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CHAINS OF VARIETIES

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Summary. If $\mathfrak{V}$ is a variety of groups that can be defined by n-variable laws and $\mathfrak{V}^{(m)}$ is the variety all of whose m-generator groups are in $\mathfrak{V}$ then there corresponds the chain: $\mathfrak{V}^{(1)} \geqq \mathfrak{V}^{(2)} \geqq \ldots \geqq \mathfrak{V}^{(n)} = \mathfrak{V}$. In this paper such chains are investigated to determine which of the inclusions are proper for certain varieties $\mathfrak{V}$. In particular the inclusions are shown to be all proper for the varieties $\mathfrak{N}_c{}^{(c)}$, $(\mathfrak{N}_c\mathfrak{A})^{(2c)}$, $\mathfrak{C}$, where $\mathfrak{N}_c$ is the variety of nilpotent-of-class-c groups, $\mathfrak{A}$ is the abelian variety and $\mathfrak{C} = (\mathfrak{C}^{(5)})$ is the variety of centre-by-metabelian groups. For $\mathfrak{A}\mathfrak{N}_c$ ($c \geqq 3$) the inclusions are likewise proper but for $\mathfrak{V} = (\mathfrak{A}\mathfrak{N}_2 \wedge \mathfrak{N}_6)$ the corresponding chain is: $\mathfrak{V}^{(1)} > \mathfrak{V}^{(2)} > \mathfrak{V}^{(3)} > \mathfrak{V}^{(4)} = \mathfrak{V}^{(5)} > \mathfrak{V}^{(6)} = \mathfrak{V}$. The remainder of the paper is devoted to the study of $\mathfrak{N}_{n+k}^{(n)}$-groups.

1. Introduction. Let $\mathfrak{V}$ be a variety of groups that can be defined by n-variable laws for some $n \geqq 1$ and consider the chain

$$\mathfrak{V}^{(1)} \geqq \mathfrak{V}^{(2)} \geqq \ldots \geqq \mathfrak{V}^{(n)} = \mathfrak{V}, \tag{1.1}$$

where $\mathfrak{V}^{(m)}$ is the variety of all those groups whose m-generator subgroups belong to $\mathfrak{V}$. For $\mathfrak{N}_c$, the variety of nilpotent-of-class-c groups, it is known that $\mathfrak{N}_2{}^{(1)} > \mathfrak{N}_2{}^{(2)} > \mathfrak{N}_2{}^{(3)} = \mathfrak{N}_2$ (Levi-Van der Waerden [**8**]) and $\mathfrak{N}_c{}^{(c)} = \mathfrak{N}_c{}^{(c+1)} = \mathfrak{N}_c$ ($c \geqq 3$) (Heineken [**5**], Macdonald [**10**]). For $\mathfrak{M}$, the metabelian variety, we have $\mathfrak{M}^{(1)} > \mathfrak{M}^{(2)} > \mathfrak{M}^{(3)} > \mathfrak{M}^{(4)} = \mathfrak{M}$ (B. H. Neumann [**14**]; c.f. Theorem 4.2 for an alternative proof). Further related results may be found in Macdonald [**11**].

In this paper we construct a series of examples which enable us to determine the chain (1.1) for certain varieties which can be defined by single (complex) commutator words. For instance we show that if $\mathfrak{V} = \mathfrak{N}_c (c \geqq 3)$ then $\mathfrak{V}^{(1)} > \mathfrak{V}^{(2)} > \ldots > \mathfrak{V}^{(c)} = \mathfrak{V}^{(c+1)} = \mathfrak{V}$ (Theorem 3.5); if $\mathfrak{V} = \mathfrak{N}_c\mathfrak{A} (c \geqq 2)$ then $\mathfrak{V}^{(1)} > \mathfrak{V}^{(2)} > \ldots > \mathfrak{V}^{(2c+1)} = \mathfrak{V}^{(2c+2)} = \mathfrak{V}$ (Theorem 4.1); if $\mathfrak{V} = \mathfrak{C}$, the centre-by-metabelian variety, then $\mathfrak{V}^{(1)} > \mathfrak{V}^{(2)} > \ldots > \mathfrak{V}^{(5)} = \mathfrak{V}$ (Theorem 4.3) and if $\mathfrak{V} = \mathfrak{A}\mathfrak{N}_c (c \geqq 3)$ then $\mathfrak{V}^{(1)} > \mathfrak{V}^{(2)} > \ldots > \mathfrak{V}^{(2c)} = \mathfrak{V}^{(2c+1)} = \mathfrak{V}^{(2c+2)} = \mathfrak{V}$ (Theorem 4.5). In contrast to these inclusions we show that if $\mathfrak{V} = \mathfrak{A}\mathfrak{N}_2 \wedge \mathfrak{N}_6$ then $\mathfrak{V}^{(1)} > \mathfrak{V}^{(2)} > \mathfrak{V}^{(3)} > \mathfrak{V}^{(4)} = \mathfrak{V}^{(5)} > \mathfrak{V}^{(6)} = \mathfrak{V}$ (Theorem 4.8). To the authors' knowledge this type of chain has not been known pre-

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viously for varieties of groups. The remainder of the paper is devoted to the study of some general properties of $\mathfrak{N}_{n+k}^{(n)}$-groups where time and again we refer to our examples to show that the results obtained are to some extent best possible. For instance we show that if $G = F_\infty(\mathfrak{N}_{n+1}^{(n)})$ $(n \geqq 3)$, then $G'' \leqq \zeta_{n-1}(G)$ but $G'' \nleqq \zeta_{n-2}(G)$ (Theorem 6.1).

