ON COMMUTATIVE SQUARES

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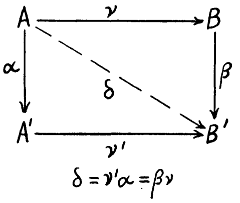
The following elementary facts about certain commutative diagrams, called "squares," are stated and proved in terms of abelian groups and their homomorphisms. However, they are valid for arbitrary abelian categories and can be proved also for them. This does not need to be shown, since every abelian category can be embedded into the category of abelian groups with preservation of exact sequences according to a result due to S. Lubkin **(1)**. Proofs are often omitted or given only for one half of a theorem, the other half being dual to the first. Generalizations to a larger class of diagrams containing all finite commutative diagrams are possible.

DEFINITION 1. *The commutative square*

(1)

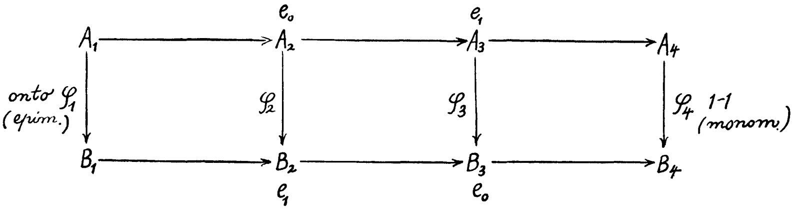


is called "smooth," if

$$\operatorname{Ker}\beta = \nu \operatorname{Ker}\alpha \qquad \textit{and} \qquad \operatorname{Im}\alpha = \nu'^{-1}\operatorname{Im}\beta.$$

PROPOSITION 1 ("Four-lemma"). *In the commutative diagram*

(2)



the middle square is smooth, that is,

$$\operatorname{Ker}\phi_3 = \nu \operatorname{Ker}\phi_2 \qquad \textit{and} \qquad \operatorname{Im}\phi_2 = \mu^{-1}\operatorname{Im}\phi_3.$$

Here "e_0" {"e_1"} *means the part* "Im ⊂ Ker" {"Im ⊃ Ker"} *of "exact."*

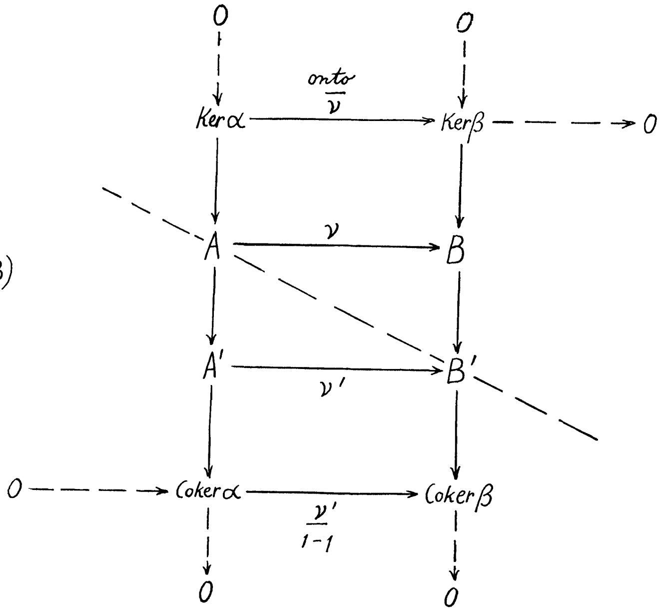
Remark. For the first (second) relation the condition "e_0" on the bottom (top) is superfluous. "e_0" at A_2 {B_3} implies "e_0," hence full exactness at B_2 {A_3}.

Received August 7, 1961. The author is indebted to Professor Saunders MacLane and Professor Helmut Röhrl for advice and criticism.

"Smoothness" of the square (1) is equivalent to embeddability in a special diagram (3) of type (2), that is, with $\bar{\nu}$ onto and $\underline{\nu}'$ 1–1 and exactness throughout. Application of the four-lemma gives

$$\text{Ker}\ \nu' = \alpha\ \text{Ker}\ \nu \qquad \text{and} \qquad \text{Im}\ \nu = \beta^{-1}\ \text{Im}\ \nu',$$

(3)



so the "smoothness" property of (1) is *symmetric* to the dotted diagonal. *The four-lemma is nothing but the statement of this symmetry*, since its hypotheses imply immediately

$$\text{Ker}\ \mu = \phi_2\ \text{Ker}\ \nu \qquad \text{and} \qquad \text{Im}\ \nu = \phi_3{}^{-1}\ \text{Im}\ \mu.$$

Hence for every smooth square (1),

$$\text{Ker}\ \beta = \nu\ \text{Ker}\ \alpha, \qquad \text{Ker}\ \nu' = \alpha\ \text{Ker}\ \nu$$

and

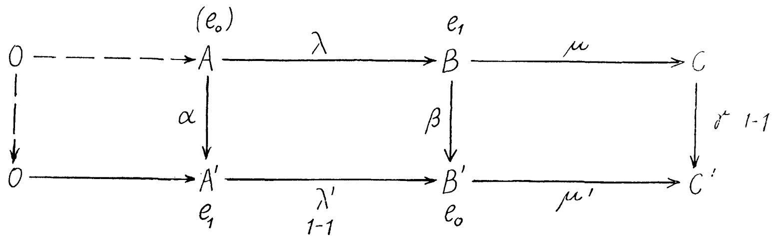
$$\text{Im}\ \alpha = \nu'^{-1}\ \text{Im}\ \beta, \qquad \text{Im}\ \nu = \beta^{-1}\ \text{Im}\ \nu';$$

moreover

$$\text{Ker}\ \delta = \text{Ker}\ \alpha \cup \text{Ker}\ \nu \qquad \text{and} \qquad \text{Im}\ \delta = \text{Im}\ \nu' \cap \text{Im}\ \beta.$$

COROLLARY a. *Let* (4) *be a commutative diagram:*

(4)



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Then

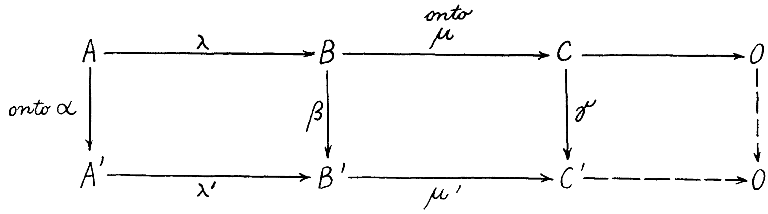
$$\operatorname{Ker} \beta = \lambda \operatorname{Ker} \alpha \qquad \textit{and} \qquad \operatorname{Im} \alpha = \lambda'^{-1} \operatorname{Im} \beta;$$

therefore

α 1–1 (*and* γ 1–1) *implies* β 1–1, $\quad$ (γ 1–1 *and*) β *onto implies* α *onto.*

COROLLARY a′ (dual to *a*). *Let* (4′) *be a commutative diagram*:

(4′)



Then

$$\operatorname{Ker} \gamma = \mu \operatorname{Ker} \beta, \qquad \operatorname{Im} \beta = \mu'^{-1} \operatorname{Im} \gamma;$$

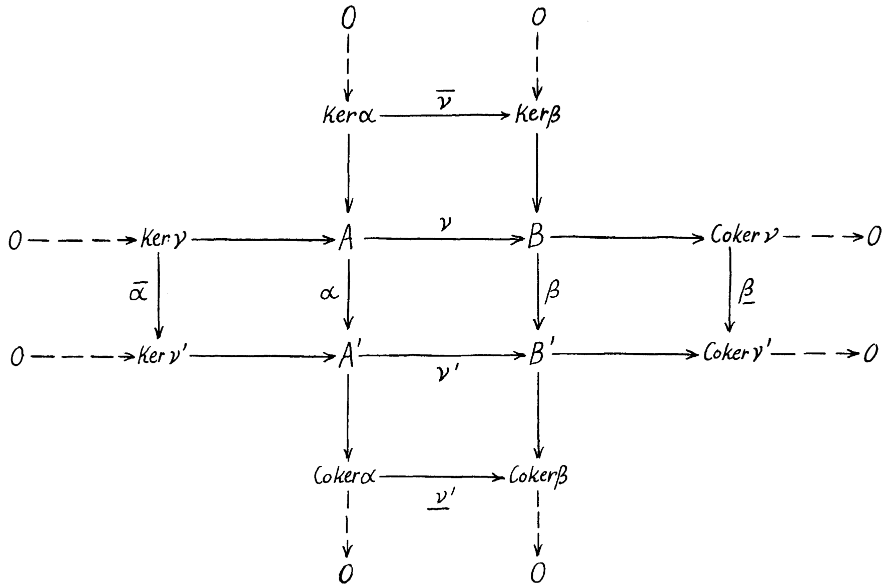
and therefore

(α *onto and*) β 1–1 *implies* γ 1–1, $\quad$ (α *onto and*) γ *onto implies* β *onto.*

Remark. Omitting the dotted arrows and the trivial assumptions "(e_0)," this can be regarded as a "three-lemma," which contains the useful "five-lemma" and saves sometimes additional diagram-chasing (see for example **2**, p. 5).

DEFINITION 2. *The commutative square* (1) *is "small" at* A $\{B'\}$, *if in the enlarged diagram* (5) *the induced map* $\bar{\nu}$ $\{\underline{\nu}'\}$ *is* 1–1 $\{$*onto*$\}$. *If it has both properties,*

(5)



then it is "small." It is called "tight" (at A $\{B'\}$), if it is both smooth and small (at A $\{B'\}$).

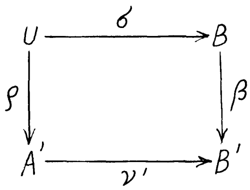
Now $\bar{\nu}$ 1–1 $\{\underline{\nu}'$ onto$\}$ is respectively equivalent to $\bar{\alpha}$ 1–1 $\{\underline{\beta}$ onto$\}$, since it means simply $\text{Ker}\,\alpha \cap \text{Ker}\,\nu = 0$ $\{\text{Im}\,\nu' \cup \text{Im}\,\beta = B'\}$; hence "smallness" is a *symmetric* property too.

DEFINITION 3. *The "upper completion" of the diagram*

$$A' \xrightarrow{\nu'} B' \xleftarrow{\beta} B$$

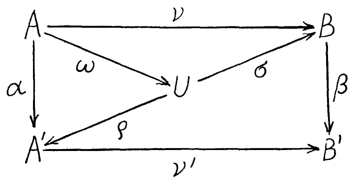
is the diagram (6) *with the subgroup $U \subset A' \oplus B$ and maps ρ, σ such that $(a', b) \in U$ if and only if $\nu'(a') = \beta(b)$ and $\rho(a', b) = a'$, $\sigma(a', b) = b$.*

(6)



It has the following universality properties: (a) it is commutative, that is, $\beta\sigma = \nu'\rho$; (b) given any two maps $\nu : A \to B$ and $\alpha : A \to A'$ such that $\beta\nu = \nu'\alpha$, there is a unique $\omega : A \to U$ defined by $\omega(a) = (\alpha(a), \nu(a)) \in U$ such that $\nu = \sigma\omega$ and $\alpha = \rho\omega$ (see (7)).

