_d(-is;\infty) = \frac{\delta + (c + (\sigma^2/2)\rho)is + \lambda[1 - \mathcal{F}f_X(s)]}{\psi_U(-is) - \delta}. \tag{3.18}$$

For $\phi_c(u;\infty)$, its Laplace transform and Fourier transform are given by

$$\mathcal{L}\phi_c(s;\infty) = \frac{\lambda[\mathcal{L}\omega(\rho) - \mathcal{L}\omega(s)]}{\psi_U(s) - \delta}, \quad \mathcal{F}\phi_c(s;\infty) = \frac{\lambda[\mathcal{L}\omega(\rho) - \mathcal{F}\omega(s)]}{\psi_U(-is) - \delta}, \tag{3.19}$$

where $\omega(u) = \int_u^\infty w(x-u) f_X(x)\, dx$, $u \ge 0$. Furthermore, since $\phi_c(0;\infty) = 0$, we have

$$\mathcal{L}\phi'_c(s;\infty) = s\mathcal{L}\phi_c(s;\infty) = \frac{\lambda s[\mathcal{L}\omega(\rho) - \mathcal{L}\omega(s)]}{\psi_U(s) - \delta} \tag{3.20}$$

and

$$\mathcal{F}\phi'_c(s;\infty) = \mathcal{L}\phi'_c(-is;\infty) = \frac{-\lambda is[\mathcal{L}\omega(\rho) - \mathcal{F}\omega(s)]}{\psi_U(-is) - \delta}. \tag{3.21}$$

Now, using the COS method, we have

$$\phi_d(u;\infty) \approx \tilde{\phi}_d(u;\infty) := \sum_{k=0}^{K}{}' B_{\phi_d,k} \cos\left(k\pi\frac{u}{a}\right), \quad 0 \le u \le a, \tag{3.22}$$

$$\phi_c(u;\infty) \approx \tilde{\phi}_c(u;\infty) := \sum_{k=0}^{K}{}' B_{\phi_c,k} \cos\left(k\pi\frac{u}{a}\right), \quad 0 \le u \le a, \tag{3.23}$$

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$$\phi_d'(u;\infty) \approx \tilde{\phi}_d'(u;\infty) := \sum_{k=0}^{K}{}' B_{\phi_d',k} \cos\left(k\pi\frac{u}{a}\right), \quad 0 \le u \le a, \tag{3.24}$$

$$\phi_c'(u;\infty) \approx \tilde{\phi}_c'(u;\infty) := \sum_{k=0}^{K}{}' B_{\phi_c',k} \cos\left(k\pi\frac{u}{a}\right), \quad 0 \le u \le a, \tag{3.25}$$

where, for $k = 0, 1, \ldots$, the COS coefficients are given by

$$B_{\phi_d,k} := \frac{2}{a}\mathrm{Re}\left\{\mathcal{F}\phi_d\left(\frac{k\pi}{a};\infty\right)\right\}, \quad B_{\phi_c,k} := \frac{2}{a}\mathrm{Re}\left\{\mathcal{F}\phi_c\left(\frac{k\pi}{a};\infty\right)\right\}, \tag{3.26}$$

$$B_{\phi_d',k} := \frac{2}{a}\mathrm{Re}\left\{\mathcal{F}\phi_d'\left(\frac{k\pi}{a};\infty\right)\right\}, \quad B_{\phi_c',k} := \frac{2}{a}\mathrm{Re}\left\{\mathcal{F}\phi_c'\left(\frac{k\pi}{a};\infty\right)\right\}. \tag{3.27}$$

Finally, by the dividends-penalty identities (2.6) and (2.7), we obtain

$$\tilde{\phi}_d(u;b) = \tilde{\phi}_d(u;\infty) - \tilde{V}(u;b)\tilde{\phi}_d'(b;\infty), \quad 0 \le u \le b, \tag{3.28}$$

and

$$\tilde{\phi}_c(u;b) = \tilde{\phi}_c(u;\infty) - \tilde{V}(u;b)\tilde{\phi}_c'(b;\infty), \quad 0 \le u \le b. \tag{3.29}$$

Remark 5. Again, the approximating error can be obtained by Lemma 1. Suppose all the conditions in Remark 4 hold true. Furthermore, suppose that $|\phi_d'(0+;\infty)| < \infty$, $|\phi_d''(0+;\infty)| < \infty$, $|\phi_d'(a;\infty)| < \infty$, $|\phi_d''(a;\infty)| < \infty$ and

$$\int_0^\infty |\phi_d''(u;\infty)|\, du < \infty, \quad \int_0^\infty |\phi_d'''(u;\infty)|\, du < \infty,$$

then we can obtain from Lemma 1 and Remark 4 that

$$|\tilde{\phi}_d(u;b) - \phi_d(u;b)| \le C \cdot (\mathcal{E}_{\phi_d}(a,K) + \mathcal{E}_h(a,K)), \tag{3.30}$$

where $\mathcal{E}_h(a,K)$ is defined in (3.15), and

$$\mathcal{E}_{\phi_d}(a,K) = aK^{-1} + Ka^{-1}\int_a^\infty \phi_d(u;\infty)\, du + Ka^{-1}\int_a^\infty |\phi_d'(u;\infty)|\, du. \tag{3.31}$$

Similarly, suppose that $|\phi_c'(0+;\infty)| < \infty$, $|\phi_c''(0+;\infty)| < \infty$, $|\phi_c'(a;\infty)| < \infty$, $|\phi_c''(a;\infty)| < \infty$ and

$$\int_0^\infty |\phi_c''(u;\infty)| du < \infty, \quad \int_0^\infty |\phi_c'''(u;\infty)|\, du < \infty.$$

Then, we can obtain

$$|\tilde{\phi}_c(u;b) - \phi_c(u;b)| \le C \cdot (\mathcal{E}_{\phi_c}(a,K) + \mathcal{E}_h(a,K)), \tag{3.32}$$

where

$$\mathcal{E}_{\phi_c}(a,K) = aK^{-1} + Ka^{-1}\int_a^\infty \phi_c(u;\infty)\, du + Ka^{-1}\int_a^\infty |\phi_c'(u;\infty)|\, du. \tag{3.33}$$

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4. The estimation method

In this section, we show how to estimate $V(u;b)$, $\phi_d(u;b)$ and $\phi_c(u;b)$ from a random sample on the surplus process. We assume that the premium rate c is known, but the Poisson intensity λ, the claim size density function f_X and the diffusion parameter σ are all unknown.

Suppose that the surplus process can be observed in the long time interval $[0,T]$, and the following random sample on the claim number and individual claim sizes during $[0,T]$ is available,

$$\{n_T,\ X_1,\ldots,X_{n_T}\},$$

where n_T denotes the number of claims received up to time T and it is assumed to be strictly positive w.l.o.g. In addition, suppose that the surplus process U_t^b, the aggregate claims process S_t and the aggregate dividends process $D(t)$ can be observed at a sequence of discrete time points so that the following sample is available

$$\{(U_{k\Delta}^b, S_{k\Delta}, D(k\Delta)) :\ k = 0,1,2,\ldots,n\},$$

where $\Delta > 0$ is a fixed sampling step satisfying $n\Delta = T$.

First, we use formula (3.13) to propose an estimator for the expected present value of dividend payments before ruin. It is easily seen that we need to estimate the following quantities,

$$\rho,\ \ \psi_U',\ \ \tilde{h}_+,\ \ \tilde{g}_+.$$

Since ρ is the positive root of equation $\psi_U(s) = \delta$, we should estimate the Laplace exponent $\psi_U(s)$ by formula (2.1). We estimate the Poisson intensity by $\hat{\lambda} = n_T/T$, and estimate the Laplace transform $\mathcal{L}f_X(s)$ by $\widehat{\mathcal{L}f}_X(s) = (1/n_T)\sum_{j=1}^{n_T} e^{-sX_j}$. It remains to estimate the diffusion parameter σ. Since $U_t^b = U_t^\infty - D(t)$, then we have

$$\sigma B_t = U_t^\infty - u - ct + S_t = U_t^b + D(t) + S_t - u - ct.$$

For $k = 1,\ldots,n$, set

$$Z_k = [U_{k\Delta}^b - U_{(k-1)\Delta}^b] + [D(k\Delta) - D((k-1)\Delta)] + [S_{k\Delta} - S_{(k-1)\Delta}] - c\Delta,$$

which are available since the premium rate c is known. Now, we estimate σ^2 by

$$\hat{\sigma}^2 = \frac{1}{n\Delta}\sum_{k=1}^{n} Z_k^2.$$

It is easily seen that $\hat{\lambda}$, $\widehat{\mathcal{L}f}_X(s)$ and $\hat{\sigma}^2$ are all unbiased estimators. Now using these estimators, we can estimate the Laplace exponent $\psi_U(s)$ and its derivative $\psi_U'(s)$ as follows,

$$\hat{\psi}_U(s) = \frac{\hat{\sigma}^2}{2}s^2 + cs - \hat{\lambda}(1 - \widehat{\mathcal{L}f}_X(s)),$$
$$\hat{\psi}_U'(s) = \hat{\sigma}^2 s + c - \frac{1}{T}\sum_{j=1}^{n_T} X_j e^{-sX_j}.$$

Since ρ is the positive root of equation $\psi_U(s) = \delta$, we define its estimator, denoted by $\hat{\rho}$, to be the positive root of the following estimating equation

$$\hat{\psi}_U(s) = \delta. \tag{4.1}$$

Then, we can estimate $\psi_U'(\rho)$ by $\hat{\psi}_U'(\hat{\rho})$.

