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EVEN EVERY JOIN-EXTENSION SOLVES A UNIVERSAL PROBLEM

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Abstract

Generalization to arbitrary join-extensions of a poset of a recent characterization of those which are complete lattices.

Schmidt (1974) devotes a just published article to showing that a complete lattice extension of a partially ordered set P, every element x of which is the sup of the elements $P \cap (x]$ it dominates in P, may be characterized as that complete lattice extension of P to which extend uniquely as complete sup morphisms all isotone maps, of P into complete lattices, which have only the $P \cap (x]$ as inverse images of principal ideals.

The restriction to complete lattices is superfluous: Every partially ordered extension E in which P is join-dense admits a unique extension, to an isotone map preserving the sups of the $P \cap (x]$, of every isotone map on P which sends these on subsets having a sup in the image: E can thus be uniquely mapped over P into every extension in which the $P \cap (x]$ have sups; and this characterizes it as extension. Moreover, the unique isotone extension will preserve all sups existing in E just when the inverse images of principal ideals contain, with any set of $P \cap (x]$, also their least upper bound under inclusion, whenever this exists. The dual property can be used to characterize sub-extensions of the MacNeille completion. Extensions of the identity, from a partially ordered group to join-semilattice-extensions, which preserve sups of the finite or bounded subsets, are also known from abstract treatments of classical ideal theory: cf. Jaffard (1960).

Let P be a subset of a partially ordered set E. The set $P \cap (x]$ of elements of P dominated by $x \in E$ is *initial*; i.e. it is closed under passage to smaller elements in P (it is also closed under those sups of E contained in P, but this

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property is not intrinsic to P). The extension E is thus mapped on a family of initial subsets of P — a map which is isotone (order preserving) — in fact, a complete inf morphism in that it sends every existing inf in E on the intersection of the images. In addition, it order embeds P — more generally, the sups in E of subsets of P — into the set of $P \cap (x]$ ordered by inclusion: for $x \in E$ dominates just those subsets of P contained in $P \cap (x]$. Indeed, restricted to the sups of subsets of P, it is even a *complete sup morphism* in that it sends every sup existing in E (by the associativity of sup this will also be the sup of a subset of P) on a smallest containing $P \cap (x]$ (which might properly contain the union of the images). This so ordered set may itself be regarded as a partially ordered extension of P via the identification of the latter's elements with their embedded images, the principal initial subsets they generate; as such it is a *join-extension* i.e. one consisting exclusively of sups of subsets of P; conversely, every join-extension is isomorphic to one such, Schmidt (1974).

The identity on P extends (uniquely) to a (necessarily isotone) map from one join-extension E to another F so as to send sups of the $P \cap (x]$ for $x \in E$ to their sups in F, whenever these exist: thus when each $P \cap (x]$ is contained in a smallest $P \cap (y]$ for $y \in F$; moreover it could be extended to a map back from F to E preserving the sup of $P \cap (x]$ only if y could be sent back on x in order preserving fashion over P, thus if $P \cap (y]$ where also contained in $P \cap (x]$. This would make $P \cap (y] = P \cap (x]$; and if it held for all $y \in F$, the identity on P would extend, uniquely so as to preserve sups of the $P \cap (x]$, to an isomorphism of E with F. Somewhat more generally, if F is an arbitrary (not necessarily a join-) extension of P into and from which the identity on P can be isotonely extended so as to preserve sups of the $P \cap (x]$ for $x \in E$, then E is isomorphic to a subextension of F onto which F can be retracted (by following the map of F on E with the embedding of E in F); and properly insofar as the image of E does not exhaust F. Since the identity on F preserves everything, this permits characterizing E among such F (as in Schmidt(1974))as that extension to which the identity on P extends uniquely. [By construing sup as an infinitary algebraic operation, this may be obtained by specialization from a similar characterization of extensions E generated by a partial algebra P.].