(7)



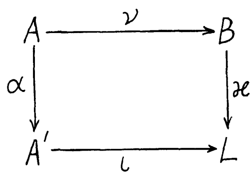
Therefore $\beta\sigma = \nu'\rho = [\nu', \beta]_r$ can be regarded as the *least common right-multiple* of ν' and β.

DEFINITION 3′. *The "lower completion" of the diagram*

$$A' \xleftarrow{\alpha} A \xrightarrow{\nu} B$$

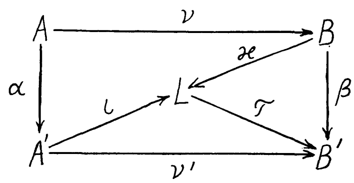
is the diagram (6′) *with the factor group $L = A' \oplus B$ $(\alpha(a), -\nu(a))$ for all $a \in A$ and maps ι, κ such that $\iota(a') = (a', 0)$, $\kappa(b) = (0, b)$.*

(6′)



The lower completion has the following universality properties: (a) it commutes, that is $\iota\alpha = \kappa\nu$; (b) given any two maps $\nu' : A' \to B'$ and $\beta : B \to B'$ with $\beta\nu = \nu'\alpha$ there is a unique $\tau : L \to B'$ defined by $\tau(a', b) = \nu'(a') + \beta(b)$ such that $\nu' = \tau\iota$ and $\beta = \tau\kappa$ (see ((7'))).

(7')



Hence $\kappa\nu = \iota\alpha = [\alpha, \nu]_l$ can be regarded as the *least common left-multiple* of α and ν.

Both completions are *unique up to isomorphism.*

PROPOSITION 2. *The upper* {*lower*} *completion* (6) {(6')} *is tight at* U {L}.

Proof. (a) (6) is smooth, that is, $\operatorname{Ker}\beta = \sigma \operatorname{Ker}\rho$ and $\operatorname{Im}\rho = \nu'^{-1}\operatorname{Im}\beta$. If $\beta(b) = 0$, then $U \ni (0, b) \in \operatorname{Ker}\rho$ and $b = \sigma(0, b)$, so $\operatorname{Ker}\beta \subset \sigma\operatorname{Ker}\rho$. If $\nu'(a') = \beta(b)$, then $(a', b) \in U$ with $a' = \rho(a', b)$, so $\operatorname{Im}\rho \supset \nu'^{-1}\operatorname{Im}\beta$.

(b) (6) is small at U: If $(a', b) \in \operatorname{Ker}\rho \cap \operatorname{Ker}\sigma$, then $\rho(a', b) = a' = 0$ and $\sigma(a', b) = b = 0$, so $(a', b) = 0$.

COROLLARY a. *If* ν' {β} *onto, then* σ {ρ} *onto.*

Proof. $\operatorname{Im}\sigma = \beta^{-1}\operatorname{Im}\nu'$ {$\operatorname{Im}\rho = \nu'^{-1}\operatorname{Im}\beta$}.

Thus the l.c.m. $[\nu', \beta]_r$ is epimorphic if and only if both ν' and β are. (This follows also from $\operatorname{Im}[\nu', \beta]_r = \operatorname{Im}\beta \cap \operatorname{Im}\nu'$.)

COROLLARY b. *If* ν' {β} 1–1, *then and only then* σ {ρ} 1–1.

Proof. $\operatorname{Ker}\sigma \subset \rho^{-1}\operatorname{Ker}\nu' \cap \operatorname{Ker}\sigma$ {$\operatorname{Ker}\rho \subset \sigma^{-1}\operatorname{Ker}\beta \cap \operatorname{Ker}\rho$} $= \operatorname{Ker}\rho \cap \operatorname{Ker}\sigma = 0$ and $\operatorname{Ker}\nu' = \rho\operatorname{Ker}\sigma$ {$\operatorname{Ker}\beta = \sigma\operatorname{Ker}\rho$} (or $\operatorname{Ker}\sigma = \operatorname{Ker}\nu' \oplus 0$ {$\operatorname{Ker}\rho = 0 \oplus \operatorname{Ker}\beta$} in obvious notation).

Thus $[\nu', \beta]_r$ is monomorphic if and only if both ν' and β are. (This follows also from $\operatorname{Ker}[\nu', \beta]_r = \operatorname{Ker}\rho \cup \operatorname{Ker}\sigma$ or $= \operatorname{Ker}\nu' \oplus \operatorname{Ker}\beta$.)

COROLLARY c. *If* $\nu' = \beta$, *then* ρ *and* σ *onto.*

Proof. $\operatorname{Im}\rho = \nu'^{-1}\operatorname{Im}\beta = \operatorname{Im}\sigma = \beta^{-1}\operatorname{Im}\nu' = \beta'^{1}\operatorname{Im}\beta = B$.

Consequently $\rho = \sigma$ is isomorphic and $[\beta, \beta]_r = \beta$ if and *only if* $\nu' = \beta$ 1–1. (If $\rho = \sigma$, then $\operatorname{Ker}\rho = \operatorname{Ker}\sigma = \operatorname{Ker}\rho \cap \operatorname{Ker}\sigma = 0$, so $\rho = \sigma$ 1–1 and ν', β are both 1–1; if $\rho = \sigma$ onto, then $\nu' = \beta$ because $\nu'\rho = \beta\sigma$. If $[\beta, \beta]_r = \beta$, that is, σ 1–1, then $\nu' = \beta$ 1–1.)

COROLLARY a'. *If* α {ν} 1–1, *then* κ {ι} 1–1.

Proof. $\operatorname{Ker}\kappa = \nu\operatorname{Ker}\alpha$ {$\operatorname{Ker}\iota = \alpha\operatorname{Ker}\nu$}.

Thus the l.c.m. $[\alpha, \nu]_l$ is monomorphic if and only if both α and ν are. (This follows also from $\mathrm{Ker}[\alpha, \nu]_l = \mathrm{Ker}\,\alpha \cup \mathrm{Ker}\,\nu$.)

COROLLARY b′. *If and only if* α $\{\nu\}$ *onto, then* κ $\{\iota\}$ *onto.*

Proof. $\mathrm{Im}\,\alpha = \iota^{-1}\,\mathrm{Im}\,\kappa$ $\{\mathrm{Im}\,\nu = \kappa^{-1}\,\mathrm{Im}\,\iota\}$ and $\mathrm{Im}\,\kappa \supset \iota\,\mathrm{Im}\,\alpha \cup \mathrm{Im}\,\kappa$ $\{\mathrm{Im}\,\iota \supset \kappa\,\mathrm{Im}\,\nu \cup \mathrm{Im}\,\iota\} = \mathrm{Im}\,\iota \cup \mathrm{Im}\,\kappa = L$ (or $\mathrm{Im}\,\iota = A' \oplus \mathrm{Im}\,\nu$ $\{\mathrm{Im}\,\kappa = \mathrm{Im}\,\alpha \oplus B\}$).

Thus $[\alpha, \nu]_l$ is epimorphic if and only if both α and ν are. (This follows also from $\mathrm{Im}[\alpha, \nu] = \mathrm{Im}\,\iota \cap \mathrm{Im}\,\kappa$ or $= \mathrm{Im}\,\alpha \oplus \mathrm{Im}\,\nu$.)

COROLLARY c′. *If* $\alpha = \nu$, *then* ι *and* κ 1–1.

Proof. $\mathrm{Ker}\,\iota = \alpha\,\mathrm{Ker}\,\nu = \mathrm{Ker}\,\kappa = \nu\,\mathrm{Ker}\,\alpha = \alpha\,\mathrm{Ker}\,\alpha = 0$.

Hence $\iota = \kappa$ is isomorphic and $[\alpha, \alpha]_l = \alpha$ if and *only if* $\alpha = \nu$ onto. (If $\iota = \kappa$, then $\mathrm{Im}\,\iota = \mathrm{Im}\,\kappa = \mathrm{Im}\,\iota \cup \mathrm{Im}\,\kappa = L$, so $\iota = \kappa$ onto and α, ν are both onto; if $\iota = \kappa$ 1–1, then $\alpha = \nu$ because $\iota\alpha = \kappa\nu$. If $[\alpha, \alpha]_l = \alpha$, that is, ι onto, then $\nu = \alpha$ onto.)

Remark. For completions see **(3)**. In **(4)** least common multiples are defined only for monomorphic or epimorphic pairs.

PROPOSITION 3. *The square* (1) *is smooth if and only if the map* $\omega: A \to U$ $\{\tau: L \to B'\}$ *in* (7) $\{(7')\}$ *is onto* $\{$1–1$\}$. *So* ω *onto if and only if* τ 1–1.

Proof. Suppose ω onto. If $\beta(b) = 0$, then $U \supset (0, b) = \omega(a) = (\alpha(a), \nu(a))$; hence $b = \nu(a)$ with $\alpha(a) = 0$: $\mathrm{Ker}\,\beta \subset \nu\,\mathrm{Ker}\,\alpha$. If $\nu'(a') = \beta(b)$, then $U \supset (a', b) = \omega(a) = (\alpha(a), \nu(a))$; hence $a' = \alpha(a)$: $\nu'^{-1}\,\mathrm{Im}\,\beta \subset \mathrm{Im}\,\alpha$. Suppose (1) is smooth. Given $(a', b) \in$ U, $\nu'(a') = \beta(b)$ or $a' = \nu'^{-1}\,\mathrm{Im}\,\beta = \mathrm{Im}\,\alpha$, thus $a' = \alpha(a)$ and $\beta(b) = \nu'\alpha(a) = \beta\nu(a)$ or $\beta(b - \nu(a)) = 0$, that is, $b - \nu(a) \in \mathrm{Ker}\,\beta = \nu\,\mathrm{Ker}\,\alpha$, so $b - \nu(a) = \nu(\bar{a})$ with $\alpha(\bar{a}) = 0$; hence $(a', b) = (\alpha(a + \bar{a}), \nu(a + \bar{a})) = \omega(a + \bar{a})$: ω onto.