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In order to estimate $\tilde{h}_+$ and $\tilde{g}_+$, we should first estimate the Fourier transforms $\mathcal{F}h_+(s)$ and $\mathcal{F}g_+(s)$. It follows from (3.7) and (3.9) that they can be estimated by the following estimators

$$\widehat{\mathcal{F}h_+}(s) = \frac{\delta}{\hat{\psi}'_U(\hat{\rho})} \cdot \frac{1}{-is-\hat{\rho}} - \frac{\delta}{\hat{\psi}_U(-is)-\delta}, \quad s \in \mathbb{R}, \tag{4.2}$$

$$\widehat{\mathcal{F}g_+}(s) = \frac{\delta}{\hat{\psi}'_U(\hat{\rho})} \cdot \frac{\hat{\rho}}{-is-\hat{\rho}} + \frac{i\delta s}{\hat{\psi}_U(-is)-\delta}, \quad s \in \mathbb{R}. \tag{4.3}$$

Then, using formulas (3.10) and (3.11), we propose the following estimators for $\tilde{h}_+$ and $\tilde{g}_+$,

$$* \, \hat{h}_+(x) := \sum_{k=0}^{K}{}' \hat{B}_{h_+,k} \cos\left(k\pi \frac{x}{a}\right), \quad 0 \le x \le a, \tag{4.4}$$

and

$$\hat{g}_+(x) := \sum_{k=0}^{K}{}' \hat{B}_{g_+,k} \cos\left(k\pi \frac{x}{a}\right), \quad 0 \le x \le a, \tag{4.5}$$

where the COS coefficients are given by

$$\hat{B}_{h_+,k} := \frac{2}{a}\mathrm{Re}\left\{\widehat{\mathcal{F}h_+}\left(\frac{k\pi}{a}\right)\right\}, \quad \hat{B}_{g_+,k} := \frac{2}{a}\mathrm{Re}\left\{\widehat{\mathcal{F}g_+}\left(\frac{k\pi}{a}\right)\right\}, \quad k = 0, 1, \ldots. \tag{4.6}$$

Now, using formula (3.13), we obtain the following estimator of $V(u;b)$,

$$\hat{V}(u;b) := \frac{\delta/\hat{\psi}'_U(\hat{\rho}) - e^{-\hat{\rho}u}\hat{h}_+(u)}{\delta\hat{\rho}e^{\hat{\rho}(b-u)}/\hat{\psi}'_U(\hat{\rho}) - e^{-\hat{\rho}u}\hat{g}_+(b)}, \quad 0 \le u \le b < a. \tag{4.7}$$

Next, we consider how to estimate the expected discounted penalty function $\phi_d(u;b)$. To this end, we shall use the dividends-penalty identity (2.6). Note that the Fourier transform of $\phi_d(u;\infty)$ and its derivative can be estimated by

$$\widehat{\mathcal{F}\phi_d}(s;\infty) = \frac{(\hat{\sigma}^2/2)(-is-\hat{\rho})}{\hat{\psi}_U(-is)-\delta}, \quad \widehat{\mathcal{F}\phi'_d}(s;\infty) = \frac{\delta + (c + (\hat{\sigma}^2/2)\hat{\rho})is + \hat{\lambda}[1 - \widehat{\mathcal{F}f_X}(s)]}{\hat{\psi}_U(-is)-\delta}. \tag{4.8}$$

So we can estimate $\phi_d(u;\infty)$ and $\phi'_d(u;\infty)$ as follows,

$$\hat{\phi}_d(u;\infty) := \sum_{k=0}^{K}{}' \hat{B}_{\phi_d,k} \cos\left(k\pi \frac{u}{a}\right), \quad \hat{\phi}'_d(u;\infty) := \sum_{k=0}^{K}{}' \hat{B}_{\phi'_d,k} \cos\left(k\pi \frac{u}{a}\right), \quad 0 \le u \le a, \tag{4.9}$$

where, for $k = 0, 1, \ldots,$

$$\hat{B}_{\phi_d,k} := \frac{2}{a}\mathrm{Re}\left\{\widehat{\mathcal{F}\phi_d}\left(\frac{k\pi}{a};\infty\right)\right\}, \quad \hat{B}_{\phi'_d,k} := \frac{2}{a}\mathrm{Re}\left\{\widehat{\mathcal{F}\phi'_d}\left(\frac{k\pi}{a};\infty\right)\right\}. \tag{4.10}$$

By the dividends-penalty identity (2.6), we propose the following estimator for $\phi_d(u;b)$,

$$\hat{\phi}_d(u;b) = \hat{\phi}_d(u;\infty) - \hat{V}(u;b)\hat{\phi}'_d(b;\infty), \quad 0 \le u \le b. \tag{4.11}$$

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Finally, we estimate the expected discounted penalty function $\phi_c(u;b)$. It is easily seen that

$$\mathcal{L}\omega(s) = \int_0^\infty \int_0^x e^{-su} w(x-u)\,du f_X(x)\,dx, \quad \mathcal{F}\omega(s) = \int_0^\infty \int_0^x e^{isu} w(x-u)\,du f_X(x)\,dx,$$

from which we obtain the following estimators for $\mathcal{L}\omega(\rho)$ and $\mathcal{F}\omega(s)$,

$$\widehat{\mathcal{L}\omega}(\hat{\rho}) = \frac{1}{n_T}\sum_{j=1}^{n_T}\int_0^{X_j} e^{-\hat{\rho}u} w(X_j-u)\,du, \quad \widehat{\mathcal{F}\omega}(s) = \frac{1}{n_T}\sum_{j=1}^{n_T}\int_0^{X_j} e^{isu} w(X_j-u)\,du. \tag{4.12}$$

Now the Fourier transforms $\mathcal{F}\phi_c(s;\infty)$ and $\mathcal{F}\phi_c'(s;\infty)$ can be estimated by

$$\widehat{\mathcal{F}\phi_c}(s;\infty) = \frac{\hat{\lambda}[\widehat{\mathcal{L}\omega}(\hat{\rho}) - \widehat{\mathcal{F}\omega}(s)]}{\hat{\psi}_U(-is)-\delta}, \quad \widehat{\mathcal{F}\phi_c'}(s;\infty) = \frac{-\hat{\lambda}is[\widehat{\mathcal{L}\omega}(\hat{\rho}) - \widehat{\mathcal{F}\omega}(s)]}{\hat{\psi}_U(-is)-\delta}. \tag{4.13}$$

Then $\phi_c(u;\infty)$ and $\phi_c'(u;\infty)$ are estimated as follows,

$$\hat{\phi}_c(u;\infty) := \sum_{k=0}^{K}{}' \hat{B}_{\phi_c,k}\cos\left(k\pi\frac{u}{a}\right), \quad \hat{\phi}_c'(u;\infty) := \sum_{k=0}^{K}{}' \hat{B}_{\phi_c',k}\cos\left(k\pi\frac{u}{a}\right), \quad 0 \le u \le a, \tag{4.14}$$

where, for $k = 0, 1, \ldots,$

$$\hat{B}_{\phi_c,k} := \frac{2}{a}\mathrm{Re}\left\{\widehat{\mathcal{F}\phi_c}\left(\frac{k\pi}{a};\infty\right)\right\}, \quad \hat{B}_{\phi_c',k} := \frac{2}{a}\mathrm{Re}\left\{\widehat{\mathcal{F}\phi_c'}\left(\frac{k\pi}{a};\infty\right)\right\}. \tag{4.15}$$

By the dividends-penalty identity (2.7), we propose the following estimator for $\phi_c(u;b)$,

$$\hat{\phi}_c(u;b) = \hat{\phi}_c(u;\infty) - \hat{V}(u;b)\hat{\phi}_c'(b;\infty), \quad 0 \le u \le b < a. \tag{4.16}$$

5. Consistency properties

In this section, we derive the consistency properties of our estimators when the observation interval $[0,T]$ is very large. First, it is easily seen that

$$\hat{\lambda} - \lambda = O_p(T^{-1/2}), \quad \hat{\sigma}^2 - \sigma^2 = O_p(T^{-1/2}).$$

Here, the notation $O_p(1)$ denotes a sequence that is bounded in probability; or more generally in our paper, for a given sequence of $\{R_{T,K,a}\}$, $O_p(R_{T,K,a})$ denotes a sequence that is bounded in probability at rate $R_{T,K,a}$. The following result on the estimator $\hat{\rho}$ is also well known. See, for example, Zhang [29].

