ON POLYNOMIALS WITH REAL ZEROS

Manzoor Ahmad

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Let

$$P(x) = c \prod_{j=1}^{n} (x-x_j)$$

be a polynomial of degree n with real and non-negative zeros $x_1 \leq x_2 \leq \ldots \leq x_n$. The zeros x_j will be said to have <u>extent</u> 1 if

$$\max_{1 \leq j \leq n} x_j = x_n = 1 .$$

Let $\xi_1 \leq \xi_2 \leq \ldots \leq \xi_{n-1}$ be the zeros of the derived polynomial $p'(x)$. The zeros $\xi_1, \xi_2, \ldots, \xi_{n-1}$ are real and non-negative, and moreover their extent can be at most equal to the extent of the zeros $x_1, x_2, \ldots, x_n$. The two can indeed be equal. For if the extent of the zeros x_j is 1 and 1 is a multiple zero of $p'(x)$ then $\xi_{n-1} = 1$. However it is not quite clear how small ξ_{n-1} can be if $x_n = 1$. The extent ξ_{n-1} of the zeros of $p'(x)$ is less than 1 only if 1 is not a multiple zero of $p(x)$. So let us suppose that $p(x)$ has a simple zero at $x = 1$. Consequently x_{n-1} is the largest zero of $p(x)/(x-1)$ or equivalently the largest zero of $p(x)$ in $0 < x < 1$ and it follows by Rolle's theorem that $p'(x)$ has a zero in the interval $(x_{n-1}, 1)$. Thus the extent ξ_{n-1} of the zeros of $p'(x)$ is greater than x_{n-1} and it remains to see how small it can be.

Since ξ_{n-1} satisfies $x_{n-1} < \xi_{n-1} < x_n = 1$ and

$$0 = \frac{p'(\xi_{n-1})}{p(\xi_{n-1})} = \sum_{j=1}^{n} (\xi_{n-1} - x_j)^{-1}$$

we see easily that if $x_{n-1} > 0$ by decreasing x_{n-1} we might decrease ξ_{n-1}. Thus the minimum extent is attained if

$$x_1 = \dots = x_{n-1} = 0, \quad x_n = 1.$$

THEOREM 1. Let $p(x)$ be a polynomial of degree n with real non-negative zeros. If the extent of the zeros of $p(x)$ is 1 then the extent of the zeros of $p'(x)$ is at least $(n-1)/n$. The result is sharp.

The problem which we have just considered is closely connected with a question raised by A. Meir and A. Sharma [1] (see also [2], [3]) which is as follows.

Let $0 \leq x_1 \leq \dots \leq x_n \leq 1$ be the zeros of a polynomial $p(x)$ of degree n, let $x_n - x_1$ be the span of $p(x)$. How small can the span of $p'(x)$ be if $x_1 = 0$ and $x_n = 1$? From the above calculations it is clear that there is a zero of $p'(x)$ in $[\frac{(n-1)}{n}, 1]$ and another one in $[0, \frac{1}{n}]$. Thus the span of the zeros of $p'(x)$ cannot be smaller than $\frac{n-2}{n}$. This bound can be easily improved. The example

$$p(x) = x(x-\frac{1}{2})^{n-2}$$

shows that the span of the zeros of $p'(x)$ can be as small as $\sqrt{\frac{n-2}{n}}$ which is approximately $1-\frac{1}{n}$ if n is large. It has been conjectured by Meir and Sharma that the polynomial $x(x-\frac{1}{2})^{n-2}(x-1)$ is indeed extremal, i.e. if the span of the zeros of $p(x)$ is 1 then the span of the zeros of $p'(x)$ cannot be smaller than $\sqrt{1-\frac{2}{n}}$.

From our earlier argument it is clear that the extremal polynomial for this problem cannot have a multiple zero at any of the points 0 or 1. So let $p(x) = x(x-1)\,q(x)$ where $q(0) \neq 0$, $q(1) \neq 0$. Now let ξ_{n-1} be the largest zero of $p'(x)$ and ξ_1 the smallest. We suppose that n_1 and n_2 denote respectively the number of zeros of $q(x)$ in $(\frac{1}{2}, 1)$ and in $(0, \frac{1}{2}]$. Since $n_1 + n_2 = n-2$ one of the two numbers n_1, n_2 is $\leq (n-2)/2$. We may suppose that $n_1 \leq (n-2)/2$. For otherwise we can consider $p(1-x)$ instead of $p(x)$.

Note that ξ_{n-1} cannot be smaller than the larger of the two roots of the equation

$$\frac{n_2+1}{x} + \frac{2n_1}{2x-1} = \frac{1}{1-x},$$

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i.e.

$$\xi_{n-1} \geq \frac{3n - n_1 - 2 + \sqrt{(n+n_1+2)^2 - 8n}}{4n} .$$

On the other hand ξ_1 is at most equal to the smaller of the two roots of the equation

$$\frac{1}{x} = \frac{2n_2}{1-2x} + \frac{n_1+1}{1-x}$$

i.e.

$$\xi_1 \leq \frac{2n - n_1 - \sqrt{4n^2 - 4nn_1 + n_1^2 - 8n}}{4n} .$$

Hence

(1) $$\xi_{n-1} - \xi_1 \geq \frac{n - 2 + \sqrt{4n^2 - 4nn_1 + n_1^2 - 8n} + \sqrt{(n+n_1 + 2)^2 - 8n}}{4n}$$

The quantity on right hand side of (1) increases as n_1 increases from 0 to $\frac{(n-2)}{2}$ and then starts decreasing. Thus

$$\xi_{n-1} - \xi_1 > \frac{1}{2}\{1 - \frac{2}{n} + \sqrt{1-\frac{2}{n}}\} .$$

We have therefore proved the following

THEOREM 2. *If the span of a polynomial* $p(x)$ *of degree* n *is* 1 *then the span of* $p'(x)$ *cannot be smaller than*

$$\frac{1}{2}\{1 - \frac{2}{n} + \sqrt{1 - \frac{2}{n}}\} .$$

The lower bound given by Theorem 2 can be further improved.

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University of Montreal

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