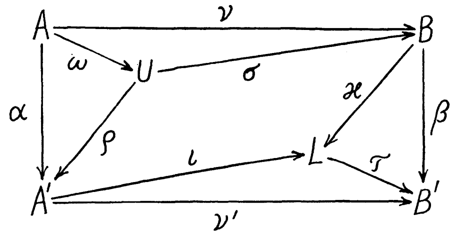
PROPOSITION 4. *The square* (1) *is small at* A $\{B'\}$ *if and only if the map* $\omega: A \to U$ $\{\tau: L \to B'\}$ *in* (7) $\{(7')\}$ *is* 1–1 $\{$*onto*$\}$.

Proof. For $a \in A$, $\omega(a) = (\alpha(a), \nu(a)) = (0, 0) = 0$ is equivalent to $a \in \mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\nu$.

Proofs can be given also by means of Proposition 2, which is restated in:

PROPOSITION 5. *The square* (1) *is its own upper* $\{$*lower*$\}$ *completion, that is, the map* ω $\{\tau\}$ *in* (7) $\{(7')\}$ *is isomorphic if and only if it is tight at* A $\{B'\}$.

PROPOSITION 6. (1) *is smooth if and only if the inner square in the diagram* (8) *built by the maps* ρ, σ, ω *and* ι, κ, τ *from the upper and lower completions* (7), (7′) *is commutative.*

(8) 

Proof. "If" follows from ω onto or τ 1–1. Assume commutativity. If $\nu'(a') = \beta(b)$, then $(a', b) \in U$ and $\iota\rho(a', b) = (a', 0) = \kappa\sigma(a', b) = (0, b)$ in L, that is, $(a', -b) = (\alpha(a), -\nu(a))$ for some $a \in A$, so $a' = \alpha(a)$ and $\nu'^{-1} \operatorname{Im} \beta \subset \operatorname{Im} \alpha$. If $\beta(b) = 0$, then $(0, b) \in U$ and similarly $(0, -b) = (\alpha(a), -\nu(a))$ with $a \in A$, hence $b = \nu(a)$ with $\alpha(a) = 0$ or $\operatorname{Ker} \beta \subset \nu \operatorname{Ker} \alpha$.

DEFINITION 4. *In any commutative diagram* (8) *the inner square* $\mathfrak{J}$ *is called a "squeeze" of the outer square* $\mathfrak{A}$, *the map* ω $\{\tau\}$ *the upper* $\{$*lower*$\}$ *"squeezing map."*

PROPOSITION 7. *If* $\mathfrak{A}$ *is smooth, then every squeeze* $\mathfrak{J}$ *of* $\mathfrak{A}$ *is smooth* (*and conversely*).

PROPOSITION 8. *If* $\mathfrak{J}$ *is smooth,* ω *onto and* τ 1–1, *then* $\mathfrak{A}$ *is smooth.*

PROPOSITION 9. *If* $\mathfrak{A}$ *is smooth and* $\mathfrak{J}$ *small* (*hence tight*) *at* U $\{L\}$, *then* ω *onto* $\{\tau$ 1–1$\}$.

PROPOSITION 10. *If* $\mathfrak{A}$ *is small at* A $\{B'\}$, *then* ω 1–1 $\{\tau$ *onto*$\}$ (*and conversely*).

COROLLARY. *If* $\mathfrak{A}$ *is small at* A $\{B'\}$ *and* ω *onto* $\{\tau$ 1–1$\}$ (*hence isomorphic*), *then* $\mathfrak{J}$ *is small at* U $\{L\}$.

PROPOSITION 11. *If* $\mathfrak{J}$ *is small at* U $\{L\}$, *and* ω 1–1 $\{\tau$ *onto*$\}$, *then* $\mathfrak{A}$ *is small at* A $\{B'\}$.

Remark. Propositions (7)–(9) $\{$(10), (11)$\}$ follow from Proposition (3) $\{$(4)$\}$ and conversely. Direct proofs are as easily available.

THEOREM 1. *For any smooth square* (1) *the square built by the maps* ρ, σ *and* ι, κ *from* (7) *and* (7′) *is its own completion, that is, it is tight and is connected commutatively with the original square by the epimorphism* ω *and the monomorphism* τ. *It is the only tight squeeze of* (1) *up to isomorphism, as well as of any other squeeze of* (1).

PROPOSITION 12 (see diagram (9)). *Given two squeezes* $\mathfrak{J}$ *and* $\bar{\mathfrak{J}}$ *of* $\mathfrak{A}$ *so that* $\bar{\mathfrak{J}}$ *is small at* $\bar{U}$ $\{\bar{L}\}$ *and* ω *onto* $\{\tau$ 1–1$\}$. *Then there is a unique* $\epsilon\colon U \to \bar{U}$ $\{\eta\colon \bar{L} \to L\}$ *such that* (9) *commutes; hence* $\bar{\mathfrak{J}}$ *is a squeeze of* $\mathfrak{J}$ *if both conditions hold.*

Proof. Given $u \in U$, $u = \omega(a)$, *define* $\epsilon(u) = \bar{\omega}(a)$. This is consistent since for $u = \omega(a_1)$, $\omega(a - a_1) = 0$. Hence $\nu(a - a_1) = \sigma\omega(a - a_1) = \bar{\sigma}\bar{\omega}(a - a_1) = 0$

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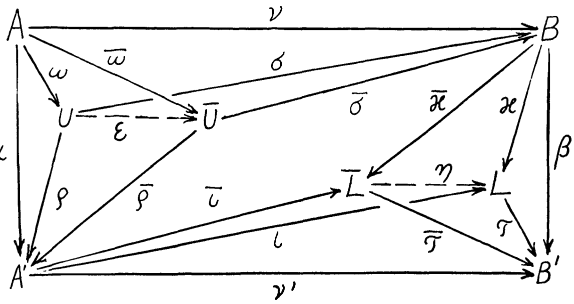
(9) 

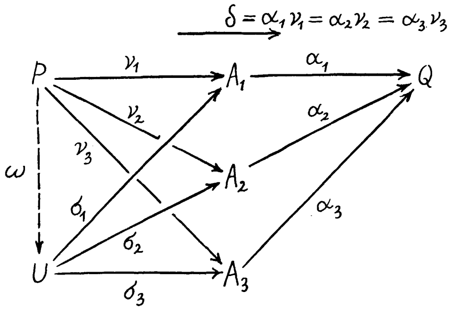
and $\alpha(a - a_1) = \rho\omega(a - a_1) = \bar{\rho}\bar{\omega}(a - a_1) = 0$. Thus $\bar{\omega}(a - a_1) \in \operatorname{Ker} \bar{\rho} \cap \operatorname{Ker} \bar{\sigma} = 0$, that is, $\bar{\omega}(a) = \bar{\omega}(a_1)$.

This again proves the uniqueness of the squeezing described in Theorem 1.

THEOREM 2. *Every commutative square* (1) *can be squeezed uniquely with epimorphic upper and monomorphic lower squeeze map to a small square obtained from the maps* ρ, σ *and* ι, κ *in* (7) *and* (7′) *by restricting* U *to* $\operatorname{Im} \omega$ *and* L *to* $L/\operatorname{Ker} \tau$. *If and only if* (1) *is smooth, this restriction is the original inner square of* (8) *and tight. If and only if* (1) *is small, this squeeze is* (1) *itself.*

Remark. In Theorem 1 the inner square is a squeeze of any squeeze of (1), whereas here this can be said only of those squeezes of (1) with epimorphic upper and monomorphic lower squeezing map. The commutativity of the inner square is ensured by the restriction process.

DEFINITION 5. *The* l.c.m. $[\alpha_1, \alpha_2, \alpha_3]_r$ *of three maps* $\alpha_i: A_i \to Q$ *with common range* Q *is their upper completion* (10) *with the subgroup* $U \subset A_1 \oplus A_2 \oplus A_3$ *and maps* σ_i *such that* $(a_1, a_2, a_3) \in U$ *if and only if* $\alpha_1(a_1) = \alpha_2(a_2) = \alpha_3(a_3)$ *and* $\sigma_i(a_1, a_2, a_3) = a_i$ *for* $i = 1, 2, 3$.

(10) 

DEFINITION 5′. *The* l.c.m. $[\nu_1, \nu_2, \nu_3]_l$ *of three maps* $\nu_i: P \to A_i$ *with common domain* P *is the lower completion* (10′) *with the factor group* $L = A_1 \oplus A_2 \oplus A_3$ $(\nu_i(p), 0, \nu_j(p))$ *for all* $p \in P$ *and maps* ι_i *such that* $\iota_i(a_i) = (0, a_i, 0)$ *for* $1 \leqslant i < j \leqslant 3$.

In both cases we have *universality*: To every commutative diagram of the

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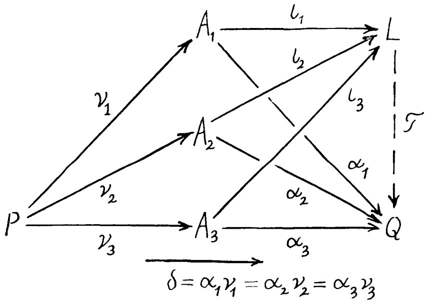
(10′)



same shape (the upper {lower} part of (10) {(10′)} with P and ν_i {Q and α_i} instead of U and σ_i {L and ι_i}, $i = 1, 2, 3$) there is a unique mapping $\omega: P \to U$ {$\tau: L \to Q$} such that (10) {(10′)} commutes. This gives:

THEOREM 3. *For mappings* $\alpha_i: A_i \to Q$, $\nu_i: P \to A_i$, $i = 1, 2, 3$, *the respective associativity laws*

$$[[\alpha_1, \alpha_2]_\tau, \alpha_3]_\tau = [\alpha_1, [\alpha_2, \alpha_3]_\tau]_\tau = [\alpha_1, \alpha_2, \alpha_3]_\tau$$

and

$$[[\nu_1, \nu_2]_\iota, \nu_3]_\iota = [\nu_1, [\nu_2, \nu_3]_\iota]_\iota = [\nu_1, \nu_2, \nu_3]_\iota$$

hold up to isomorphisms.