Lemma 2. *Suppose that $c > \lambda EX$ and $EX^2 < \infty$, then for each $\delta > 0$, we have $\hat{\rho} - \rho = O_p(T^{-1/2})$.*

Suppose the conditions in Lemma 2 hold true. For the estimator $\hat{\psi}_U'(\hat{\rho})$, note that

$$\hat{\psi}_U'(\hat{\rho}) - \psi_U'(\rho) = (\hat{\sigma}^2\hat{\rho} - \sigma^2\rho) - \left(\frac{1}{T}\sum_{j=1}^{n_T} X_j e^{-\hat{\rho}X_j} - \lambda E[Xe^{-\rho X}]\right). \tag{5.1}$$

Lemma 2 and $\hat{\sigma}^2 - \sigma^2 = O_p(T^{-1/2})$ imply that $(\hat{\sigma}^2\hat{\rho} - \sigma^2\rho) = O_p(T^{-1/2})$. By Proposition 4 in Xie and Zhang [27], we obtain

$$\frac{1}{T}\sum_{j=1}^{n_T} X_j e^{-\hat{\rho}X_j} - \lambda E[Xe^{-\rho X}] = O_p(T^{-1/2}).$$

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Hence, it follows from (5.1) that

$$\hat{\psi}'_U(\hat{\rho}) - \psi'_U(\rho) = O_p(T^{-1/2}). \tag{5.2}$$

Recall that ρ is the root of equation $\psi_U(s) = \delta$, then we have

$$\begin{aligned}\psi_U(-is) - \delta &= \psi_U(-is) - \psi_U(\rho)\\ &= (-is-\rho)\left(\frac{\sigma^2}{2}(-is+\rho) + c - \lambda\frac{\mathcal{L}f_X(-is) - \mathcal{L}f_X(\rho)}{-is-\rho}\right)\\ &= (-is-\rho)l(s),\end{aligned} \tag{5.3}$$

where $l(s) = (\sigma^2/2)(-is+\rho) + c + \lambda\int_0^\infty e^{isx}\int_0^x e^{-(\rho+is)y}\,dy\,f_X(x)\,dx$. Note that

$$\begin{aligned}|l(s)| &\geq \left|\frac{\sigma^2}{2}(-is+\rho)+c\right| - \left|\lambda\int_0^\infty e^{isx}\int_0^x e^{-(\rho+is)y}\,dy\,f_X(x)\,dx\right|\\ &\geq \left|\frac{\sigma^2}{2}(-is+\rho)+c\right| - \lambda EX\end{aligned} \tag{5.4}$$

$$\geq \frac{\sigma^2}{2}\rho + c - \lambda EX > c - \lambda EX > 0. \tag{5.5}$$

Similarly, since $\hat{\rho}$ is the root of equation $\hat{\psi}_U(s) = \delta$, we have

$$\begin{aligned}\hat{\psi}_U(-is) - \delta &= (-is-\hat{\rho})\left(\frac{\hat{\sigma}^2}{2}(-is+\hat{\rho}) + c - \hat{\lambda}\frac{\widehat{\mathcal{L}f}_X(-is) - \widehat{\mathcal{L}f}_X(\hat{\rho})}{-is-\hat{\rho}}\right)\\ &= (-is-\hat{\rho})\hat{l}(s),\end{aligned} \tag{5.6}$$

where $\hat{l}(s) = (\hat{\sigma}^2/2)(-is+\hat{\rho}) + c + (1/T)\sum_{j=1}^{n_T} e^{isX_j}\int_0^{X_j} e^{-(\hat{\rho}+is)y}\,dy$. Furthermore, note that

$$\begin{aligned}\hat{l}(s) &\geq \left|\frac{\hat{\sigma}^2}{2}(-is+\hat{\rho})+c\right| - \left|\frac{1}{T}\sum_{j=1}^{n_T} e^{isX_j}\int_0^{X_j} e^{-(\hat{\rho}+is)y}\,dy\right|\\ &\geq \left|\frac{\hat{\sigma}^2}{2}(-is+\hat{\rho})+c\right| - \frac{1}{T}\sum_{j=1}^{n_T} X_j\end{aligned} \tag{5.7}$$

$$\geq \frac{\hat{\sigma}^2}{2}\hat{\rho} + c - \frac{1}{T}\sum_{j=1}^{n_T} X_j > c - \frac{1}{T}\sum_{j=1}^{n_T} X_j, \tag{5.8}$$

which together with (5.6) gives

$$|\hat{\psi}_U(-is) - \delta| \geq |is+\hat{\rho}| \cdot \left(c - \frac{1}{T}\sum_{j=1}^{n_T} X_j\right). \tag{5.9}$$

The following lemma shows the uniform convergence of $1/(\hat{\psi}_U(-is)-\delta)$ and $s/(\hat{\psi}_U(-is)-\delta)$, which plays an important role in studying the convergence of our estimators.

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Lemma 3. *Suppose that $c > \lambda EX$, $EX^2 < \infty$ and $a = o(K)$. Then, we have*

$$\sup_{s\in\mathcal{S}}\left|\frac{1}{\hat{\psi}_U(-is)-\delta}-\frac{1}{\psi_U(-is)-\delta}\right| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right) \tag{5.10}$$

and

$$\sup_{s\in\mathcal{S}}\left|\frac{s}{\hat{\psi}_U(-is)-\delta}-\frac{s}{\psi_U(-is)-\delta}\right| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right). \tag{5.11}$$

Proof. First, we study the uniform convergence of $1/(\hat{\psi}_U(-is)-\delta)$. By (5.3) and (5.5), we have for any real number s,

$$|\psi_U(-is)-\delta| \geq |is+\rho|\cdot[c-\lambda EX], \tag{5.12}$$

and by (5.6) and (5.8) we have for any real number s,

$$|\hat{\psi}_U(-is)-\delta| \geq |is+\hat{\rho}|\cdot\left(c-\frac{1}{T}\sum_{j=1}^{n_T}X_j\right). \tag{5.13}$$

Recall the condition $c > \lambda EX$. In the remainder of this proof, suppose that $c-(1/T)\sum_{j=1}^{n_T}X_j > 0$, since $(1/T)\sum_{j=1}^{n_T}X_j$ converges to λEX a.s.

Now using (5.12) and (5.13) and the following result

$$\begin{aligned}\psi_U(-is)-\hat{\psi}_U(-is) &= \left(\frac{\sigma^2}{2}-\frac{\hat{\sigma}^2}{2}\right)(-is)^2-(\lambda-\hat{\lambda})+\lambda\mathcal{L}f_X(-is)-\hat{\lambda}\widehat{\mathcal{L}f}_X(-is)\\ &= \frac{1}{2}(\hat{\sigma}^2-\sigma^2)s^2+(\hat{\lambda}-\lambda)-\left(\frac{1}{T}\sum_{j=1}^{n_T}e^{isX_j}-\lambda Ee^{isX}\right),\end{aligned} \tag{5.14}$$