2. Notation. Unless otherwise specified all notation is standard and follows that of Hanna Neumann [**15**].

3. Examples. Let $n \geqq 2$ be a fixed positive integer and let $A(n)$ be the ring of polynomials in non-commuting variables $X_{n+1} \cup Y_{n+1}$ over Z, where $X_{n+1} = \{x_1, \ldots, x_{n+1}\}$ and $Y_{n+1} = \{y_1, \ldots, y_{n+1}\}$. Let $B(n)$ be the basic ideal of $A(n)$; that is the ideal generated by $X_{n+1} \cup Y_{n+1}$. We are interested in the ring $B(n)$ and certain ideals of $B(n)$; but in order to describe these ideals we need to explain certain terms. A monomial of length $m (m > 0)$ in the ring $B(n)$ is an element of the form $z_1 \ldots z_m$ in $B(n)$ with $z_i \in X_{n+1} \cup Y_{n+1}$, $i = 1, \ldots, m$. We say $z_1 \ldots z_m$ has a repeated x-entry to mean that for some k, l satisfying $1 \leqq k < l \leqq m$, $z_k = z_l \in X_{n+1}$. We say $z_1 \ldots z_m$ has r y-entries to mean that the number of $z_i, (i = 1, \ldots, m)$, such that $z_i \in Y_{n+1}$ is precisely r. For each positive integer k, we define five ideals of $B(n)$ as follows:

J$(n, k, 1)$ = The ideal of $B(n)$ generated by all monomials of length $n + k + 2$ in $B(n)$.

$J(n, k, 2)$ = The ideal of $B(n)$ generated by all monomials of length $n + k + 1$ in $B(n)$ with a repeated x-entry.

$J(n, k, 3)$ = The ideal of $B(n)$ generated by all monomials of length $n + k + 1$ in $B(n)$ in which the number of y-entries is different from k.

$J(n, k, 4)$ = The ideal of $B(n)$ generated by all elements of $B(n)$ of the form $z_1 \ldots z_r + z_{1\sigma} \ldots z_{r\sigma}$ where $r = n + k + 1$, $z_i \in X_{n+1} \cup Y_{n+1}$, $(i = 1, \ldots, r)$, and σ is any odd permutation of $\{1, \ldots, r\}$ fixing those indices j for which $z_j \in Y_{n+1}$.

$J(n, k)$ = The ideal of $B(n)$ generated by $J(n, k, 1)$, $J(n, k, 2)$, $J(n, k, 3)$ and $J(n, k, 4)$.

The rings that we shall need are the quotients

$$R(n, k) = B(n)/J(n, k).$$

We will say that a monomial $z_1 \ldots z_{n+k+1}$ in $R(n, k)$ is in canonical form if, whenever $z_i = x_{\alpha_i}$, $z_j = x_{\alpha_j}$, then $i < j$ if and only if $\alpha_i < \alpha_j$ for all $i, j \in \{1, \ldots, n + k + 1\}$, $\alpha_i, \alpha_j \in \{1, \ldots, n + 1\}$. By making repeated use of $J(n, k, 4)$, we can clearly reduce every monomial of weight $n + k + 1$ that is not zero in $R(n, k)$ to its canonical form. Thus the additive group of $R(n, k)$ is the free abelian group, freely generated by all distinct monomials of weight $1, \ldots, n + k$ in variables $X_{n+1} \cup Y_{n+1}$ together with those monomials of weight $n + k + 1$ which have k y-entries and are in canonical form. In par-

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ticular $R(n, k)^{n+k+1}$ is freely generated, as an abelian group, by monomials of weight $n + k + 1$ with k y-entries and in canonical form.

For $\rho_i \in R(n, k)$ we define the ring commutator $\langle \rho_1, \rho_2 \rangle = \rho_1\rho_2 - \rho_2\rho_1$, and, inductively for $m > 2$, $\langle \rho_1, \ldots, \rho_m \rangle = \langle \langle \rho_1, \ldots, \rho_{m-1} \rangle, \rho_m \rangle$ defines the left-normed ring commutator of weight m. In order to reduce complication in notation, we shall occasionally use the semicolon to separate the commutator signs. For instance we shall write $\langle \rho_1, \rho_2; \rho_3, \rho_4 \rangle$ to mean $\langle \langle \rho_1, \rho_2 \rangle, \langle \rho_3, \rho_4 \rangle \rangle$. A complex ring commutator of weight m in $\rho_1, \ldots, \rho_r$ is any expression of the form $\langle t_1, \ldots, t_{i_1}; t_{i_1+1}, \ldots, t_{i_2}; \ldots; \ldots, t_m \rangle$ where $t_i \in \{\rho_1, \ldots, \rho_r\}$. We shall denote by $\langle \rho_{1,(r)}\rho_2 \rangle$ the expression $\langle \rho_1, \rho_2, \ldots, \rho_2 \rangle$ where ρ_2 occurs $r > 0$ times.

Where there is no ambiguity we shall write R for $R(n, k)$. Since $R^{n+k+2} = 0$, for the purpose of studying R^{n+k+1}, we may assume that each $\rho_i \in R$ is of the form

$$(3.1) \qquad \begin{cases} \rho_i = \zeta_i + \eta_i \quad \text{where } \zeta_i = \sum_{j=1}^{n+1} \zeta_{ij} x_j, \ \eta_i \quad \sum_{t=1}^{n+1} \eta_{ij} y_j; \\ \zeta_{ij}, \eta_{ij} \in Z. \end{cases}$$

LEMMA 3.1. *In* $R(n, k)$, $\rho_1 \ldots \rho_{n+k+1} = 0$ *if for some* $1 \leqq r < s \leqq n + k + 1$ $\rho_r = \rho_s = \sum_{j=1}^{n+1} \zeta_{rj} x_j$.