Thus the statements of the Corollary to Proposition 2 immediately extend to this case. The analogies can be carried further:

DEFINITION 6. *The upper {lower} diagram in* (10) {(10′)} *is called "smooth" or "small" at* P {Q} *according as* ω *onto* {τ 1–1} *or* ω 1–1 {τ *onto*}, *respectively. It is called "tight" at* P {Q}, *if it has both properties.*

PROPOSITION 13. *Smallness at* P *in* (10) *is equivalent to*

$$\operatorname{Ker} \nu_1 \cap \operatorname{Ker} \nu_2 \cap \operatorname{Ker} \nu_3 = 0,$$

smoothness at P *with*

$$\supset \begin{cases} K_1: \operatorname{Ker} \alpha_1 [= \nu_1 \operatorname{Ker} \delta = \nu_1(\operatorname{Ker} \nu_2 \cup \operatorname{Ker} \nu_3)] = \nu_1(\operatorname{Ker} \nu_2 \cap \operatorname{Ker} \nu_3) \\ K_2: \operatorname{Ker} \alpha_2 [= \nu_2 \operatorname{Ker} \delta = \nu_2(\operatorname{Ker} \nu_1 \cup \operatorname{Ker} \nu_3)] = \nu_2(\operatorname{Ker} \nu_1 \cap \operatorname{Ker} \nu_3) \\ K_3: \operatorname{Ker} \alpha_3 [= \nu_3 \operatorname{Ker} \delta = \nu_3(\operatorname{Ker} \nu_1 \cup \operatorname{Ker} \nu_2)] = \nu_3(\operatorname{Ker} \nu_1 \cap \operatorname{Ker} \nu_2) \\ K_\delta: \operatorname{Ker} \delta [= \operatorname{Ker} \nu_1 \cup \operatorname{Ker} \nu_2 \cup \operatorname{Ker} \nu_3] \\ \qquad = (\operatorname{Ker} \nu_1 \cap \operatorname{Ker} \nu_2) \cup (\operatorname{Ker} \nu_1 \cap \operatorname{Ker} \nu_3) \cup (\operatorname{Ker} \nu_2 \cap \operatorname{Ker} \nu_3) \end{cases}$$

and

$$\subset \begin{cases} I_1: \operatorname{Im} \nu_1 [= \alpha_1^{-1} \operatorname{Im} \delta] = \alpha_1^{-1}(\operatorname{Im} \alpha_2 \cap \operatorname{Im} \alpha_3) \\ I_2: \operatorname{Im} \nu_2 [= \alpha_2^{-1} \operatorname{Im} \delta] = \alpha_2^{-1}(\operatorname{Im} \alpha_1 \cap \operatorname{Im} \alpha_3) \\ I_3: \operatorname{Im} \nu_3 [= \alpha_3^{-1} \operatorname{Im} \delta] = \alpha_3^{-1}(\operatorname{Im} \alpha_1 \cap \operatorname{Im} \alpha_2) \\ I_\delta: \operatorname{Im} \delta = \operatorname{Im} \alpha_1 \cap \operatorname{Im} \alpha_2 \cap \operatorname{Im} \alpha_3. \end{cases}$$

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PROPOSITION 13′. *In* (10′) *smallness at* Q *is equivalent to*

$$\operatorname{Im}\alpha_1 \cup \operatorname{Im}\alpha_2 \cup \operatorname{Im}\alpha_3 = Q,$$

smoothness with

$$\begin{cases} K_1'\colon \operatorname{Ker}\alpha_1[=\nu_1 \operatorname{Ker}\delta] = \nu_1(\operatorname{Ker}\nu_2 \cup \operatorname{Ker}\nu_3) \\ K_2'\colon \operatorname{Ker}\alpha_2[=\nu_2 \operatorname{Ker}\delta] = \nu_2(\operatorname{Ker}\nu_1 \cup \operatorname{Ker}\nu_3) \\ K_3'\colon \operatorname{Ker}\alpha_3[=\nu_3 \operatorname{Ker}\delta] = \nu_3(\operatorname{Ker}\nu_1 \cup \operatorname{Ker}\nu_2) \\ K_\delta'\colon \operatorname{Ker}\delta = \operatorname{Ker}\nu_1 \cup \operatorname{Ker}\nu_2 \cup \operatorname{Ker}\nu_3 \end{cases}$$

and

$$\begin{cases} I_1'\colon \operatorname{Im}\nu_1[=\alpha_1^{-1}(\operatorname{Im}\alpha_2 \cap \operatorname{Im}\alpha_3) = \alpha_1^{-1}\operatorname{Im}\delta] = \alpha_1^{-1}(\operatorname{Im}\alpha_2 \cup \operatorname{Im}\alpha_3) \\ I_2'\colon \operatorname{Im}\nu_2[=\alpha_2^{-1}(\operatorname{Im}\alpha_1 \cap \operatorname{Im}\alpha_3) = \alpha_2^{-1}\operatorname{Im}\delta] = \alpha_2^{-1}(\operatorname{Im}\alpha_1 \cup \operatorname{Im}\alpha_3) \\ I_3'\colon \operatorname{Im}\nu_3[=\alpha_3^{-1}(\operatorname{Im}\alpha_1 \cap \operatorname{Im}\alpha_2) = \alpha_3^{-1}\operatorname{Im}\delta] = \alpha_3^{-1}(\operatorname{Im}\alpha_1 \cup \operatorname{Im}\alpha_2) \\ I_\delta'\colon \operatorname{Im}\delta[=\operatorname{Im}\alpha_1 \cap \operatorname{Im}\alpha_2 \cap \operatorname{Im}\alpha_3] \\ \qquad = (\operatorname{Im}\alpha_1 \cup \operatorname{Im}\alpha_2) \cap (\operatorname{Im}\alpha_1 \cup \operatorname{Im}\alpha_3) \cap (\operatorname{Im}\alpha_2 \cup \operatorname{Im}\alpha_3). \end{cases}$$

Remark. Replacing throughout "=" by the indicated inclusion signs "$\supset$" and "$\subset$" we obtain statements that follow already from commutativity. Obviously smoothness at P does not imply smoothness at Q, that is, the self-duality "ω onto if and only if τ 1–1" in Proposition 3, true for squares, no longer holds.

PROPOSITION 14. *Condition* K_i *and* I_δ *implies* I_i; I_i *and* K_δ *implies* K_i; I_i *implies* I_δ; K_i *and* K_j *implies* K_δ *for* $1 \leqslant i < j \leqslant 3$. *Dually* K_i' *and* I_δ' *implies* I_i'; I_i' *and* K_δ' *implies* K_i'; I_i' *and* I_j' *implies* I_δ'; K_i' *implies* K_δ'.

COROLLARY. *The conditions* K_1, K_2, K_3, and I_δ $\{K_1', K_2', K_2',$ *and* $I_\delta'\}$ *or* I_1, I_2, I_3, *and* K_δ $\{I_1', I_2', I_3',$ *and* $K_\delta'\}$ *or* K_i, K_j, I_l $\{I_i', I_j', K_l'\}$ *with* $1 \leqslant i \neq j \neq l \leqslant 3$ *alone imply the others, that is, smoothness at* P $\{Q\}$.

Remark. The corresponding facts for the square (1) are in obvious notation: K_β and I_δ implies I_ν, $K_{\nu'}$ and I_δ implies I_α; similarly I_α and K_δ implies $K_{\nu'}$, I_ν and K_δ implies K_β. Hence K_β, $K_{\nu'}$, and I_δ alone, or I_α, I_ν, and K_δ alone, as well as K_β and I_α alone, or $K_{\nu'}$ and I_ν alone, imply full smoothness. The last statement is the four-lemma.

Now the commutative diagrams of given shape, say

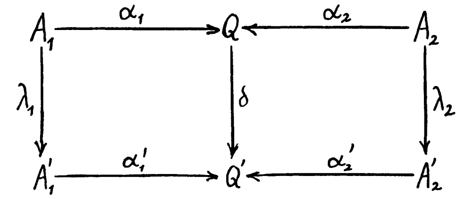
$$\mathfrak{D}\colon A_1 \xrightarrow{\alpha_1} Q \xleftarrow{\alpha_2} A_2,$$

form an abelian category (see **3**), whose maps $\omega\colon \mathfrak{D} \to \mathfrak{D}'$ with

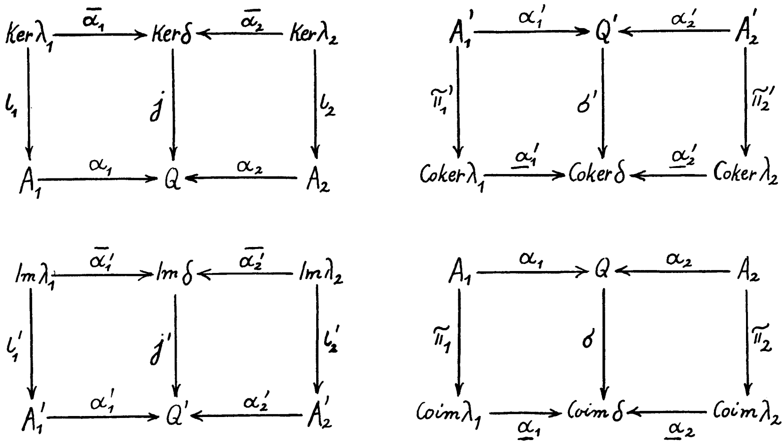
$$\mathfrak{D}'\colon A_1' \xrightarrow{\alpha_1'} Q \xleftarrow{\alpha_2'} A_2'$$

are triples $\omega = (\lambda_1, \delta, \lambda_2)$ such that the diagram (11) commutes.

The sum of two mappings $\omega = (\lambda_1, \lambda, \delta_2)$ and $\tau\colon (\mu_1, \epsilon, \mu_2)\colon \mathfrak{D} \to \mathfrak{D}'$ is defined by $\omega + \tau = (\lambda_1 + \mu_1, \delta + \epsilon, \lambda_2 + \mu_2)\colon \mathfrak{D} \to \mathfrak{D}'$. ω is monomorphic {epimorphic}, that is, $\omega\phi = 0$ $\{\phi\omega = 0\}$ implies $\phi = 0$ if and only if $\lambda_1, \lambda_2, \delta$ are all 1–1 {onto}.

(11) 

Ker ω, Coker ω, Im $\omega \simeq$ Coim ω together with their natural injection (ι_1, j, ι_2), projection $(\pi_1{}', \sigma', \pi_2{}')$, injection $(\iota_1{}', j', \iota_2{}')$ and projection (π_1, σ, π_2) are defined in obvious manner from the respective induced diagrams (12).