we can obtain

$$\begin{aligned}&\sup_{s\in\mathcal{S}}\left|\frac{1}{\hat{\psi}_U(-is)-\delta}-\frac{1}{\psi_U(-is)-\delta}\right|\\ &= \sup_{s\in\mathcal{S}}\frac{|\psi_U(-is)-\hat{\psi}_U(-is)|}{|\psi_U(-is)-\delta|\cdot|\hat{\psi}_U(-is)-\delta|}\\ &\leq \sup_{s\in\mathcal{S}}\frac{|\frac{1}{2}(\hat{\sigma}^2-\sigma^2)s^2+(\hat{\lambda}-\lambda)-((1/T)\sum_{j=1}^{n_T}e^{isX_j}-\lambda Ee^{isX})|}{|is+\rho|\cdot|is+\hat{\rho}|\cdot(c-\lambda EX)\cdot(c-(1/T)\sum_{j=1}^{n_T}X_j)}\\ &\leq \sup_{s\in\mathcal{S}}\frac{\left|\frac{1}{2}(\hat{\sigma}^2-\sigma^2)s^2\right|}{|is+\rho|\cdot|is+\hat{\rho}|\cdot(c-\lambda EX)\cdot(c-(1/T)\sum_{j=1}^{n_T}X_j)}\\ &\quad+\sup_{s\in\mathcal{S}}\frac{|(\hat{\lambda}-\lambda)-((1/T)\sum_{j=1}^{n_T}e^{isX_j}-\lambda Ee^{isX})|}{|is+\rho|\cdot|is+\hat{\rho}|\cdot(c-\lambda EX)\cdot(c-(1/T)\sum_{j=1}^{n_T}X_j)}\\ &\leq \frac{\frac{1}{2}|\hat{\sigma}^2-\sigma^2|}{(c-\lambda EX)\cdot(c-(1/T)\sum_{j=1}^{n_T}X_j)}+\frac{\sup_{s\in\mathcal{S}}|(\hat{\lambda}-\lambda)-((1/T)\sum_{j=1}^{n_T}e^{isX_j}-\lambda Ee^{isX})|}{\rho\hat{\rho}\cdot(c-\lambda EX)\cdot(c-(1/T)\sum_{j=1}^{n_T}X_j)}.\end{aligned} \tag{5.15}$$

The convergence rate $\hat{\sigma}^2-\sigma^2 = O_p(T^{-1/2})$ implies that

$$\frac{\frac{1}{2}|\hat{\sigma}^2-\sigma^2|}{(c-\lambda EX)\cdot(c-(1/T)\sum_{j=1}^{n_T}X_j)} = O_p(T^{-1/2}). \tag{5.16}$$

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It follows from Lemma 3 in Xie and Zhang [27] that, for $a = o(K)$,

$$\sup_{s\in\mathcal{S}}\left|(\hat{\lambda}-\lambda)-\left(\frac{1}{T}\sum_{j=1}^{n_T}e^{isX_j}-\lambda Ee^{isX}\right)\right| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right), \tag{5.17}$$

leading to

$$\frac{\sup_{s\in\mathcal{S}}|(\hat{\lambda}-\lambda)-((1/T)\sum_{j=1}^{n_T}e^{isX_j}-\lambda Ee^{isX})|}{\rho\hat{\rho}\cdot(c-\lambda EX)\cdot(c-(1/T)\sum_{j=1}^{n_T}X_j)} = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right),$$

which together with (5.15), (5.16) yields (5.10).

Next, we prove (5.11). By (5.3) and (5.6), we have

$$\begin{aligned}\frac{s}{\hat{\psi}_U(-is)-\delta}-\frac{s}{\psi_U(-is)-\delta} &= \frac{s[\psi_U(-is)-\hat{\psi}_U(-is)]}{[\psi_U(-is)-\delta]\cdot[\hat{\psi}_U(-is)-\delta]}\\ &= \frac{s(\frac{1}{2}(\hat{\sigma}^2-\sigma^2)s^2+(\hat{\lambda}-\lambda)-((1/T)\sum_{j=1}^{n_T}e^{isX_j}-\lambda Ee^{isX}))}{(-is-\rho)(-is-\hat{\rho})l(s)\hat{l}(s)},\end{aligned} \tag{5.18}$$

which gives

$$\begin{aligned}&\sup_{s\in\mathcal{S}}\left|\frac{s}{\hat{\psi}_U(-is)-\delta}-\frac{s}{\psi_U(-is)-\delta}\right|\\ &\le \sup_{s\in\mathcal{S}}\left|\frac{\frac{1}{2}(\hat{\sigma}^2-\sigma^2)s^3}{(-is-\rho)(-is-\hat{\rho})l(s)\hat{l}(s)}\right|+\sup_{s\in\mathcal{S}}\left|\frac{s((\hat{\lambda}-\lambda)-((1/T)\sum_{j=1}^{n_T}e^{isX_j}-\lambda Ee^{isX}))}{(-is-\rho)(-is-\hat{\rho})l(s)\hat{l}(s)}\right|\\ &\le \frac{1}{2}\cdot\sup_{s\in\mathcal{S}}\left|\frac{(\hat{\sigma}^2-\sigma^2)s}{l(s)\hat{l}(s)}\right|+\frac{1}{\rho}\cdot\sup_{s\in\mathcal{S}}\left|\frac{(\hat{\lambda}-\lambda)-((1/T)\sum_{j=1}^{n_T}e^{isX_j}-\lambda Ee^{isX})}{l(s)\hat{l}(s)}\right|.\end{aligned} \tag{5.19}$$

It follows from (5.4) that

$$\sup_{s\in\mathcal{S}}|s/l(s)| \le \sup_{s\in\mathcal{S}}\left|\frac{s}{|(\sigma^2/2)(-is+\rho)+c|-\lambda EX}\right| \le C,$$

which together with (5.8) and the convergence rate $\hat{\sigma}^2-\sigma^2 = O_p(T^{-1/2})$ gives

$$\sup_{s\in\mathcal{S}}\left|\frac{(\hat{\sigma}^2-\sigma^2)s}{l(s)\hat{l}(s)}\right| \le \frac{C}{c-(1/T)\sum_{j=1}^{n_T}X_j}|\hat{\sigma}^2-\sigma^2| = O_p(T^{-1/2}). \tag{5.20}$$

By (5.5), (5.8) and (5.17), we obtain

$$\begin{aligned}&\sup_{s\in\mathcal{S}}\left|\frac{(\hat{\lambda}-\lambda)-((1/T)\sum_{j=1}^{n_T}e^{isX_j}-\lambda Ee^{isX})}{l(s)\hat{l}(s)}\right|\\ &\le \frac{\sup_{s\in\mathcal{S}}|(\hat{\lambda}-\lambda)-((1/T)\sum_{j=1}^{n_T}e^{isX_j}-\lambda Ee^{isX})|}{(c-\lambda EX)\cdot(c-(1/T)\sum_{j=1}^{n_T}X_j)} = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right).\end{aligned} \tag{5.21}$$

By combining (5.19)–(5.21), we obtain (5.11). □

In order to derive the estimation error of $\hat{V}(u;b)$, we should first consider the estimation errors of $\hat{h}_+$ and $\hat{g}_+$. We have the following results.

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Proposition 1. *Suppose that $c > \lambda EX$, $EX^2 < \infty$ and $a = o(K)$. Then, we have*

$$\sup_{0\le x\le a} |\hat{h}_+(x) - \tilde{h}_+(x)| = O_p\left((K/a)\sqrt{\frac{\log(K/a)}{T}}\right) \tag{5.22}$$

and

$$\sup_{0\le x\le a} |\hat{g}_+(x) - \tilde{g}_+(x)| = O_p\left((K/a)\sqrt{\frac{\log(K/a)}{T}}\right). \tag{5.23}$$

Proof. We only prove (5.22), since (5.23) can be proved similarly by using (5.11). First, it is easily seen that

$$\sup_{0\le x\le a} |\hat{h}_+(x) - \tilde{h}_+(x)| \le \sum_{k=0}^{K} |\hat{B}_{h_+,k} - B_{h_+,k}| \le \frac{2(K+1)}{a} \sup_{s\in\mathcal{S}} |\widehat{\mathcal{F}h_+}(s) - \mathcal{F}h_+(s)|, \tag{5.24}$$

where $\mathcal{S} = \{k\pi/a : k = 0, 1, \ldots, K\}$. For each $s \in \mathcal{S}$, we have

$$\widehat{\mathcal{F}h_+}(s) - \mathcal{F}h_+(s) = I_h(s) + II_h(s),$$

where

$$I_h(s) = \left(\frac{\delta}{\hat{\psi}'_U(\hat{\rho})} \cdot \frac{1}{-is - \hat{\rho}} - \frac{\delta}{\psi'_U(\rho)} \cdot \frac{1}{-is - \rho}\right), \quad II_h(s) = \left(\frac{\delta}{\psi_U(-is) - \delta} - \frac{\delta}{\hat{\psi}_U(-is) - \delta}\right).$$

By Lemma 2 and (5.2), we can easily prove that

$$\sup_{s\in\mathcal{S}} |I_h(s)| = O_p(T^{-1/2}).$$

By Lemma 3, we have

$$\sup_{s\in\mathcal{S}} |II_h(s)| = \delta \cdot \sup_{s\in\mathcal{S}} \left|\frac{1}{\hat{\psi}_U(-is) - \delta} - \frac{1}{\psi_U(-is) - \delta}\right| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right).$$

Hence, we have

$$\sup_{s\in\mathcal{S}} |\widehat{\mathcal{F}h_+}(s) - \mathcal{F}h_+(s)| \le \sup_{s\in\mathcal{S}} |I_h(s)| + \sup_{s\in\mathcal{S}} |II_h(s)| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right).$$

which together with (5.24) completes the proof. □

By the convergence rate given in (5.2), Lemma 2 and Proposition 1, we can easily obtain the following result.