Proof. On expanding $\rho_1 \ldots \rho_{n+k+1}$ as a linear combination of monomials, and deleting those terms that lie in $J(n, k, 2)$ and $J(n, k, 3)$ we are left with a term in $J(n, k, 4)$; because, for every monomial $z_1 \ldots z_{r-1} x_i z_{r+1} \ldots z_{s-1} x_j z_{s+1} \ldots z_{n+k+1}$, $z_i \in X_{n+1} \cup Y_{n+1}$ in the expansion we also have

$$z_1 \ldots z_{r-1} x_j z_{r+1} \ldots z_{s-1} x_i z_{s+1} \ldots z_{n+k+1}$$

in the expansion.

LEMMA 3.2. *In* $R(n, k)$, $\rho_1 \ldots \rho_{n+k+1} = 0$ *if* $|\{\rho_1, \ldots, \rho_{n+k+1}\}| \leqq n$.

Proof. In the expansion of $\rho_1 \ldots \rho_{n+k+1}$ as a linear combination of monomials only terms which involve precisely k y-entries need be considered. Let $\Lambda \subseteq \{1, 2, \ldots, n + k + 1\}$ and $|\Lambda| = k$, and let t_Λ denote the linear combination of those monomials $z_1 \ldots z_{n+k+1}$ in the expansion of $\rho_1 \ldots \rho_{n+k+1}$ such that $z_i \in Y_{n+1}$ if and only if $i \in \Lambda$. By 3.1, $t_\Lambda = \sigma_1 \ldots \sigma_{n+k+1}$ where $\sigma_i = \zeta_i$ if $i \notin \Lambda$ and $\sigma_i = \eta_i$ if $i \in \Lambda$. Since $|\{\rho_1, \ldots, \rho_{n+k+1}\}| \leqq n$, there exist integers r, s such that $1 \leqq r < s \leqq n + k + 1$, $r, s \notin \Lambda$ and $\rho_r = \rho_s$. Thus $\zeta_r = \zeta_s$. By Lemma 3.1, $t_\Lambda = 0$. Since $\rho_1 \ldots \rho_{n+k+1} = \sum_\Lambda t_\Lambda$, we conclude that $\rho_1 \ldots \rho_{n+k+1} = 0$.

Since any complex ring commutator of weight m in $\rho_1, \ldots, \rho_r$ can be expressed as a homogeneous polynomial of degree m in $\rho_1, \ldots, \rho_r$, we obtain the following result as an immediate corollary to Lemma 3.2.

LEMMA 3.3. *A complex ring commutator of weight* $n + k + 1$ *in* $\rho_1, \ldots, \rho_n$ *is* 0 *in* $R(n, k)$.

By a result of Magnus [**13**, Chapter 5], the elements $1 + \rho$, $\rho \in R(n, k)$ generate a nilpotent group of class $n + k + 1$ under multiplication. We denote this group by $G(n, k)$. If we denote by the square brackets the usual group commutator, then observe that for $1 + \rho_i \in G(n, k)$,

$$(3.2)\quad [1 + \rho_1, \ldots, 1 + \rho_{i_1}; 1 + \rho_{i_1+1}, \ldots, 1 + \rho_{i_2}; \ldots; \ldots, 1 + \rho_{n+k+1}] \\ = 1 + \langle \rho_1, \ldots, \rho_{i_1}; \rho_{i_1+1}, \ldots, \rho_{i_2}; \ldots; \ldots, \rho_{n+k+1} \rangle$$

in $R(n, k)$. In particular,

$$(3.3)\quad [1 + \rho_1, \ldots, 1 + \rho_{n+k+1}] = 1 + \langle \rho_1, \ldots, \rho_{n+k+1} \rangle$$

in $R(n, k)$. From this observation and Lemma 3.3. we obtain the following

LEMMA 3.4. $G(n, k) \in \mathfrak{N}_{n+k}^{(n)}$ $(n > 1, k > 0)$.

However, $G(n, k)$ is not nilpotent of class $n + k$, for we next prove the following

LEMMA 3.5. $G(n, k) \notin \mathfrak{N}_{n+k}^{(n+1)}$ $(n > 1, k > 0)$.

Proof. It suffices to find a non-trivial $n + k + 1$ weight left-normed group commutator in $G(n, k)$, or, equivalently, in view of (3.3), to find a non-trivial ring commutator in $R(n, k)$ of weight $n + k + 1$ involving $n + 1$ elements. Let

$$c = \langle x_2, x_3, {}_{(k+1)}(x_1 + y_1), x_4, \ldots, x_{n+1} \rangle$$

and $c_i = \langle x_2, x_{3,(i)}y_1, x_{1,(k-i)}y_1, x_4, \ldots, x_{n+1} \rangle$, $i = 0, \ldots, k$. By virtue of Lemma 3.1 and the fact that ring commutators are multilinear with respect to their components in $R(n, k)$, $c = \sum_{i=0}^{k} c_i$. If $k > 2$ and $1 < i \leqq k$, then the monomial $y_1^{k-1}x_1y_1x_2 \ldots x_{n+1}$ does not occur in the expansion of c_i as a sum of monomials in canonical form. For, $\langle x_2, x_3, y_1, y_1 \rangle = (x_2x_3 - x_3x_2)y_1^2 - 2y_1(x_2x_3 - x_3x_2) + y_1^2(x_2x_3 - x_3x_2)$. Thus $y_1^{k-1}x_1y_1x_2 \ldots x_{n+1}$ does not occur in $\langle (x_2x_3 - x_3x_2)y_1^2 - 2y_1(x_2x_3 - x_3x_2) + y_1^2(x_2x_3 - x_3x_2)), z_3, \ldots, z_{k+1}, x_4, \ldots, x_{n+1} \rangle$ where one of the z_i is equal to x_1 and the rest equal to y_1. Thus the coefficient of the nomomial $y_1^{k-1}x_1y_1x_2 \ldots x_{n+1}$ in the expansion of c is the same as the coefficient in the expansion of $c_0 + c_1$. Observe that