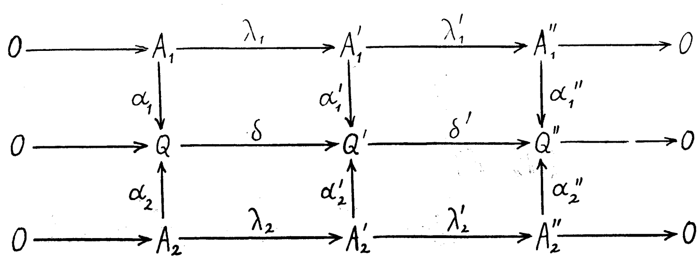
(12) 

Given $\mathfrak{D}$, define $T(\mathfrak{D}) = U$ as the top group in the upper completion of $\mathfrak{D}$, that is, $(a_1, a_2) \in U \subset A_1 \oplus A_2$ if and only if $\alpha_1(a_1) = \alpha_2(a_2)$. For $\omega\colon \mathfrak{D} \to \mathfrak{D}'$ in $\mathfrak{A}$ define a mapping $T(\omega)\colon U \to U' = T(\mathfrak{D}')$ by $T(\omega)(a_1, a_2) = (\lambda_1(a_1), \lambda_2(a_2)) \in U'$. Define $S(\mathfrak{D}) = Q/\text{Im}\,\alpha_1 \cup \text{Im}\,\alpha_2$ and $S(\omega) : S(\mathfrak{D}) \to S(\mathfrak{D}') = Q'/\text{Im}\,\alpha_1{}' \cup \text{Im}\,\alpha_2{}'$ by $S(\omega)(q) = \delta(q)$. This is consistent, since $\delta(\alpha_i(a_i)) = \alpha_i{}'\lambda_i(a_i) \equiv 0$ for $i = 1, 2$.

THEOREM 4. *T and S are additive covariant functors and S is the first right satellite of T. The higher satellites are trivial. For any exact sequence*

$$0 \to \mathfrak{D} \xrightarrow{\omega} \mathfrak{D}' \xrightarrow{\omega'} \mathfrak{D}'' \to 0,$$

namely

(13) 

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the sequence

$$0 \to T(\mathfrak{D}) \xrightarrow{T(\omega)} T(\mathfrak{D}') \xrightarrow{T(\omega')} T(\mathfrak{D}'') \xrightarrow{\theta} S(\mathfrak{D}) \xrightarrow{S(\omega)} S(\mathfrak{D}') \xrightarrow{S(\omega')} S(\mathfrak{D}'') \to 0$$

is exact, with a connecting homomorphism θ *defined as follows:*

For $(a_1'', a_2'') \in U'' = T(\mathfrak{D}'')$, that is $\alpha_1''(a_1'') = \alpha_2''(a_2'')$, find $a_1'' = \lambda_1'(a_1')$, $a_2'' = \lambda_2'(a_2')$. Then

$$\alpha_1''\lambda_1'(a_1') - \alpha_2''\lambda_2'(a_2') = \delta'(\alpha_1'(a_1') - \alpha_2'(a_2')) = 0.$$

Hence $\alpha_1'(a_1') - \alpha_2'(a_2') = \delta(q)$ with q unique. Set $\theta(a_1'', a_2'') = q$.

Proof. θ is well defined, since for $a_1'' = \lambda_1'(\bar{a}_1')$, $a_2'' = \lambda_2'(\bar{a}_2')$, $\lambda_1'(\bar{a}_1' - a_1) = 0$ and $\lambda_2'(\bar{a}_2' - a_2') = 0$; hence $\bar{a}_1' - a_1' = \lambda_1(a_1)$, $\bar{a}_2' - a_2' = \lambda_2(a_2)$, and $\alpha_1'(\bar{a}_1') - \alpha_2'(\bar{a}_2') = \alpha_1'(a_1') - \alpha_2'(a_2') + \alpha_1'\lambda_1(a_1) - \alpha_2'\lambda_2(a_2) = \delta(q) + \delta(\alpha_1(a_1) - \alpha_2(a_2))$. But $\alpha_1(a_1) - \alpha_2(a_2) \equiv 0$. Now

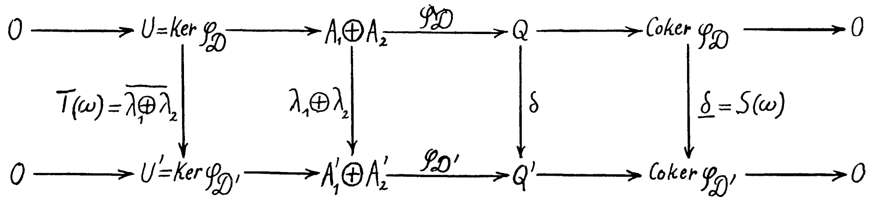
$$T(\mathfrak{D}) = U = \operatorname{Ker} \phi_{\mathfrak{D}} \quad \text{and} \quad S(\mathfrak{D}) = \operatorname{Coker} \phi_{\mathfrak{D}}$$

with the "associated map" $\phi_{\mathfrak{D}}: A_1 \oplus A_2 \to Q$ given by

$$\phi_{\mathfrak{D}}(a_1, a_2) = \alpha_1(a_1) - \alpha_2(a_2).$$

$T(\omega)$ and $S(\omega)$ are the induced maps in diagram (14). Thus the "kernel-cokernel-lemma" (see **3**) provides a proof. Direct calculation gives another.

(14)

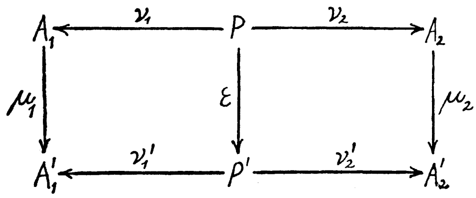


Dually in the abelian category $\mathfrak{B}$ formed by the diagrams of shape

$$\mathfrak{C}: A_1 \xleftarrow{\nu_1} P \xrightarrow{\nu_2} A_2$$

define $F(\mathfrak{C}) = L = A_1 \oplus A_2/(\nu_1(p), -\nu_2(p))$ for $p \in P$, the bottom group of the lower completion of $\mathfrak{C}$, and for $\tau: \mathfrak{C} \to \mathfrak{C}'$ in $\mathfrak{B}$, namely $\tau = (\mu_1, \epsilon, \mu_2)$ such that (11′) commutes, the map $F(\tau): L \to L' = F(\mathfrak{C}')$ by $F(\tau)(a_1, a_2) = (\mu_1(a_1), \mu_2(a_2))$, consistent since $F(\tau)(\nu_1(p), -\nu_2(p)) = (\nu_1'\epsilon(p), -\nu_2'\epsilon(p)) \equiv 0$ in L'. Define $G(\mathfrak{C}) = \operatorname{Ker} \nu_1 \cap \operatorname{Ker} \nu_2$ and the map $G(\tau): \operatorname{Ker} \nu_1 \cap \operatorname{Ker} \nu_2 \to \operatorname{Ker} \nu_1' \cap \operatorname{Ker} \nu_2' = G(\mathfrak{C}')$ by $G(\tau)(p) = \epsilon(p) \in \operatorname{Ker} \nu_1' \cap \operatorname{Ker} \nu_2'$.

(11′)



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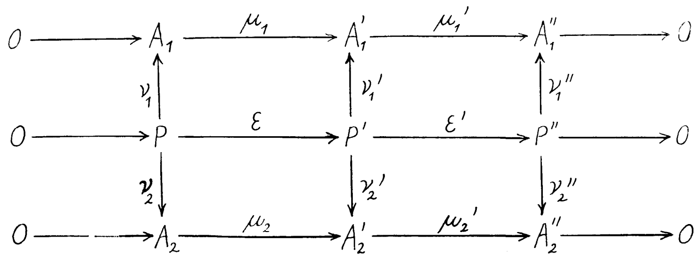
THEOREM 4′. *F and G are additive covariant functors and G is the first left satellite of F. The higher satellites are trivial. For any exact sequence*

$$0 \to \mathfrak{S} \xrightarrow{\tau} \mathfrak{S}' \xrightarrow{\tau'} \mathfrak{S}'' \to 0,$$

namely:

(13′)



the sequence

$$0 \to G(\mathfrak{S}) \xrightarrow{G(\tau)} G(\mathfrak{S}') \xrightarrow{G(\tau')} G(\mathfrak{S}'') \xrightarrow{\vartheta} F(\mathfrak{S}) \xrightarrow{F(\tau)} F(\mathfrak{S}') \xrightarrow{F(\tau')} F(\mathfrak{S}'') \to 0$$

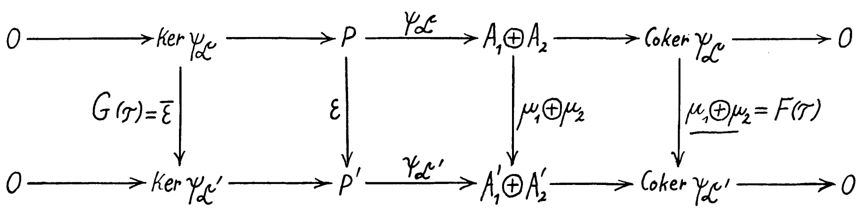
is exact, with connecting homomorphism ϑ defined as follows: For $p'' \in \operatorname{Ker} \nu_1''$ $\cap \operatorname{Ker} \nu_2''$ find $p'' = \epsilon'(p')$; then $\mu_1'\nu_1'(p') = 0$ and $\mu_2'\nu_2'(p') = 0$, and hence $\nu_1'(p') = \mu_1(a_1)$ and $\nu_2'(p') = \mu_2(a_2)$ with unique a_1, a_2. Set $\vartheta(p'') = (a_1, -a_2)$.

Proof. ϑ is well defined. For $p'' = \epsilon'(\bar{p}')$, $\epsilon'(\bar{p}' - p') = 0$; hence $\bar{p}' = p' + \epsilon(p)$ and $\nu_1'(\bar{p}') = \mu_1(a_1) + \mu_1(\nu_1(p))$, $\nu_2'(\bar{p}') = \mu_2(a_2) + \mu_2(\nu_2(p))$. But $(\nu_1(p), -\nu_2(p)) \equiv 0$ in L. With an "associated map" $\psi_{\mathfrak{S}}: P \to A_1 \oplus A_2$ defined by $\psi_{\mathfrak{S}}(p) = (\nu_1(p), -\nu_2(p))$

$$F(\mathfrak{S}) = \operatorname{Coker} \psi_{\mathfrak{S}} \quad \text{and} \quad G(\mathfrak{S}) = \operatorname{Ker} \psi_{\mathfrak{S}}.$$

$F(\tau)$ and $G(\tau)$ are the induced maps in (14′).