In addition to its mappability over the identity on P into any extension having sups for its $P \cap (x]$, a join-, or indeed any, extension E of P admits extensions of those isotone maps ϕ_0 from P to a partially ordered set F which send each of its $P \cap (x]$ on a subset having a sup $\phi(x)$ in F. This ϕ may be characterized as that isotone extension of ϕ_0 sending every $x \in E$ on a smallest possible image in F; alternatively, as the composition, of the canonical map from E to the set of $P \cap (x]$, with the unique extension of ϕ_0 from the image of P which preserves the sup of each $P \cap (x]$. Suppose S is any subset of E with a

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sup, x_0. Since ϕ is isotone, $\phi(x_0)$ is always an upper bound for the $\phi(x)$, $x \in S$; it can fail to be the sup only if $\phi(x) \leqq$ some $y \in F$ for which $\phi(x_0) \nleqq y$; and given that $\phi(x)$ is the sup of $\phi_0(P \cap (x])$, this comes to $P \cap (x] \subset \phi_0^{-1}(y]$ failing to imply $P \cap (x_0] \subset \phi_0^{-1}(y]$. Calling an initial subset of P for which this implication *does* hold, for every S in a family $\boldsymbol{S}$ of subsets of E, an $\boldsymbol{S}$*-ideal*, we may summarize this as a

THEOREM. *Let ϕ_0 be an isotone map from P sending every $P \cap (x]$, for x in an extension E of P, on a subset having a sup in the image. In order for the map which sends every $x \in E$ on this sup to preserve the sups in E of a family $\boldsymbol{S}$ of its subsets, it is necessary and sufficient that every $\phi_0^{-1}(y]$ be an $\boldsymbol{S}$-ideal.*

For the special case $E = P$, this reduces to Schmidt's Theorem 1. At the same time, it yields the harder part of his Theorem 2, inasmuch as every $P \cap (x]$ is an $\boldsymbol{S}$-ideal when $\boldsymbol{S}$ consists of subsets of sups in E of subsets of P.

Combining, we obtain the *Initial Characterization of Join-Extensions*: every isotone ϕ_0 on P for which $\phi_0(P \cap (x])$ has a sup for every x in a join-extension E, extends uniquely to E as an isotone map preserving sups of the $P \cap (x]$; sups of subsets for which the $\phi_0^{-1}(y]$ are ideals will also be preserved. Conversely, let F be an arbitrary extension of P having sups for the $P \cap (x]$, and suppose the identity on P extends to a map from F to E preserving these sups: then E is isomorphic (qua extension of P) to the sub-extension of these sups. Alternatively, if the identity on P extends to a self-map of F onto and preserving these sups, which is unique in any class of self-maps preserving these sups and including the identity on F, then F is isomorphic to E.

A dual characterization is also available for sub-extensions of the MacNeille completion M of P. The $P \cap (x]$ of M are just those initial subsets which contain all lower bounds to their upper bounds (in P): hence if such a $P \cap (x]$ — or more generally any subset of it having x for sup — has a y for sup in any other extension of P, then $P \cap (y] \subseteq P \cap (x]$; consequently the extension of the identity on P so as to preserve a set of sups in M of subsets of P, is necessarily an isomorphism into the other extension. Combining this with our previous remarks on extendability of maps, we obtain the *Final Characterization of Sub-Extensions of the MacNeille Completion*: every isotone $\phi_0 : Q \to P$ extends to an isotone ϕ from an extension F of Q into the subextension $E \subset M$ of P consisting of all sups in M of $\phi_0(Q \cap (y])$ for $y \in F$; moreover ϕ can be chosen to preserve all sups existing in F of the $Q \cap (y]$. Conversely, if the identity on P extends to an isotone $\phi : E \to F$ preserving sups (of subsets of P) which exhaust $E \subset M$, then ϕ is an isomorphism of this subextension of M with its image in F — alternatively, if F is also exhausted by these sups, it is isomorphic to E.

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