$$z_1 \ldots z_r(\langle x_i, x_j, x_k \rangle)z_{r+4} \ldots z_{n+k+1} = 0 \quad \text{if} \quad z_i \in X_{n+1} \cup Y_{n+1},$$

for it is equal to

$$z_1 \ldots z_r(x_ix_jx_k - x_jx_ix_k - x_kx_ix_j + x_kx_jx_i)z_{r+4} \ldots z_{n+k+1}$$

which is in $J(n, k, 4)$. Thus $c_0 = 0$. Now observe that the coefficient of $y_1^{k-1}x_1y_1x_2 \ldots x_{n+1}$ in the expansion of c_1 as linear combination of nomomials in canonical form is the same as that of each of the following commutators:

$$\langle 2x_1y_1x_2x_{3,(k-1)}y_1, x_4, \ldots, x_{n+1} \rangle,$$
$$\langle -2y_1x_1y_1x_2x_{3,(k-2)}y_1, x_4, \ldots, x_{n+1} \rangle, \ldots,$$
$$\langle (-1)^{k-1}2y_1^{k-1}x_1y_1x_2x_3, x_4, \ldots, x_{n+1} \rangle, \ldots,$$
$$\langle (-1)^{k-1}2y_1^{k-1}x_1y_1x_2 \ldots x_n, x_{n+1} \rangle.$$

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In each of these the coefficient of $y_1^{k-1}x_1y_1x_2 \ldots x_{n+1}$ is $2 \cdot (-1)^{k-1} \neq 0$, so that $c \neq 0$ in $R(n, k)$.

THEOREM 3.6. $\mathfrak{N}_c^{(1)} > \mathfrak{N}_c^{(2)} > \ldots > \mathfrak{N}_c^{(c)} = \mathfrak{N}_c^{(c+1)} = \mathfrak{N}_c$ $(c \geqq 3)$.

Proof. Since $\mathfrak{N}_c^{(1)}$ is the variety of all groups, $\mathfrak{N}_c^{(1)} > \mathfrak{N}_c^{(2)}$; and the equality $\mathfrak{N}_c^{(c)} = \mathfrak{N}_c^{c+1}$ is due to Heineken and Macdonald as mentioned in the introduction. The inclusions $\mathfrak{N}_c^{(m)} > \mathfrak{N}_c^{(m+1)}$ $(2 \leqq m \leqq c - 1)$ follow from Lemmas 3.4 and 3.5 by considering $G(m, c - m)$.

Let $H(n, k)$ be the subgroup of $G(n, k)$ generated by $1 + x_1 + y_1$, $1 + x_2, \ldots, 1 + x_{n+1}$. Then $H(n, k)$ is a finitely generated torsion free group of class precisely $n + k + 1$ all of whose n generator subgroups are of class at most $n + k$. It follows by a well-known result of Gruenberg [**1**] that $H(n, k)$ is residually a finite p-group for every prime p. Thus we obtain the following generalization of a result of Gupta-Gupta-Newman [**3**].

THEOREM 3.7. *For any integers $n \geqq 2$, $k \geqq 1$ and every prime p, there is a finite p-group of nilpotency class precisely $n + k + 1$, all of whose n-generator subgroups are nilpotent of class at most $n + k$.*

Remark 1. Theorem 3.7 can also be proved independent of Gruenberg's result by replacing the algebra $A(n)$ by $A^*(n) = Z_{p^t}[X_{n+1} \cup Y_{n+1}]$, where p^t does not divide $k + 2$, and using arguments similar to above except that Z_{p^t} replaces Z wherever it occurs.

4. The chain problem. In the previous section we showed that for $c \geqq 3$, $\mathfrak{N}_c^{(1)} > \ldots > \mathfrak{N}_c^{(c)}$. In this section similar results for $\mathfrak{N}_c\mathfrak{A}$ $(c \geqq 2)$, $\mathfrak{A}\mathfrak{N}_c$ $(c \geqq 2)$ and $\mathfrak{C}$ will be obtained. In addition we give an alternative proof of B. H. Neumann's result that $\mathfrak{M}^{(1)} > \ldots > \mathfrak{M}^{(4)}$, where $\mathfrak{M}$ is the variety of metabelian groups.

THEOREM 4.1. *Let* $\mathfrak{B} = \mathfrak{N}_c\mathfrak{A}$ $(c \geqq 2)$. *Then*

$$\mathfrak{B}^{(1)} > \mathfrak{B}^{(2)} > \ldots > \mathfrak{B}^{(2c+1)} = \mathfrak{B}^{(2c+2)} = \mathfrak{B}.$$

Proof. The equality $\mathfrak{B}^{(2c+1)} = \mathfrak{B}$ is due to Macdonald [**11**] and $\mathfrak{B}^{(1)} > \mathfrak{B}^{(2)}$ is obvious. To prove $\mathfrak{B}^{(m)} > \mathfrak{B}^{(m+1)}$ $(2 \leqq m \leqq 2c)$, we consider again the group $G(m, 2c - m + 1)$ which by Lemma 3.3 belongs to $\mathfrak{B}^{(m)}$ and we show that a certain commutator of weight $2c + 2$ in $m + 1$ variables is non-zero in $R(m, 2c - m + 1)$. For the following argument we set $\rho_i = x_i + y_i$ for all i considered.