(14′)



Remark. For the right half of the diagram (10), built by the maps $\alpha_1, \alpha_2, \alpha_3$, an "associated map" can be similarly defined, namely $\chi: A_1 \oplus A_2 \oplus A_3 \to Q \oplus Q \oplus Q$ given by $\chi(a_1, a_2, a_3) = (\alpha_1(a_1) - \alpha_2(a_2), \alpha(a_1) - \alpha_3(a_3), \alpha_2(a_2) - \alpha_3(a_3))$; then $\operatorname{Ker} \chi = U$ gives the upper completion and $\operatorname{Coker} \chi$ its right satellite. Similarly, for the left half of the diagram (10′), built by the maps ν_1, ν_2, ν_3, the associated map $\xi: P \oplus P \oplus P \to A_1 \oplus A_2 \oplus A_3$ is defined by $\xi(p_1, p_2, p_3) = (\nu_1(p_2 + p_3), \nu_2(p_1 - p_2), \nu_3(-p_1 - p_2))$. Thus $\operatorname{Coker} \xi = L$ gives the lower completion and $\operatorname{Ker} \xi$ its satellite. More generally to any

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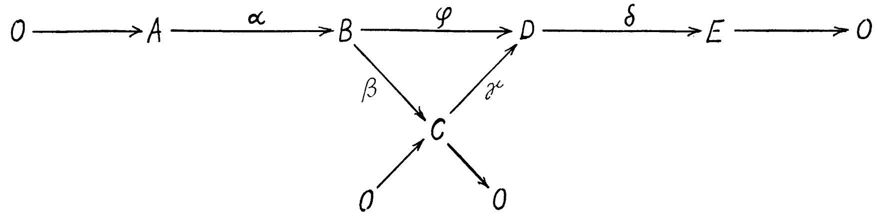
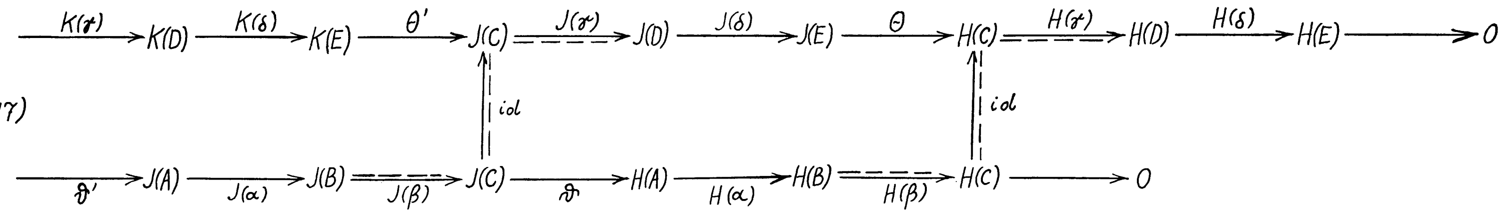
(16)



(17)



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finite (and infinite) diagram (not necessarily of l.c.m.-type) there corresponds an upper and lower associated map, which gives its upper and lower completion, respectively, with satellite.

PROPOSITION 15 (see **3**). *If the triangle*

$$\begin{array}{ccc} B & \xrightarrow{\quad\phi\quad} & D \\ & {\scriptstyle\beta}\searrow \quad \nearrow{\scriptstyle\gamma} & \\ & C & \end{array}$$

is commutative, the sequence

$$(15)\quad 0 \to \operatorname{Ker}\beta \xrightarrow{i} \operatorname{Ker}\phi \xrightarrow{\bar\beta} \operatorname{Ker}\gamma \xrightarrow{\epsilon} \operatorname{Coker}\beta \xrightarrow{\gamma} \operatorname{Coker}\phi \xrightarrow{\pi} \operatorname{Coker}\gamma \to 0$$

with natural injection i, projection π, and induced maps $\bar\beta$, γ, ϵ (by identity) is exact.

Proof. $\phi = \gamma\beta$ implies $\operatorname{Ker}\phi = \beta^{-1}\operatorname{Ker}\gamma \supset \operatorname{Ker}\beta$, $\operatorname{Im}\phi = \gamma \operatorname{Im}\beta \subset \operatorname{Im}\gamma$ and $\beta\operatorname{Ker}\phi = \operatorname{Im}\beta \cap \operatorname{Ker}\gamma \subset \operatorname{Ker}\gamma$, $\gamma^{-1}\operatorname{Im}\phi = \operatorname{Ker}\gamma \cup \operatorname{Im}\beta \supset \operatorname{Im}\beta$.

Let H be any additive right-exact functor with left satellites $J, K, \ldots$, $\phi: B \to D$ any map, $C = \operatorname{Im}\phi$, $\alpha: A = \operatorname{Ker}\phi \to B$ and $\delta: D \to E = \operatorname{Coker}\phi$ the natural injection and projection respectively. The canonical splitting (16) into short exact sequences provides a diagram (17) with both rows exact, where the vertical maps are identities and $\theta, \theta', \vartheta, \vartheta'$ are the connecting homomorphisms.

Since $H(\phi) = H(\gamma)H(\beta)$ and $J(\phi) = J(\gamma)J(\beta)$, the sequences

$$0 \to \operatorname{Ker} H(\beta) \xrightarrow{i} \operatorname{Ker} H(\phi) \xrightarrow{\overline{H(\beta)}} \operatorname{Ker} H(\gamma) \xrightarrow{\epsilon} \overset{=0}{\operatorname{Coker}} H(\beta) \xrightarrow{H(\gamma)} \operatorname{Coker} H(\phi) \xrightarrow{\pi} \operatorname{Coker} H(\gamma) \to 0,$$

(18)

$$0 \to \operatorname{Ker} J(\beta) \xrightarrow{j} \operatorname{Ker} J(\phi) \xrightarrow{\overline{J(\beta)}} \operatorname{Ker} J(\gamma) \xrightarrow{\eta} \operatorname{Coker} J(\beta) \xrightarrow{J(\gamma)} \operatorname{Coker} J(\phi) \xrightarrow{\rho} \operatorname{Coker} J(\gamma) \to 0$$

are exact. Together with (17) they induce the commutative diagram (19) with exact lines ———, – – –, - - - -. In particular:

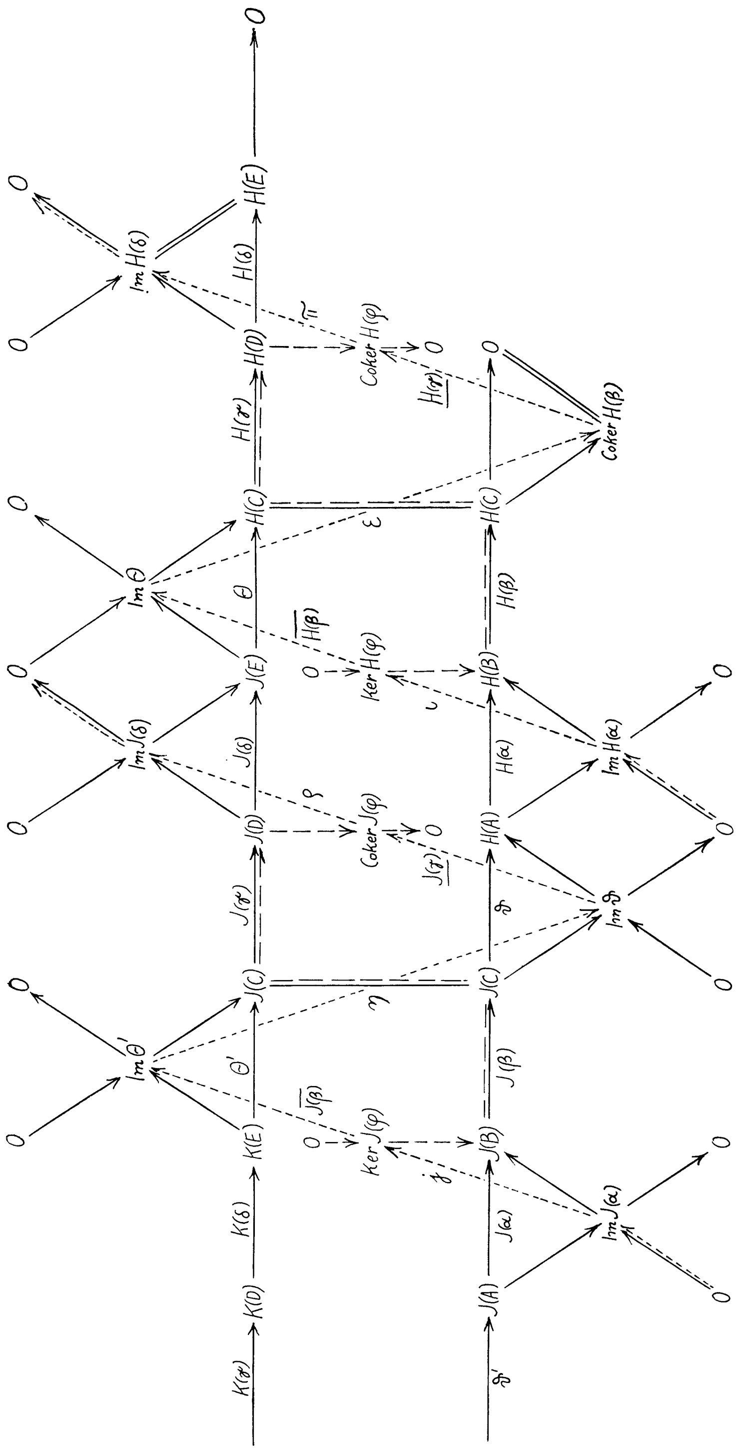
(a) $\operatorname{Ker} H(\phi)$ has a subgroup $\operatorname{Im} H(\alpha)[= \operatorname{Ker} H(\beta) \simeq \operatorname{Coker}\vartheta]$ with factor group $\operatorname{Coker} J(\delta)[\simeq \operatorname{Ker} H(\gamma) = \operatorname{Im}\theta]$.

(b) $\operatorname{Coker} H(\phi) \simeq H(E)[= \operatorname{Im} H(\delta) \simeq \operatorname{Coker} H(\gamma)]$.

Similarly in case $K = 0$, that is, $\theta' = 0$ and $\vartheta' = 0$, hence $\operatorname{Ker} J(\gamma) = \operatorname{Im}\theta' = 0$:

(c) $\operatorname{Ker} J(\phi) \simeq J(A)[\simeq \operatorname{Im} J(\alpha) = \operatorname{Ker} J(\beta)]$.

(d) $\operatorname{Coker} J(\phi)$ has a subgroup $\operatorname{Coker} J(\beta)[\simeq \operatorname{Im}\vartheta = \operatorname{Ker} H(\alpha)]$ with factor group $\operatorname{Coker} J(\gamma)[\simeq \operatorname{Im} J(\delta) = \operatorname{Ker}\theta]$.