Proposition 2. *Suppose that $c > \lambda EX$, $EX^2 < \infty$ and $a = o(K)$. Then, we have*

$$\hat{V}(u; b) - \tilde{V}(u; b) = O_p\left((K/a)\sqrt{\frac{\log(K/a)}{T}}\right). \tag{5.25}$$

Next, we derive the convergence rate for the estimators of the expected discounted penalty functions. It follows from formulas (4.11) and (4.16) that we should first study the consistency properties of $\hat{\phi}_d(u; \infty)$ and $\hat{\phi}'_d(u; \infty)$.

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Proposition 3. *Suppose that $c > \lambda EX$, $EX^2 < \infty$ and $a = o(K)$. Then, we have*

$$\sup_{0\le u\le a} |\hat{\phi}_d(u;\infty) - \tilde{\phi}_d(u;\infty)| = O_p\left((K/a)\sqrt{\frac{\log(K/a)}{T}}\right), \tag{5.26}$$

$$\sup_{0\le u\le a} |\hat{\phi}'_d(u;\infty) - \tilde{\phi}'_d(u;\infty)| = O_p\left((K/a)\sqrt{\frac{\log(K/a)}{T}}\right). \tag{5.27}$$

Proof. First, we prove (5.26). By formulas (3.22) and (4.9), we can obtain

$$\begin{aligned}\sup_{0\le u\le a} |\hat{\phi}_d(u;\infty) - \tilde{\phi}_d(u;\infty)| &\le \sum_{k=0}^{K} |\hat{B}_{\phi_d,k} - B_{\phi_d,k}| \\ &\le \frac{2(K+1)}{a} \cdot \sup_{s\in\mathcal{S}} |\widehat{\mathcal{F}\phi}_d(s;\infty) - \mathcal{F}\phi_d(s;\infty)|. \end{aligned} \tag{5.28}$$

By (3.19) and (4.8), we obtain

$$\begin{aligned}&\sup_{s\in\mathcal{S}} \left|\widehat{\mathcal{F}\phi}_d(s;\infty) - \mathcal{F}\phi_d(s;\infty)\right| \\ &\quad = \sup_{s\in\mathcal{S}} \left|(\frac{(\hat{\sigma}^2/2)(-is-\hat{\rho})}{\hat{\psi}_U(-is)-\delta} - \frac{(\sigma^2/2)(-is-\rho)}{\psi_U(-is)-\delta}\right| \\ &\quad \le \sup_{s\in\mathcal{S}} \left|\frac{(\hat{\sigma}^2/2)s}{\hat{\psi}_U(-is)-\delta} - \frac{(\sigma^2/2)s}{\psi_U(-is)-\delta}\right| + \sup_{s\in\mathcal{S}} \left|\frac{(\hat{\sigma}^2/2)\hat{\rho}}{\hat{\psi}_U(-is)-\delta} - \frac{(\sigma^2/2)\rho}{\psi_U(-is)-\delta}\right|. \end{aligned} \tag{5.29}$$

Furthermore, by Lemma 3, we have

$$\begin{aligned}&\sup_{s\in\mathcal{S}} \left|\frac{(\hat{\sigma}^2/2)s}{\hat{\psi}_U(-is)-\delta} - \frac{(\sigma^2/2)s}{\psi_U(-is)-\delta}\right| \\ &\quad \le \frac{\hat{\sigma}^2}{2} \cdot \sup_{s\in\mathcal{S}} \left|\frac{s}{\hat{\psi}_U(-is)-\delta} - \frac{s}{\psi_U(-is)-\delta}\right| + \frac{|\hat{\sigma}^2-\sigma^2|}{2} \cdot \sup_{s\in\mathcal{S}} \left|\frac{s}{\psi_U(-is)-\delta}\right| \\ &\quad \le \frac{\hat{\sigma}^2}{2} \cdot \sup_{s\in\mathcal{S}} \left|\frac{s}{\hat{\psi}_U(-is)-\delta} - \frac{s}{\psi_U(-is)-\delta}\right| + \frac{|\hat{\sigma}^2-\sigma^2|}{2} \cdot \sup_{s\in\mathcal{S}} \left|\frac{s}{(-is-\rho)(c-\lambda EX)}\right| \\ &\quad = O_p(1)\cdot O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right) + O_p(T^{-1/2}) \\ &\quad = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right). \end{aligned} \tag{5.30}$$

Similarly, we have

$$\begin{aligned}&\sup_{s\in\mathcal{S}} \left|\frac{(\hat{\sigma}^2/2)\hat{\rho}}{\hat{\psi}_U(-is)-\delta} - \frac{(\sigma^2/2)\rho}{\psi_U(-is)-\delta}\right| \\ &\quad \le \frac{\hat{\sigma}^2\hat{\rho}}{2} \cdot \sup_{s\in\mathcal{S}} \left|\frac{1}{\hat{\psi}_U(-is)-\delta} - \frac{1}{\psi_U(-is)-\delta}\right| + \frac{|\hat{\sigma}^2\hat{\rho}-\sigma^2\rho|}{2} \cdot \sup_{s\in\mathcal{S}} \left|\frac{1}{(-is-\rho)(c-\lambda EX)}\right| \end{aligned}$$

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$$= O_p(1) \cdot O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right) + O_p(T^{-1/2})$$
$$= O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right). \tag{5.31}$$

Hence, (5.29) gives

$$\sup_{s\in\mathcal{S}} |\widehat{\mathcal{F}\phi}_d(s;\infty) - \mathcal{F}\phi_d(s;\infty)| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right),$$

which together with (5.28) yields (5.26).

Next, we prove (5.27). Again, we can obtain

$$\sup_{0\le u\le a} |\hat{\phi}'_d(u;\infty) - \tilde{\phi}'_d(u;\infty)| \le \frac{2(K+1)}{a} \cdot \sup_{s\in\mathcal{S}} |\widehat{\mathcal{F}\phi}'_d(s;\infty) - \mathcal{F}\phi'_d(s;\infty)|, \tag{5.32}$$

where

$$\widehat{\mathcal{F}\phi}'_d(s;\infty) - \mathcal{F}\phi'_d(s;\infty) = \frac{\delta + (c+\hat{\sigma}^2/2)is + \hat{\lambda}[1-\widehat{\mathcal{F}f}_X(s)]}{\hat{\psi}_U(-is)-\delta} - \frac{\delta + (c+\sigma^2/2)is + \lambda[1-\mathcal{F}f_X(s)]}{\psi_U(-is)-\delta}.$$

Furthermore, we have

$$\sup_{s\in\mathcal{S}}\left|\widehat{\mathcal{F}\phi}'_d(s;\infty) - \mathcal{F}\phi'_d(s;\infty)\right| \le \sup_{s\in\mathcal{S}}\left|\frac{\delta+\hat{\lambda}[1-\widehat{\mathcal{F}f}_X(s)]}{\hat{\psi}_U(-is)-\delta} - \frac{\delta+\lambda[1-\mathcal{F}f_X(s)]}{\psi_U(-is)-\delta}\right| + \sup_{s\in\mathcal{S}}\left|\frac{(c+\hat{\sigma}^2/2)is}{\hat{\psi}_U(-is)-\delta} - \frac{(c+\sigma^2/2)is}{\psi_U(-is)-\delta}\right|.$$

By Lemma 3 and (5.17) and using the same arguments in (5.30) and (5.31), we can obtain

$$\sup_{s\in\mathcal{S}}\left|\frac{\delta+\hat{\lambda}[1-\widehat{\mathcal{F}f}_X(s)]}{\hat{\psi}_U(-is)-\delta} - \frac{\delta+\lambda[1-\mathcal{F}f_X(s)]}{\psi_U(-is)-\delta}\right| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right)$$

and

$$\sup_{s\in\mathcal{S}}\left|\frac{(c+\hat{\sigma}^2/2)is}{\hat{\psi}_U(-is)-\delta} - \frac{(c+\sigma^2/2)is}{\psi_U(-is)-\delta}\right| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right).$$

Hence, we obtain

$$\sup_{s\in\mathcal{S}}\left|\widehat{\mathcal{F}\phi}'_d(s;\infty) - \mathcal{F}\phi'_d(s;\infty)\right| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right),$$

which together with (5.32) gives (5.27). □

Next, we derive the convergence rates for $\hat{\phi}_c(u;\infty)$ and $\hat{\phi}'_c(u;\infty)$.