Case 1. $m = 2c$. We look at the coefficient of $y_2x_1x_2 \ldots x_{m+1}$ in the expansion of $t = \langle \rho_1, \rho_2; \rho_2, \rho_3; \rho_4, \rho_5; \ldots; \rho_m, \rho_{m+1} \rangle$ as a linear combination of monomials in canonical form. Since the coefficient of $y_2x_1x_2x_3$ in the expansion of $\langle \rho_1, \rho_2; \rho_2, \rho_3 \rangle$ is equal to -4, the coefficient of $y_2x_1 \ldots x_{m+1}$ in t is $-4 \cdot 2^{((m+1)-3)/2} = -2^{c+1}$ since $m = 2c$.

Case 2. m even and $2 \leqq m < 2c$. We look at the coefficient of $y_1y_2(y_1y_3)^n y_3x_1 \ldots x_{m+1}$ in the expansion of

$$t = \langle \rho_1, \rho_2, {}_{(n+1)} \langle \rho_1, \rho_3 \rangle; \rho_2, \rho_3; \rho_4, \rho_5; \ldots; \rho_m, \rho_{m+1} \rangle$$

as a linear combination of monomials in canonical form. Here $n = (2c - m - 2)/2$. Notice that the coefficient of $y_1y_2(y_1y_3)^n y_3x_1 \ldots x_{m+1}$ in t is the same as that in the expansion of each of the following commutators:

$$\langle y_1y_{2,(n)}(y_1y_3), \langle \rho_1, \rho_3 \rangle, \langle \rho_2, \rho_3 \rangle, \ldots, \langle \rho_m, \rho_{m+1} \rangle \rangle,$$
$$\langle y_1y_2(y_1y_3)^n, \langle \rho_1, \rho_3 \rangle, \langle \rho_2, \rho_3 \rangle, \langle \rho_4, \rho_5 \rangle, \ldots, \langle \rho_m, \rho_{m+1} \rangle,$$
$$\langle y_1y_2(y_1y_3)^n, - y_3x_1, 2x_2x_3, 2x_4x_5, \ldots, 2x_mx_{m+1} \rangle.$$

In all these cases, the coefficient of $y_1y_2(y_1y_3)^n y_3x_1 \ldots x_{m+1}$ is $-2^{m/2}$.

Case 3. m odd. In this case let

$$t = \langle \rho_1, \rho_2;_{(n+1)} \langle \rho_1, \rho_3 \rangle; \rho_2, \rho_4; \rho_5, \rho_6; \ldots; \rho_m, \rho_{m+1} \rangle$$

where $n = (2c - m - 1)/2$. The coefficient of $y_1y_2(y_1y_3)^n x_1 \ldots x_{m+1}$ in the expansion of t as a linear combination of monomials in canonical form is the same as in each of the following commutators:

$$\langle y_1y_{2,(n)}(y_1y_3), 2x_1x_3, 2x_2x_4, 2x_5x_6, \ldots, 2x_mx_{m+1} \rangle,$$
$$\langle 4y_1y_2(y_1y_3)^n x_1x_3x_2x_4, 2x_5x_6, \ldots, 2x_mx_{m+1} \rangle,$$
$$\langle -4y_1y_2(y_1y_3)^n x_1x_2x_3x_4, 2x_5x_6, \ldots, 2x_mx_{m+1} \rangle.$$

In each case the coefficient is $-2^{(m+1)/2}$.

Thus in each of the three cases $t \neq 0$ and hence $\mathfrak{V}^{(m)} > \mathfrak{V}^{(m+1)}$ for all m satisfying $2 \leqq m \leqq 2c$.

As a further application of our techniques we give an alternative proof of the following theorem.

THEOREM 4.2 (B.H. Neuman [**14**]). $\mathfrak{M}^{(2)} > \mathfrak{M}^{(3)} > \mathfrak{M}^{(4)}$.

Proof. To show $\mathfrak{M}^{(2)} > \mathfrak{M}^{(3)}$ it suffices to show that $G(2, 1) \notin \mathfrak{N}^{(3)}$; for $G(2, 1) \in \mathfrak{N}_3{}^{(2)}$ by Lemma 3.4 and $\mathfrak{N}_3{}^{(2)} \subseteq \mathfrak{M}^{(2)}$. In the expansion of $\langle x_1 + y_1, x_2; x_1 + y_1, x_3 \rangle$ as a linear combination of monomials in canonical form, the coefficient of $y_1x_1x_2x_3$ is -4; for it is the same as the coefficient of $y_1x_1x_2x_3$ in $\langle 2x_1x_2 + y_1x_2; 2x_1x_3 + y_1x_3 \rangle$.

To show that $\mathfrak{M}^{(3)} > \mathfrak{M}^{(4)}$ we consider $R^*(2, 1) = R(2, 1)/I_4$ where I_4 is the ideal $\{4\rho; \rho \in R(2, 1)\}$, and the corresponding group $G^*(2, 1) = 1 + R^*(2, 1)$ under multiplication. Since $\langle y_1, x_1; x_2, x_3 \rangle = 2y_1x_1x_2x_3 - 2x_1y_1x_2x_3 - 2x_1x_2y_1x_3 + 2x_1x_2x_3y_1 \neq 0$ in $R^*(2, 1)$, it follows that

$$G^*(2, 1) \notin \mathfrak{M}^{(4)} = \mathfrak{M}.$$

To show that $G^*(2, 1) \in \mathfrak{M}^{(3)}$, it suffices by a result of Macdonald [**10**] to show that $\langle \rho_1, \rho_2; \rho_1, \rho_3 \rangle = 0$ for all $\rho_i \in R^*(2, 1)$, $i = 1, 2, 3$. Write $\rho_i = \zeta_i + \eta_i$ (see 3.1) and use Lemma 3.1 to obtain

$$\langle \rho_1, \rho_2; \rho_1, \rho_3 \rangle = \langle \zeta_1, \rho_2; \eta_1, \rho_3 \rangle + \langle \eta_1, \rho_2; \zeta_1, \rho_3 \rangle + \langle \eta_1, \rho_2; \eta_1\rho_3 \rangle.$$