(19)

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Assume the case of abelian groups with $H = (\ )\otimes X$ and $J = \mathrm{Tor}(\ ,X)$ for some fixed group X. The definition of "$\otimes$," "Tor," and of the connecting homomorphism θ in terms of generators and relations (see **3**) allows an explicit description of the isomorphism

$$\kappa: \mathrm{Ker}\, H(\phi)/\mathrm{Im}\, H(\alpha) \overset{\sim}{\leftarrow} J(E)/\mathrm{Im}\, J(\delta):$$

Given $e \in E$ and $x \in X$ with $he = 0$ and $hx = 0$ for some integer $h > 0$, find $e = \delta(d)$; then $\delta(hd) = 0$; and hence $hd = \phi(b)$. Put $\kappa\tau_h(e, x) = b \otimes x$ mod $\mathrm{Im}\, H(\alpha)$. This is well defined: For $e = \delta(d')$, $\delta(d - d') = 0$; thus $hd' = \phi(b - h\bar{b})$ and $(b - h\bar{b})\otimes x = b \otimes x - \bar{b}\otimes hx = b\otimes x$. $hd = \phi(b')$ gives $b - b' = \alpha(a)$, so $b'\otimes x = b\otimes x + H(\alpha)(a\otimes x)$. For $\tau_h(d, x) \in J(D)$, that is, $hd = 0$ and $hx = 0$, $J(\delta)\tau_h(d, x) = \tau_h(e, x)$ with $e = \delta(d)$ and $hd = \phi(0)$; hence $\kappa\tau_h(d, x) = 0 \otimes x = 0$.

Since $\mathrm{Ker}\, H(\phi) = H(\beta)^{-1}\,\mathrm{Im}\,\theta$ and $\mathrm{Im}\, H(\alpha)$ is generated by the symbols $\alpha(a)\otimes x$, that is, by those $b\otimes x$ with $\phi(b) = 0$, the definition of θ and ϑ shows that:

$\mathrm{Ker}\, H(\phi)$ is generated within $H(B) = B\otimes X$ by the symbols $b\otimes x$ with $\phi(b) = hd$ and $hx = 0$ for some $h = 0, 1, 2, \ldots$ ($h = 0$ gives $\mathrm{Im}\, H(\alpha)$),

$\mathrm{Im}\,\theta$ is generated within $H(C) = C\otimes X$ by the symbols $c\otimes x$ with $\gamma(c)[= c] = hd$ and $hx = 0$ for some $h = 1, 2, 3, \ldots$ ($h = 0$ gives $0\otimes x = 0$),

$\mathrm{Im}\,\vartheta$ is generated in $A\otimes X$ by the symbols $a\otimes x$ with $\alpha(a)[=a] = hb$ and $hx = 0$ for $h = (0), 1, 2, 3, \ldots$.

Given the diagram

$$\mathfrak{G}: A_1 \xrightarrow{\alpha_1} Q \xleftarrow{\alpha_2} A_2$$

replace $\phi: B\to D$ by the associated map $\phi_{\mathfrak{D}}: A_1\oplus A_2 \to Q$ defined above, A by $U = \mathrm{Ker}\,\phi_{\mathfrak{D}} = [\alpha_1, \alpha_2]_r$, E by

$$\hat{U} = \mathrm{Coker}\,\phi_{\mathfrak{D}} = Q/\mathrm{Im}\,\alpha_1 \cup \mathrm{Im}\,\alpha_2.$$

Furthermore let

$$H(\mathfrak{D}): A_1\otimes X \xrightarrow{\alpha_1\otimes 1} Q\otimes X \xleftarrow{\alpha_2\otimes 1} A_2\otimes X$$

and

$$J(\mathfrak{D}): \mathrm{Tor}\,(A_1, X) \xrightarrow{\mathrm{Tor}\,(\alpha_1, 1)} \mathrm{Tor}\,(Q, X) \xrightarrow{\mathrm{Tor}\,(\alpha_2, 1)} \mathrm{Tor}\,(A_2, X)$$

be the induced diagrams with associated maps $H(\phi_{\mathfrak{D}}) = \phi_{\mathfrak{D}}\otimes 1$ and $J(\phi_{\mathfrak{D}}) = \mathrm{Tor}(\phi_{\mathfrak{D}}, 1)$; finally $V = \mathrm{Ker}\, H(\phi_{\mathfrak{D}}) = [\alpha_1\otimes 1, \alpha_2\otimes 1]_r$, $\hat{V} = \mathrm{Coker}\, H(\phi_{\mathfrak{D}})$, $W = \mathrm{Ker}\, J(\phi_{\mathfrak{D}}) = [\mathrm{Tor}(\alpha_1, 1),\ \mathrm{Tor}(\alpha_2, 1)]_r$ and $\hat{W} = \mathrm{Coker}\, J(\phi_{\mathfrak{D}})$, the respective least common multiples with satellites. Then $\mathrm{Im}\,\theta$ is generated within $\mathrm{Im}\,\alpha_1 \cup \mathrm{Im}\,\alpha_2 \otimes X$ by the symbols $(\alpha_1(a_1) - \alpha_2(a_2))\otimes x$ with $\alpha_1(a_1) - \alpha_2(a_2) = hq$ and $hx = 0$ for some integer $h > 0$. $\mathrm{Im}\,\vartheta$ is generated within $U\otimes X$ by the symbols $(ha_1, ha_2)\otimes x$ with $h(\alpha_1(a_1) - \alpha_2(a_2)) = 0$ and $hx = 0$ for some $h > 0$.

https://doi.org/10.4153/CJM-1963-007-6 Published online by Cambridge University Press

THEOREM 5. (a) *V is generated within* $(A_1 \oplus A_2) \otimes X \simeq (A_1 \otimes X) \oplus (A_2 \otimes X)$ *by the symbols* $(a_1, a_2) \otimes x$ *such that* $\alpha_1(a_1) - \alpha_2(a_2) = hq$ *and* $hx = 0$ *for some* $q \in Q$ *and integer* $h = 0, 1, 2, \ldots$.

It contains the subgroup $\mathrm{Im}\, U \otimes X$, *with factor group isomorphic to* $\mathrm{Tor}(\hat{U}, X)/\mathrm{Im}\,\mathrm{Tor}\,(Q, X)$ *by the mapping* κ *defined as follows: For* $h[q] = 0$, *that is,* $hq = \alpha_1(a_1) - \alpha_2(a_2) \in \mathrm{Im}\,\alpha_1 \cup \mathrm{Im}\,\alpha_2$, *put* $\kappa\tau_h([q], x) = (a_1, a_2) \otimes x$.

$\mathrm{Im}\, U \otimes X$ *in turn is isomorphic to* $U \otimes X$ *modulo the subgroup* $\mathrm{Im}\,\vartheta$.

(b) $\hat{V} \simeq \hat{U} \otimes X$ *by the canonical map* $q \otimes x \to [q] \otimes x$.

(c) $W \simeq \mathrm{Tor}(U, X)$, *that is,* $[\mathrm{Tor}\,(\alpha_1, 1), \mathrm{Tor}(\alpha_2, 1)]_r \simeq \mathrm{Tor}([\alpha_1, \alpha_2]_r, X)$ *by* $\tau_h((a_1, a_2), x) = (\tau_h(a_1, x), \tau_h(a_2, x))$, *where* $h(a_1, a_2) = 0$ *and* $\alpha_1(a_1) - \alpha_2(a_2) = 0$, *hence* $\mathrm{Tor}(\alpha_1, 1)\tau_h(a_1, x) - \mathrm{Tor}(\alpha_2, 1)\tau_h(a_2, x) = \tau_h(\alpha_1(a_1), x) - \tau_h(\alpha_2(a_2), x) = \tau_h(0, x) = 0$.

(d) $\hat{W} = \mathrm{Tor}(Q, X)/\mathrm{Im}\,\mathrm{Tor}\,(\alpha_1, 1) \cup \mathrm{Im}\,\mathrm{Tor}\,(\alpha_2, 1)$ *has a subgroup* $\mathrm{Tor}(\mathrm{Im}\,\alpha_1 \cup \mathrm{Im}\,\alpha_2, X)$ *isomorphic to* $\mathrm{Im}\,\vartheta$, *with isomorphism given by* $\tau_h(\alpha_1(a_1) - \alpha_2(a_2), x) \to (ha_1, ha_2) \otimes x$. *The corresponding factor group is isomorphic to the subgroup of* $\mathrm{Tor}\,(\hat{U}, X)$ *generated by those* $\tau_h([q], x)$ *with* $hq = 0$ *and* $hx = 0$, *via the canonical map* $\tau_h(q, x) \to \tau_h([q], x)$. *The latter has a factor group isomorphic to* $\mathrm{Im}\,\theta$ *by* $\tau_h([q], x) \to hq \otimes x$ *for* $hx = 0$ *and* $hq = \alpha_1(a_1) - \alpha_2(a_2)$.

To complete the picture given by (19) we describe its maps in terms of generators (upper symbols are generators, lower symbols images under the preceding map):

First row:

$$
\begin{array}{ccccc}
\tau_h(\alpha_1(a_1) - \alpha_2(a_2), x) & & \tau_h(q, x) & & \tau_h([q], x) \\
0 \to \mathrm{Tor}\,(\mathrm{Im}\,\alpha_1 \cup \mathrm{Im}\,\alpha_2, X) & \xrightarrow{J(\gamma)} & \mathrm{Tor}\,(Q, X) & \xrightarrow{J(\delta)} & \mathrm{Tor}\,(\hat{U}, X) \quad \to \\
0 & & \tau_h(\alpha_1(a_1) - \alpha_2(a_2), x) & & \tau_h([q], x) \text{ with } hq = 0
\end{array}
$$

$$
\begin{array}{cccccc}
& (\alpha_1(a_1) - \alpha_2(a_2)) \otimes x & & q \otimes x & & [q] \otimes x \\
\xrightarrow{\theta} & (\mathrm{Im}\,\alpha_1 \cup \mathrm{Im}\,\alpha_2) \otimes X & \xrightarrow{H(\gamma)} & Q \otimes X & \xrightarrow{H(\delta)} & \hat{U} \otimes X \to 0 \\
& (\alpha_1(a_1) - \alpha_2(a_2)) \otimes x \text{ with} & & (\alpha_1(a_1) - \alpha_2(a_2)) \otimes x & & [q] \otimes x \\
& \alpha_1(a_1) - \alpha_2(a_2) = hq,\ hx = 0 & & & &
\end{array}
$$