Proposition 4. *Suppose that $c > \lambda EX$, $EX^4 < \infty$, $a = o(K)$, and*

$$E\left(\int_0^X w(X-x)\,dx\right) < \infty, \quad E\left(\int_0^X xw(X-x)\,dx\right)^2 < \infty.$$

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Then, we have

$$\sup_{0\le u\le a} |\hat{\phi}_c(u;\infty) - \tilde{\phi}_c(u;\infty)| = O_p\left((K/a)\sqrt{\frac{\log(K/a)}{T}}\right), \tag{5.33}$$

$$\sup_{0\le u\le a} |\hat{\phi}'_c(u;\infty) - \tilde{\phi}'_c(u;\infty)| = O_p\left((K/a)\sqrt{\frac{\log(K/a)}{T}}\right). \tag{5.34}$$

Proof. First, we prove (5.33). It is easily obtained that

$$\sup_{0\le u\le a} |\hat{\phi}_c(u;\infty) - \tilde{\phi}_c(u;\infty)| \le \frac{2(K+1)}{a} \cdot \sup_{s\in\mathcal{S}} |\widehat{\mathcal{F}\phi_c}(s;\infty) - \mathcal{F}\phi_c(s;\infty)|, \tag{5.35}$$

where

$$\begin{aligned}
&\sup_{s\in\mathcal{S}}\left|\widehat{\mathcal{F}\phi_c}(s;\infty) - \mathcal{F}\phi_c(s;\infty)\right| \\
&\quad= \sup_{s\in\mathcal{S}}\left|\frac{\hat{\lambda}[\widehat{\mathcal{L}\omega}(\hat{\rho}) - \widehat{\mathcal{F}\omega}(s)]}{\hat{\psi}_U(-is)-\delta} - \frac{\lambda[\mathcal{L}\omega(\rho) - \mathcal{F}\omega(s)]}{\psi_U(-is)-\delta}\right| \\
&\quad\le \sup_{s\in\mathcal{S}}\left|\frac{\hat{\lambda}\widehat{\mathcal{L}\omega}(\hat{\rho})}{\hat{\psi}_U(-is)-\delta} - \frac{\lambda\mathcal{L}\omega(\rho)}{\psi_U(-is)-\delta}\right| + \sup_{s\in\mathcal{S}}\left|\frac{\hat{\lambda}\widehat{\mathcal{F}\omega}(s)}{\hat{\psi}_U(-is)-\delta} - \frac{\lambda\mathcal{F}\omega(s)}{\psi_U(-is)-\delta}\right|.
\end{aligned}$$

Furthermore, by Xie and Zhang [27], we have

$$\hat{\lambda}\widehat{\mathcal{L}\omega}(\hat{\rho}) - \lambda\mathcal{L}\omega(\rho) = O_p(T^{-1/2}), \quad \sup_{s\in\mathcal{S}} |\hat{\lambda}\widehat{\mathcal{F}\omega}(s) - \lambda\mathcal{F}\omega(s)| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right).$$

Then, by Lemma 3, we can obtain

$$\sup_{s\in\mathcal{S}} |\widehat{\mathcal{F}\phi_c}(s;\infty) - \mathcal{F}\phi_c(s;\infty)| = O_p\left(\sqrt{\frac{\log(K/a)}{T}}\right),$$

which together with (5.35) gives (5.33). The proof of (5.34) is similar, and we omit the detailed arguments. ☐

Now by combining the results in Propositions 2–4, we can obtain the convergence rates of $\hat{\phi}_d(u;b)$ and $\hat{\phi}_c(u;b)$.

Proposition 5. *Suppose that $c > \lambda EX$, $EX^2 < \infty$ and $a = o(K)$. Then, we have*

$$\hat{\phi}_d(u;b) - \tilde{\phi}_d(u;b) = O_p\left((K/a)\sqrt{\frac{\log(K/a)}{T}}\right). \tag{5.36}$$

Proposition 6. *Suppose that $c > \lambda EX$, $EX^4 < \infty$, $a = o(K)$, and*

$$E\left(\int_0^X w(X-x)\,dx\right) < \infty, \quad E\left(\int_0^X xw(X-x)\,dx\right)^2 < \infty.$$

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Then, we have

$$|\hat{\phi}_c(u;b) - \tilde{\phi}_c(u;b)| = O_p\left((K/a)\sqrt{\frac{\log(K/a)}{T}}\right). \tag{5.37}$$

Remark 6. Propositions 5 and 6 show that the convergence rate depend the parameters a, K and T. Here, the parameters a and K appear in the COS approximation step, while the parameter T stands for sample size in the statical estimation step. Usually, the parameter a need not to be large enough, and we can use some cumulant-based methods for selection (see Section 6).

For the series truncation parameter K, although a larger value can lead to better COS approximation, it also implies that more COS coefficients have to be estimated in the statistical estimation step. Hence, the statistical estimation error is increasing w.r.t. K. We remark that the condition $a = o(K)$ is just used for theoretical derivation (see also [27]), which means that the parameter a should be dominated by the series truncation parameter K. This condition is satisfied in our paper, since when using the cumulant-based method to choose a, we usually have $a \leq 50$, but when choosing the truncation parameter K, we usually take $K = 2^8, 2^9, 2^{10}$, etc., larger than a. Although we observe the error propagation w.r.t K, we can enlarge the sample size (i.e., take large T) to get satisfactory estimators. Anyway, if we simply look at the estimation of $\tilde{\phi}_d(u;b)$ and $\tilde{\phi}_c(u;b)$ (i.e., for fixed K and a), Propositions 5 and 6 show that we can nearly obtain $O_p(T^{-1/2})$ convergence rate.

6. Numerical results

In this section, we present some numerical results to illustrate the performance of our method. All computations are performed in MATLAB on a desktop computer with Intel(R) Core(TM) i7-6700 CPU, 3.4 GHz and a RAM of 8 GB. In this part, we set $c = 4$, $\lambda = 3$, $\sigma = 1$, $\delta = 0.1$ and $b = 30$. We consider the following three claim size density functions,

1. Exp(1): $f_X(x) = e^{-x}$, $x > 0$;
2. Erlang(2,2): $f_X(x) = 4xe^{-2x}$, $x > 0$;
3. Gamma($\frac{3}{2}, \frac{3}{2}$): $f_X(x) = \frac{3\sqrt{6x}e^{-3x/2}}{2\sqrt{\pi}}$, $x > 0$.

The choice of the truncation parameter a is selected according to the following rule (see [9]):

$$a = \kappa_1 + L\sqrt{\kappa_2 + \sqrt{\kappa_4}}, \tag{6.1}$$

where

$$\kappa_j = \frac{d^j}{ds^j}\log\left(\int e^{sx} f(x)\,dx\right)\bigg|_{s=0} = \frac{d^j}{ds^j}\log \mathcal{F}f(-is)\bigg|_{s=0}, \tag{6.2}$$

and f is a probability density function. As in Xie and Zhang [27], we set $L = 10$. When computing $\tilde{h}_+$ and $\tilde{g}_+$, we take $\mathcal{F}f = \mathcal{F}h_+$; when computing $\tilde{\phi}_d$ and $\tilde{\phi}'_d$, we take $\mathcal{F}f = \mathcal{F}\phi_d$; when computing $\tilde{\phi}_c$ and $\tilde{\phi}'_c$, we take $\mathcal{F}f = \mathcal{F}\phi_c$.

For the expected present value of total dividend payments before ruin, we easily obtain the explicit formula by (2.4) when f_X is exponential or Erlang(2,2). When f_X is Gamma($\frac{3}{2}, \frac{3}{2}$), in order to provide a benchmark, we compute the reference values by the COS method with truncation parameter $K = 2^{13}$. For the expected discounted penalty function, we set $w \equiv 1$, then $\phi_d(u;b)$ is the Laplace transform of the time of ruin by oscillation, and $\phi_c(u;b)$ is the Laplace transform of the time of ruin due to a claim. When f_X is exponential or Erlang(2,2), we can use Laplace inversion transform to compute the explicit formulas by (3.16) and (3.19) for $\phi_d(u;b)$ and $\phi_c(u;b)$, respectively. When f_X is Gamma($\frac{3}{2}, \frac{3}{2}$), we also compute the reference values by the COS method with truncation parameter $K = 2^{13}$.