Now $\langle \eta_1, \rho_2; \eta_1, \rho_3 \rangle = 0$ in $R(2, 1)$ for it lies in $J(2, 1, 3)$.
For the same reason, $\langle \xi_1, \rho_2; \eta_1, \rho_3 \rangle = \langle \xi_1, \xi_2; \eta_1, \xi_3 \rangle$ so that

$$\begin{aligned}\langle \rho_1, \rho_2; \rho_1, \rho_3 \rangle &= 2\xi_1\xi_2(\eta_1\xi_3 - \xi_3\eta_1) - 2(\eta_1\xi_3 - \xi_3\eta_1)\xi_1\xi_2 + 2(\eta_1\xi_2 - \xi_2\eta_1)\xi_1\xi_3 \\ &\quad - 2\xi_1\xi_3(\eta_1\xi_2 - \xi_2\eta_1) \\ &= 4\xi_1\xi_2\eta_1\xi_3 - 4\xi_1\xi_2\xi_3\eta_1 - 4\eta_1\xi_1\xi_2\xi_3 + 4\xi_1\eta_1\xi_2\xi_3 = 0 \text{ in } R^*(2, 1).\end{aligned}$$

Remark 2. We have B. H. Neumann's example showing $\mathfrak{M}^{(2)} > \mathfrak{M}^{(3)}$ is a 2-group. Recently, C. K. Gupta [**2**] has shown the existence of a torsion free group in $\mathfrak{M}^{(2)}$ and not in $\mathfrak{M}^{(3)}$. Note the $G(2, 1)$ is also a torsion free group, but it lacks other interesting features of C. K. Gupta's group.

We now consider the variety $\mathfrak{C}$ of centre-by-metabelian groups which is defined by the law $[x, y; u, v; w] = 1$.

THEOREM 4.3. $\mathfrak{C}^{(2)} > \mathfrak{C}^{(3)} > \mathfrak{C}^{(4)} > \mathfrak{C}^{(5)}$.

Proof. The group $G(2, 2) \in \mathfrak{C}^{(2)}$ and to show $G(2, 2) \notin \mathfrak{C}^{(3)}$, we note that in $R(2, 2)$, $\langle x_1 + y_1, x_2 + y_2; x_1 + y_1, x_3 + y_3; x_1 + y_1 \rangle \neq 0$ as the sum of the coefficients of $y_1^2x_1x_2x_3$ is 4. Similarly $G(3, 1) \in \mathfrak{C}^{(3)}$ and to show $G(3, 1) \notin \mathfrak{C}^{(4)}$ we note that in $R(3, 1)$, $\langle x_1 + y_1, x_2 + y_2; x_1 + y_1, x_3 + y_3; x_4 + y_4 \rangle \neq 0$ as the sum of the coefficients of $y_1x_1x_2x_3x_4$ is -4.

The final inequality $\mathfrak{C}^{(4)} > \mathfrak{C}^{(5)}$ requires a somewhat different approach†. Let $R_5 = Z[x_1, \ldots, x_5]/I(x_{i(1)} \ldots x_{i(6)})$, where $I(x_{i(1)} \ldots x_{i(6)})$ is the ideal generated by all monomials of length 6. Let G_5 be the multiplicative group generated by $1 + x_1, \ldots, 1 + x_5$. Then G_5 is the free nilpotent-of-class-5 group freely generated by $1 + x_i$, $i = 1, \ldots, 5$ (see for instance [**13** Chapter 5]) and the mapping $[1 + x_{i(1)}, \ldots, 1 + x_{i(5)}] \rightarrow \langle x_{i(1)}, \ldots, x_{i(5)} \rangle$ defines a homomorphism of $\gamma_5(G_5)$ onto the additive subgroup K_5 of R_5 generated by all Lie-elements of the form $\langle x_{i(1)}, \ldots, x_{i(5)} \rangle$.

The laws defining $\mathfrak{C}^{(4)}$ correspond thus to the subgroup A_5 of K_5 generated by all elements of the form

(*) $$\langle x_{i(1)}, x_{i(2)}; x_{i(3)}, x_{i(4)}; x_{i(5)} \rangle$$

with $|\{i(1), \ldots, i(5)\}| \leqq 4$ and

(**) $$\langle x_{i(1)}, x_{i(2)}; x_{i(3)}, x_{i(4)}; x_{i(5)} \rangle + \langle x_{i(1\tau)}, x_{i(2\tau)}; x_{i(3\tau)}, x_{i(4\tau)}; x_{i(5\tau)} \rangle$$

with $|\{i(1), \ldots, i(5)\}| = 5$ and τ any transposition of $\{1, \ldots, 5\}$. Thus to show $\mathfrak{C}^{(4)} > \mathfrak{C}^{(5)}$ it is enough to show that $c = \langle x_1, x_2; x_3, x_4; x_5 \rangle \notin \bar{A}_5$, where $\bar{A}_5$ is the subgroup of A_5 generated by all elements of the form (**). It follows from the work of Macdonald [**10**] that $\bar{A}_5$ contains $2c$ so that $\bar{A}_5$ is generated by all elements of the form $c + c\sigma$ where σ is any permutation of $\{1, \ldots, 5\}$ and $c\sigma = \langle x_{1\sigma}, x_{2\sigma}; x_{3\sigma}, x_{4\sigma}; x_{5\sigma} \rangle$.