Second row:

$$
\begin{array}{ccccc}
\tau_h((a_1, a_2), x) & & \tau_h((a_1, a_2), x) & & \tau_h(\alpha_1(a_1) - \alpha_2(a_2), x) \\
0 \to \mathrm{Tor}\,(U, X) & \xrightarrow{J(\alpha)} & \mathrm{Tor}\,(A_1 \oplus A_2, X) & \xrightarrow{J(\beta)} & \mathrm{Tor}\,(\mathrm{Im}\,\alpha_1 \cup \mathrm{Im}\,\alpha_2, X) \quad \to \\
0 & & \tau_h((a_1, a_2), x) \text{ with} & & \tau_h(\alpha_1(a_1) - \alpha_2(a_2), x) \\
& & \alpha_1(a_1) - \alpha_2(a_2) = 0 & & \text{with } h(a_1, a_2) = 0
\end{array}
$$

$$\begin{array}{ccccccc} & & (a_1, a_2) \otimes x \text{ with} & & & & \\ & & \alpha_1(a_1) - \alpha_2(a_2) = 0 & & (a_1, a_2) \otimes x & & (\alpha_1(a_1) - \alpha_2(a_2)) \otimes x \\ \xrightarrow{\vartheta} & & U \otimes X & \xrightarrow{H(\alpha)} & (A_1 \oplus A_2) \otimes X & \xrightarrow{H(\beta)} & (\operatorname{Im} \alpha_1 \cup \operatorname{Im} \alpha_2) \otimes X \to 0. \\ (ha_1, ha_2) \otimes x \text{ with} & & (a_1, a_2) \otimes x \text{ with} & & & & (\alpha_1(a_1) - \alpha_2(a_2)) \otimes x \\ h(\alpha_1(a_1) - \alpha_2(a_2)) = 0 & & \alpha_1(a_1) - \alpha_2(a_2) = 0 & & & & \end{array}$$

Dually, given diagram

$$\mathfrak{C}: A_1 \xleftarrow{\nu_1} P \xrightarrow{\nu_2} A_2$$

replace $\phi: B \to D$ by the associated map $\psi_{\mathfrak{C}}: P \to A_1 \oplus A_2$, A by $\check{L}^\nu = \operatorname{Ker} \psi_{\mathfrak{C}} = \operatorname{Ker} \nu_1 \cap \operatorname{Ker} \nu_2$, E by $L = \operatorname{Coker} \psi_{\mathfrak{C}} = [\nu_1, \nu_2]_l$. Then the induced diagrams

$$H(\mathfrak{C}): A_1 \otimes X \xleftarrow{\nu_1 \otimes 1} P \otimes X \xrightarrow{\nu_2 \otimes 1} A_2 \otimes X$$

and

$$J(\mathfrak{C}): \operatorname{Tor}(A_1, X) \xleftarrow{\operatorname{Tor}(\nu_1, 1)} \operatorname{Tor}(P, X) \xrightarrow{\operatorname{Tor}(\nu_2, 1)} \operatorname{Tor}(A_2, X)$$

have associated maps $H(\psi_{\mathfrak{C}}) = \psi_{\mathfrak{C}} \otimes 1$ and $J(\psi_{\mathfrak{C}}) = \operatorname{Tor}(\psi_{\mathfrak{C}}, 1)$ respectively. Their least common multiples with satellites are

$$\check{M} = \operatorname{Ker} H(\psi_{\mathfrak{C}}), \; M = \operatorname{Coker} H(\psi_{\mathfrak{C}}) = [\nu_1 \otimes 1, \nu_2 \otimes 1]_l, \; \check{N} = \operatorname{Ker} J(\psi_{\mathfrak{C}})$$

and

$$N = \operatorname{Coker} J(\psi_{\mathfrak{C}}) = [\operatorname{Tor}(\nu_1, 1), \operatorname{Tor}(\nu_2, 1)]_l.$$

Then $\operatorname{Im} \vartheta$ is generated within $\check{L} \otimes X$ by the symbols $hp \otimes x$ with $hp \in \check{L}$ and $hx = 0$, and $\operatorname{Im} \theta$ is generated within $P/\check{L} \otimes X$ by the symbols $p \otimes x$ with $(\nu_1(p), -\nu_2(p)) = h(a_1, a_2)$ and $hx = 0$.

THEOREM 5′. (a) *$\check{M}$ is generated within $P \otimes X$ by the symbols $p \otimes x$ such that $(\nu_1(p), -\nu_2(p)) = h(a_1, a_2)$ and $hx = 0$ for some $h = 0, 1, 2, \ldots$.*

It contains $\operatorname{Im}(\check{L} \otimes X)$ as a subgroup, generated by those $p \otimes x$ with $h = 0$, with corresponding factor group isomorphic to $\operatorname{Tor}(L, X)/\operatorname{Im} \operatorname{Tor}(A_1 \oplus A_2, X)$ by the map κ defined as follows: Given generator $\tau_h(l, x)$ with $l = [(a_1, a_2)]$, that is $h(a_1, a_2) = (\nu_1(p), -\nu_2(p))$ and $hx = 0$, put $\kappa\tau_h(l, x) = p \otimes x$.

$\operatorname{Im}(\check{L} \otimes X)$ itself is isomorphic to $\check{L} \otimes X$ modulo the subgroup $\operatorname{Im} \vartheta$.

(b) *$M \simeq L \otimes X$, that is $[\nu_1 \otimes 1, \nu_2 \otimes 1]_l \simeq [\nu_1, \nu_2]_l \otimes X$ via the natural map.*

(c) *$\check{N} \simeq \operatorname{Tor}(\check{L}, X)$ by inclusion: For $p \in \check{L}$ and $i = 1, 2$*

$$\tau_h(p, x) \xrightarrow{\operatorname{Tor}(\nu_i, 1)} \tau_h(\nu_i(p), x) = 0.$$

(d) *$N = [\operatorname{Tor}(\nu_1, 1), \operatorname{Tor}(\nu_2, 1)]_l \simeq \operatorname{Tor}(A_1 \oplus A_2, X)$ modulo the subgroup generated by the symbols $\tau_h((\nu_1(p), -\nu_2(p)), x)$ with $hp = 0$ and $hx = 0$, a* *subgroup isomorphic to* $\mathrm{Tor}(P, X)/\mathrm{Im}\,\mathrm{Tor}(\check{L}, X)$ *by* $\tau_h(p, x) \to \tau_h((\nu_1(p), -\nu_2(p)), x)$, *with* $\mathrm{Im}\,\mathrm{Tor}(\check{L}, X) \simeq \mathrm{Tor}(\check{L}, X)$. N *contains a subgroup isomorphic to* $\mathrm{Im}\,\vartheta$ *by* $hp \otimes x \to \tau_h((\nu_1(p), -\nu_2(p)), x)$, *where* $hp \in \check{L}$ *and* $hx = 0$. *The corresponding factor group of* N *is isomorphic by the canonical map to the subgroup of* $\mathrm{Tor}(L, X)$ *generated by the symbols* $\tau_h([a_1, a_2], x)$ *with* $h(a_1, a_2) = 0$, *and* $\mathrm{Tor}(L, X)$ *modulo this subgroup is isomorphic to* $\mathrm{Im}\,\theta$ *via* $\tau_h([a_1, a_2], x) \to [p] \otimes x$, *where* $h(a_1, a_2) = (\nu_1(p), -\nu_2(p))$.

In terms of generators the maps in (19) are:

First row:

$$\begin{array}{ccccccc}
 & \tau_h([p], x) & & \tau_h((a_1, a_2), x) & & \tau_h([a_1, a_2], x) & \\
0 \to & \mathrm{Tor}(P/\check{L}, X) & \xrightarrow{J(\gamma)} & \mathrm{Tor}(A_1 \oplus A_2, X) & \xrightarrow{J(\delta)} & \mathrm{Tor}(L, X) & \to \\
 & 0 & & \tau_h((\nu_1(p), -\nu_2(p)), x) & & \tau_h([a_1, a_2], x) & \\
 & & & \text{with } h(\nu_1(p), -\nu_2(p)) = 0 & & \text{with } h(a_1, a_2) = 0 &
\end{array}$$

$$\begin{array}{ccccccc}
 & [p] \otimes x & & (a_1, a_2) \otimes x & & [a_1, a_2] \otimes x & \\
\xrightarrow{\theta} & P/\check{L} \otimes X & \xrightarrow{H(\gamma)} & (A_1 \oplus A_2) \otimes X & \xrightarrow{H(\delta)} & L \otimes X \to 0 & \\
 & [p] \otimes x \text{ with } hx = 0 & & (\nu_1(p), -\nu_2(p)) \otimes x & & [a_1, a_2] \otimes x & \\
 & \text{and } (\nu_1(p), -\nu_2(p)) = h(a_1, a_2) & & & & &
\end{array}$$

Second row:

$$\begin{array}{ccccccc}
 & \tau_h(p, x) \text{ with} & & & & & \\
 & (\nu_1(p), -\nu_2(p)) = 0 & & \tau_h(p, x) & & \tau_h([p], x) & \\
0 \to & \mathrm{Tor}(\check{L}, X) & \xrightarrow{J(\alpha)} & \mathrm{Tor}(P, X) & \xrightarrow{J(\beta)} & \mathrm{Tor}(P/\check{L}, X) & \to \\
 & 0 & & \tau_h(p, x) \text{ with} & & \tau_h([p], x) & \\
 & & & (\nu_1(p), -\nu_2(p)) = 0 & & \text{with } hp = 0 &
\end{array}$$

$$\begin{array}{ccccccc}
 & p \otimes x \text{ with} & & & & & \\
 & (\nu_1(p), -\nu_2(p)) = 0 & & p \otimes x & & [p] \otimes x & \\
\xrightarrow{\vartheta} & \check{L} \otimes X & \xrightarrow{H(\alpha)} & P \otimes X & \xrightarrow{H(\beta)} & P/\check{L} \otimes X & \longrightarrow 0 \\
 & hp \otimes x \text{ with} & & p \otimes x \text{ with} & & [p] \otimes x & \\
 & h(\nu_1(p), -\nu_2(p) = 0 & & (\nu_1(p), -\nu_2(p)) = 0 & & &
\end{array}$$

Remark. The case of the functors $\mathrm{Hom}(X, \ )$ and $\mathrm{Hom}(\ , X)$ can be treated similarly. Thus, for example, in the above notation,

$$[\mathrm{Hom}(1, \alpha_1), \mathrm{Hom}(1, \alpha_2)]_r \simeq \mathrm{Hom}(X, [\alpha_1, \alpha_2]_r),$$

$$[\mathrm{Hom}(\nu_1, 1), \mathrm{Hom}(\nu_2, 1)]_r \simeq \mathrm{Hom}([\nu_1, \nu_2]_l, X).$$

Added in proof: The four-lemma (Proposition 1) also appears in a recent publication of D. Puppe, *Korrespondenzen in abelschen Kategorien*, Math. Ann. *148* (1962), 1–30 (see p. 10, no. 3.1). It is proved there within the framework of his theory of abelian correspondences. His conjecture (see p. 18, no. 4.18) that important parts of the theory can be formulated by assuming "quasi-exact categories" (these are abelian categories without addition) only turns out to be true: Every such category can be imbedded canonically into an $\mathfrak{J}$-category of correspondences as defined by Puppe (proof to be published soon). This provides in particular a proof of the four-lemma (as well as of other elementary constructions of homological algebra) in quasi-exact categories. An elegant proof can also be given independently of the theory of correspondences under the same assumptions.

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