First, we study the approximation performance of the COS method. In order to measure the accuracy of our method, we consider average absolute errors for $\tilde{V}(u;b)$, $\tilde{\phi}_d(u;b)$ and $\tilde{\phi}_c(u;b)$, respectively,

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Table 1. *Average absolute errors by the COS method.*

p	$\tilde{V}(u;b)$			$\tilde{\phi}_d(u;b)$			$\tilde{\phi}_c(u;b)$		
	Exp	Erlang	Gamma	Exp	Erlang	Gamma	Exp	Erlang	Gamma
6	0.034492	0.024331	0.034561	0.028865	0.026421	0.028959	0.025058	0.021115	0.011907
7	0.028245	0.017991	0.032642	0.021888	0.020943	0.023772	0.021909	0.012736	0.007867
8	0.009112	0.008653	0.018372	0.014322	0.009054	0.012048	0.012729	0.004236	0.009977
9	0.004665	0.002410	0.019538	0.004661	0.002652	0.003612	0.003372	0.001176	0.001311
10	0.001383	0.000094	0.002446	0.001312	0.000853	0.001063	0.001073	0.000757	0.000856

which are defined by

$$\frac{1}{\#\mathcal{U}}\sum_{u\in\mathcal{U}}|\tilde{V}(u;b)-V^{\mathrm{ref}}(u;b)|,\qquad \frac{1}{\#\mathcal{U}'}\sum_{u\in\mathcal{U}'}|\tilde{\phi}_d(u;b)-\phi_d^{\mathrm{ref}}(u;b)|$$

and

$$\frac{1}{\#\mathcal{U}''}\sum_{u\in\mathcal{U}''}|\tilde{\phi}_c(u;b)-\phi_c^{\mathrm{ref}}(u;b)|,$$

where $V^{\mathrm{ref}}(u;b)$, $\phi_d^{\mathrm{ref}}(u;b)$ and $\phi_c^{\mathrm{ref}}(u;b)$ are the reference values. We take $\mathcal{U}=\{0.1,0.2,\dots,30\}$, $\mathcal{U}'=\{0.1,0.2,\dots,5\}$ and $\mathcal{U}''=\{0.1,0.2,\dots,15\}$, since $\phi_d(u;b)$ is very small for $u>5$, and $\phi_c(u;b)$ is very close to zero for $u>15$. The average absolute errors are reported in Table 1, where we consider the truncation parameter $K=2^p$ for $p=6,7,8,9,10$. All results in the following tables are rounded at the sixth decimal place. In Table 1, it can be easily seen that the absolute errors decrease with p (or equivalently K) for $\tilde{V}(u;b)$, $\tilde{\phi}_d(u;b)$ and $\tilde{\phi}_c(u;b)$, which means that the large truncation parameter can reduce the error in our examples.

Next, we consider the effectiveness of our statistical estimation method, based on a random sample on the individual claim sizes $\{X_1,\dots,X_{n_T}\}$ and claim number n_T. For the parameter a, we consider the following empirical version of the cumulant-based method,

$$a=\hat{\kappa}_1+L\sqrt{\hat{\kappa}_2+\sqrt{\hat{\kappa}_4}}, \tag{6.3}$$

where

$$\hat{\kappa}_j=\frac{d^j}{ds^j}\log\widehat{\mathcal{F}f}(-is)\bigg|_{s=0},\qquad j=1,2,4. \tag{6.4}$$

When computing $\hat{h}_+$ and $\hat{g}_+$, we take $\widehat{\mathcal{F}f}=\widehat{\mathcal{F}h_+}$; when computing $\hat{\phi}_d$ and $\hat{\phi}_d'$, we take $\widehat{\mathcal{F}f}=\widehat{\mathcal{F}\phi_d}$; when computing $\hat{\phi}_c$ and $\hat{\phi}_c'$, we take $\widehat{\mathcal{F}f}=\widehat{\mathcal{F}\phi_c}$. Again, we set $L=10$.

Set the truncation parameter $K=2^{10}$ and the inter-observation interval $\Delta=1$. We perform 200 experiments and consider average absolute errors for $\hat{V}(u;b)$, $\hat{\phi}_d(u;b)$ and $\hat{\phi}_c(u;b)$, respectively, which are defined by

$$\frac{1}{\#\mathcal{U}}\sum_{u\in\mathcal{U}}\frac{1}{200}\sum_{j=1}^{200}|\hat{V}_j(u;b)-V^{\mathrm{ref}}(u;b)|,\qquad \frac{1}{\#\mathcal{U}'}\sum_{u\in\mathcal{U}'}\frac{1}{200}\sum_{j=1}^{200}|\hat{\phi}_{d,j}(u;b)-\phi_d^{\mathrm{ref}}(u;b)|$$

and

$$\frac{1}{\#\mathcal{U}''}\sum_{u\in\mathcal{U}''}\frac{1}{200}\sum_{j=1}^{200}|\hat{\phi}_{c,j}(u;b)-\phi_c^{\mathrm{ref}}(u;b)|,$$

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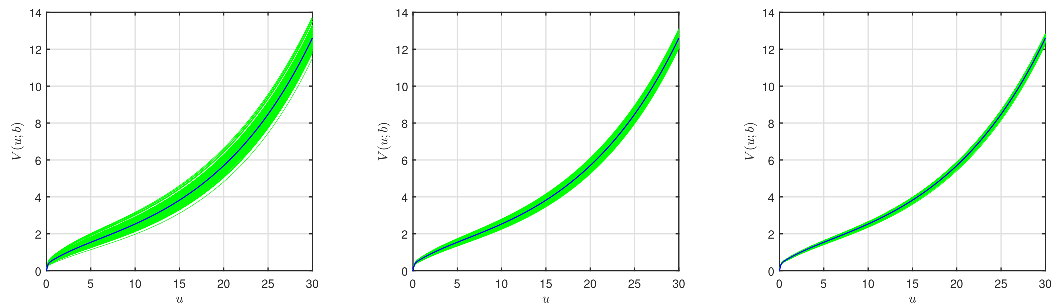
Table 2. *Empirical average absolute errors by the COS method.*

q	$\hat{V}(u;b)$			$\hat{\phi}_d(u;b)$			$\hat{\phi}_c(u;b)$		
	Exp	Erlang	Gamma	Exp	Erlang	Gamma	Exp	Erlang	Gamma
0	0.328940	0.284153	0.289102	0.003099	0.003204	0.003051	0.009850	0.006604	0.007109
1	0.204445	0.182295	0.192941	0.002494	0.002266	0.002429	0.006314	0.004809	0.005842
2	0.160831	0.126271	0.143326	0.001987	0.001715	0.001886	0.004936	0.003465	0.003803
3	0.112534	0.103669	0.093200	0.001708	0.001325	0.001405	0.003567	0.002374	0.002726
4	0.073890	0.069054	0.071132	0.001608	0.001092	0.001359	0.002663	0.001833	0.002027
5	0.053173	0.052682	0.046592	0.001482	0.000958	0.001254	0.002003	0.001473	0.001643



(a) (b) (c)

Figure 1. *Beams for estimating $V(u;b)$: 200 estimators in green, and the true in bold blue. (a) $q = 1$; (b) $q = 3$ and (c) $q = 5$.*

where $\hat{V}_j(u;b)$, $\hat{\phi}_{d,j}(u;b)$ and $\hat{\phi}_{c,j}(u;b)$ denote the jth experiment values of $\hat{V}(u;b)$, $\hat{\phi}_d(u;b)$ and $\hat{\phi}_c(u;b)$, respectively.

In Table 2, we list the empirical average absolute errors of the estimator for the three functions. For the observation interval $[0,T]$, we take $T = 1000 \times 2^q$ for $q = 0, 1, 2, 3, 4, 5$. It is obvious that the estimation errors are also decreasing with q (or equivalently the time T), that is, the approximation results become closer to reference values, which means that our estimator performs better when the sample sizes become large.

In order to show the stability of our method, we plot 200 estimated curves (green curves) and the true curves (bold blue curves) on the same picture. For the expected present value of total dividends before ruin with the exponential claim size density function, we plot the curves of $\hat{V}_j(u;b)$ as functions of u when $T = 1000 \times 2^1$, 1000×2^3 and 1000×2^5in Figure 1. It is obvious that all the lines increase with u for all the three models, which is consistent with the actual situation that the expected present value of total dividends before ruin increases with the initial surplus. Besides, as T increases, the estimator $\hat{V}(u;b)$ tends to be stable and converges to $V(u;b)$.