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†This was also proved independently by Dr. M. F. Newman whom we thank for communicating the proof. The proof given here is different.

Let B_5 be the subgroup of K_5 generated by all elements of the form $\langle x_{i(1)}, x_{i(2)}; x_{i(3)}, x_{i(4)}; x_{i(5)}\rangle$ with $|\{i(1), \ldots, i(5)\}| = 5$. Since $\langle x, y\rangle = -\langle y, x\rangle$, and $\langle x, y; z, t\rangle = -\langle z, t; x, y\rangle$ it follows that B_5 is generated by all elements of the form

$c_{ij} = \langle x_1, x_i; x_k, x_l; x_j\rangle \quad (k > l) \quad i, j \in \{2, 3, 4, 5\}$ and i, j, k, l all distinct

and

$$d_i = \langle x_2, x_i; x_k, x_l; x_1\rangle \quad (k > l) \quad i = 3, 4, 5.$$

There are 12 c_{ij}'s and 3 d_i's and we first of all note that these generate B_5 freely. Indeed let $\sum\delta_i d_i + \sum\delta_{ij}c_{ij} = 0$. To ease the notation we write $ijk\ldots$ for $x_ix_jx_k\ldots$. Then

$$\begin{aligned} c_{ij} &= (1ikl - 1ilk - i1kl + i1lk - kl1i + kli1 + lk1i - lki1)j \\ &\quad + j(1ilk - 1ikl + i1kl - i1lk + kl1i - kli1 - lk1i + lki1) \\ d_i &= (2ikl - 2ilk - i2kl + i2lk - kl2i + kli2 + lk2i - lki2)1 \\ &\quad + 1(2ilk - 2ikl + i2kl - i2lk + kl2i - kli2 - lk2i + lki2). \end{aligned}$$

Now the coefficients of 12345, 12354, 12435, 12534, 12453, 12543, 13245, 13254, 13425, 13452, 13524 and 13542 are, respectively, $\delta_3 - \xi_{25}$, $-\delta_3 - \xi_{24}$, $\delta_4 + \xi_{25}$, $\delta_5 + \xi_{24}$, $-\delta_4 - \xi_{23}$, $-\delta_5 + \xi_{23}$, $-\delta_3 - \xi_{35}$, $\delta_3 - \xi_{34}$, $-\delta_5 + \xi_{35}$, $\delta_5 - \xi_{32}$, $-\delta_4 + \xi_{34}$ and $\delta_4 + \xi_{32}$. Equating each of these to zero, we obtain $0 = \delta_3 = \delta_4 = \delta_5 = \xi_{23} = \xi_{24} = \xi_{25} = \xi_{32} = \xi_{34} = \xi_{35}$. With this knowledge we obtain the rest of ξ_{ij}'s equal to zero by looking at the coefficients of 14352, 14253, 14235, 15342, 15243 and 15234.

Now $\bar{A}_5$ is generated by $\{c + c_{ij}, d + d_k; i, j \in \{2, \ldots, 5\}, i \neq j$ and $k = 3, 4, 5\}$. If $c \in \bar{A}_5$ then

$$c = c_{25} = \sum\alpha_{ij}(c + c_{ij}) + \sum\beta_k(c + d_k)$$

implies that $-1 = 0$ which is not possible. This completes the proof of the theorem.

LEMMA 4.4. *Let* $\mathfrak{B} = (\mathfrak{A}\mathfrak{N}_c)^{(2c)}$ $(c \geqq 2)$. *Then* $\mathfrak{B}^{(1)} > \mathfrak{B}^{(2)} > \ldots > \mathfrak{B}^{(2c)} = \mathfrak{B}$.

Proof. By Lemma 3.4, $G(m, 2c - m + 1) \in \mathfrak{N}_{2c+1}^{(m)} \leqq \mathfrak{B}^{(m)}$ for $m \in \{2, \ldots, 2c - 1\}$. Thus to prove the lemma it suffices to show that $G(m, 2c - m + 1) \notin \mathfrak{B}^{(m+1)}$. As in the proof of Lemma 3.5 we show that a certain commutator in $R(m, 2c - m + 1)$ does not vanish.

Case 1 $(c \geqq m)$. In this case let

$$t = \langle \rho_{1,(c)}\rho_2; \rho_{2,(c-m+2)}\rho_3, \rho_4, \ldots, \rho_{m+1}\rangle$$

where $\rho_i = x_i + y_i$. Note that the coefficient of $y_2{}^cx_1y_3{}^{c-m+1}x_2\ldots x_{m+1}$ in the expansion of t as a linear combination of monomials in canonical form is the same as the corresponding coefficient in each of the following commutators:

$$\left\langle \left(\sum_{i=0}^{c} (-1)^i \binom{c}{i} \rho_2{}^i \rho_1 \rho_2{}^{c-i} \right); \left(\sum_{j=0}^{d} (-1)^j \binom{d}{j} \rho_3{}^j \rho_2 \rho_3{}^{d-j} \right), \rho_4, \ldots, \rho_{m+1} \right\rangle$$

where $d = c - m + 2$,

$$\langle (-1)^c y_2{}^c x_1, ((-1)^{d-1} d y_3{}^{d-1} x_2 x_3 + (-1)^d y_3{}^{d-1} x_3 x_2), x_4, \ldots, x_{m+1} \rangle,$$
$$\langle (-1)^c y_2{}^c x_1; (-1)^{d-1}(d+1) y_3{}^{d-1} x_2 x_3, x_4, \ldots, x_{m+1} \rangle.$$

In each of these the coefficient of $y_2{}^c x_1 y_3{}^{d-1} x_2 \ldots x_{m+1}$ is $(-1)^{c+d-1}(d+1) = (-1)^{m-1} \cdot (c - m + 3)$.