The functions $\phi_d(u;b)$, $\phi_c(u;b)$ and $\phi(u;b)$ are also investigated for Erlang(2,2) claim size density function. In Figure 2, we find that $\phi_d(u;b)$ is a decreasing function of the initial surplus u, which means that ruin is more likely to happen caused by oscillation when u is small. At the same time, we can also observe that as T increases, the estimator $\hat{\phi}_d(u;b)$ tends to be stable and converges to $\phi_d(u;b)$. As for $\hat{\phi}_c(u;b)$ and $\hat{\phi}(u;b)$, we plot the corresponding curves in Figures 3 and 4. Again, we can observe that the estimation becomes better as T becomes larger.

Finally, we investigate the effectiveness of our method when further using spectral filters to accelerate the decay rate of the COS coefficients. Recall that a real and symmetric function $\Gamma(s)$ is a filter of order υ if: (i) $\Gamma(0) = 1, \Gamma^{(l)}(0) = 0, 1 \le l \le \upsilon - 1$; (ii) $\Gamma(s) = 0$ for $|s| \ge 1$; (iii) $\Gamma(s) \in C^{\upsilon-1}$, $s \in \mathbb{R}$, where

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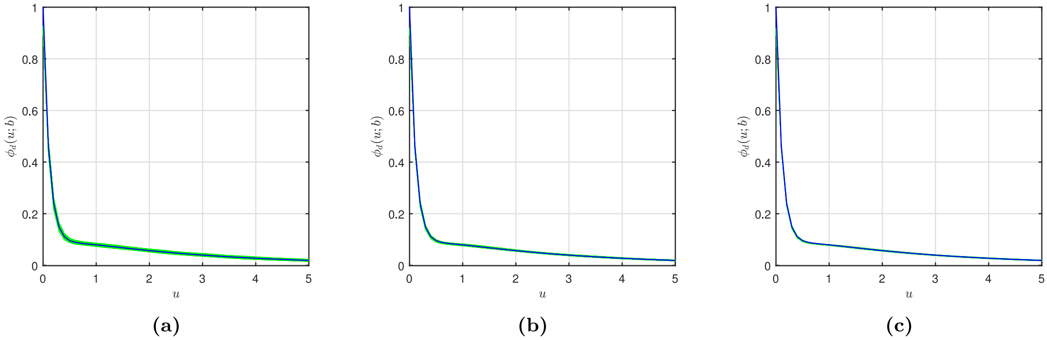

Figure 2. *Beams for estimating $\phi_d(u;b)$: 200 estimators in green, and the true in bold blue. (a) $q = 1$; (b) $q = 3$ and (c) $q = 5$.*

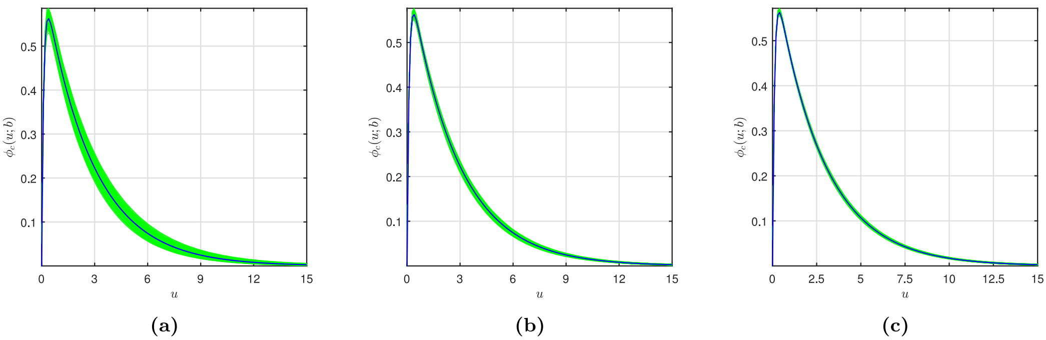

Figure 3. *Beams for estimating $\phi_c(u;b)$: 200 estimators in green, and the true in bold blue. (a) $q = 1$; (b) $q = 3$ and (c) $q = 5$.*

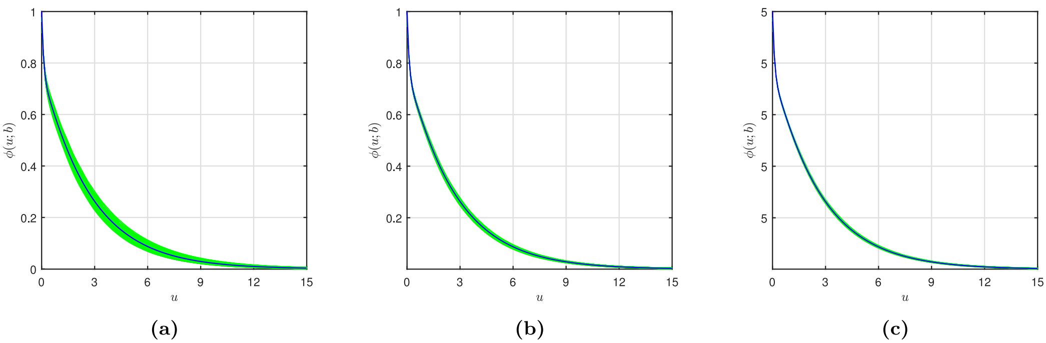

Figure 4. *Beams for estimating $\phi(u;b)$: 200 estimators in green, and the true in bold blue. (a) $q = 1$; (b) $q = 3$ and (c) $q = 5$.*

in particular $\Gamma^{(l)}(\pm 1) = 0$ for $0 \leq l \leq \upsilon - 1$. For a function f, its filter-COS approximation is simply defined by

$$\tilde{f}^{\text{filter}}(x) = {\sum_{k=0}^{K}}' \Gamma(k/K) B_{f,k} \cos\left(k\pi \frac{x - a_1}{a_2 - a_1}\right), \quad a_1 \leq x \leq a_2.$$

In the following experiments, we select an exponential filter $\Gamma(s) = \exp(-\nu s^{\upsilon})$ with $\upsilon = 6$ and $\nu = -\log \epsilon$, where ϵ represents the machine epsilon so that $\Gamma(1) = \epsilon \approx 0$. We take $\epsilon = 10^{-6}$. In Table 3, we list the absolute errors of the three functions by using the filter-COS method. Similarly, we can find that the calculation errors decrease with the truncation parameter K. Compared with Table 1, we

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Table 3. *Average absolute errors by the filter-COS method.*

p	$\tilde{V}(u;b)$			$\tilde{\phi}_d(u;b)$			$\tilde{\phi}_c(u;b)$		
	Exp	Erlang	Gamma	Exp	Erlang	Gamma	Exp	Erlang	Gamma
6	0.002814	0.001834	0.015319	0.025578	0.021668	0.022997	0.004712	0.004442	0.004535
7	0.001464	0.000923	0.004861	0.017049	0.013511	0.014764	0.003721	0.003069	0.003291
8	0.000553	0.000359	0.002026	0.007109	0.005520	0.006106	0.001669	0.001319	0.001426
9	0.000222	0.000123	0.001053	0.002798	0.001936	0.002094	0.000688	0.000463	0.000498
10	0.000033	0.000010	0.000411	0.000336	0.000190	0.000110	0.000099	0.000042	0.000028

Table 4. *Empirical average absolute errors by the filter-COS method.*

q	$\hat{V}(u;b)$			$\hat{\phi}_d(u;b)$			$\hat{\phi}_c(u;b)$		
	Exp	Erlang	Gamma	Exp	Erlang	Gamma	Exp	Erlang	Gamma
0	0.301895	0.256849	0.282391	0.002804	0.003166	0.002899	0.009251	0.006041	0.007095
1	0.211058	0.202374	0.207984	0.002044	0.002014	0.002097	0.006299	0.004475	0.005501
2	0.145835	0.140759	0.126585	0.001582	0.001493	0.001564	0.004750	0.003421	0.003824
3	0.104888	0.100684	0.097467	0.001137	0.001148	0.001020	0.003257	0.002559	0.002689
4	0.079401	0.067289	0.066253	0.000987	0.000859	0.000720	0.002331	0.001667	0.001873
5	0.051070	0.047694	0.052073	0.000754	0.000646	0.000550	0.001792	0.001222	0.001261

can find that the errors calculated by the filter-COS method are much smaller than the errors obtained without filter. In particular, we observed that for the $\tilde{V}(u;b)$ and $\tilde{\phi}_c(u;b)$ functions, the errors are significantly improved with the addition of filtering. We also list the empirical errors of the estimator for the three functions by the filter-COS method in Table 4, and almost all errors obtained by using the filter-COS method are also reduced compared with Table 2. In summary, spectral filters can accelerate the convergence rate.

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