Case 2 ($c < m < 2c$). In this case let

$$t = \langle \rho_{1,(2c+1-m)} \rho_2, \rho_3, \ldots, \rho_d; \rho_1, \rho_{d+1}, \ldots, \rho_{m+1} \rangle$$

where $d = m + 1 - c$ and once again $\rho_i = x_i + y_i$. Observe that the coefficient of $y_2{}^{2c-m} x_1 \ldots x_d y_1 x_{d+1} \ldots x_{m+1}$ in the expansion of t as a linear combination of monomials in canonical form is the same as the corresponding coefficient in the expansion of each of the following:

$$\left\langle \left(\sum_{i=1}^{e} (-1)^i \binom{e}{i} \rho_2{}^i \rho \, \rho_2{}^{e-i} \right), \rho_3, \ldots, \rho_d; \rho_1, \rho_{d+1}, \ldots, \rho_{m+1} \right\rangle$$

where $e = 2c + 1 - m$,

$$\langle ((-1)^e y_2{}^{e-1} x_2 x_1 - (-1)^e d y_2{}^{e-1} x_1 x_2), x_3, \ldots, x_d; y_1, x_{d+1}, \ldots, x_{m+1} \rangle,$$
$$\langle (-1)^{e+1}(e+1) y_2{}^{e-1} x_1 x_2 \ldots x_d; y_1 x_{d+1} \ldots z_{m+1} \rangle.$$

In each case the coefficient is $(-1)^{e+1}(e+1) = (-1)^{2c-m} \cdot (2c + 2 - m)$.

By the Heineken-Macdonald result we have $\mathfrak{A}\mathfrak{N}_c = (\mathfrak{A}\mathfrak{N}_c)^{(2c+2)} = (\mathfrak{A}\mathfrak{N}_c)^{(2c+1)} = (\mathfrak{A}\mathfrak{N}_c)^{(2c)}$ $(c \geqq 3)$. This fact together with Lemma 4.4 yields the following result.

THEOREM 4.5. *If* $\mathfrak{B} = \mathfrak{A}\mathfrak{N}_c (c \geqq 3)$ *then*

$$\mathfrak{B}^{(1)} > \mathfrak{B}^{(2)} > \ldots > \mathfrak{B}^{(2c)} = \mathfrak{B}.$$

Essentially Theorem 4.5 has been proved by considering the chain (1.1) for $\mathfrak{B} = \mathfrak{A}\mathfrak{N}_c \wedge \mathfrak{N}_{2c+2}$ $(c \geqq 3)$. We now investigate the corresponding chain for the variety $\mathfrak{B} = \mathfrak{A}\mathfrak{N}_2 \wedge \mathfrak{N}_6$ and show that it is exceptional.

LEMMA 4.6.† *If* $\mathfrak{B} = \mathfrak{A}\mathfrak{N}_2 \wedge \mathfrak{N}_6$, *then* $\mathfrak{B}^{(5)} > \mathfrak{B}^{(6)} = \mathfrak{B}$.

Proof. The proof will follow a similar argument to that used in Theorem 4.3 to show $\mathfrak{C}^{(4)} > \mathfrak{C}^{(5)}$. Here we consider $R_6 = Z[x_1, \ldots, x_6]/I(x_{i(1)} \ldots x_{i(7)})$ where $I(x_{i(1)} \ldots x_{i(7)})$ is the ideal generated by monomials of length 7. Let $b = \langle x_1, x_2, x_3; x_4, x_5, x_6 \rangle$ and $b\sigma = \langle x_{1\sigma}, x_{2\sigma}, x_{3\sigma}; x_{4\sigma}, x_{5\sigma}, x_{6\sigma} \rangle$ where σ is a

†The proof is based on a suggestion of Dr. M. F. Newman (oral communication).

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permutation of $\{1, 2, \ldots, 6\}$. Let B_1 be the additive group generated by all expressions $b\sigma + (b\sigma)\tau$ where τ is any transposition. As in the proof of Theorem 4.3 we shall show that $b \notin B_1$. Since $b + b(13) \in B_1$, by Jacobi Identity it follows that $3b \in B_1$.

Let B be the additive group generated by all commutators b_δ. Then B is freely generated by the basic Lie elements

$$\langle x_i, x_j, x_k; x_l, x_m, x_n \rangle$$

where $i > j < k$, $l > m < n$, $i > l$ (c.f. [**13**]); and

$$\langle x_{i\sigma}, x_{j\sigma}, x_{k\sigma}; x_{l\sigma}, x_{m\sigma}, x_{n\sigma} \rangle \equiv |\sigma| \langle x_i, x_j, x_k; x_l, x_m, x_n \rangle \text{ modulo } B_1.$$

Let B_2 be the subgroup of B generated by $3b$ and all $b - |\sigma|b\sigma$ where $b\sigma$ is one of the free generators of B. Clearly $b \notin B_2$ and it is enough to show that $B_1 \leqq B_2$.

As in Theorem 4.3, B_1 is generated by all $b - |\sigma|b\sigma$ where σ is any permutation of $\{1, \ldots, 6\}$. If $b\sigma = -b\sigma'$ where $b\sigma'$ is a free generator of B, then $|\sigma|b\sigma = |\sigma'|b\sigma'$. If $b\sigma$ is not a free generator or its negative then it is easily seen that $|\sigma|b\sigma = -|\sigma'|b\sigma' - |\sigma''|b\sigma''$ where $b\sigma'$ and $b\sigma''$ are free generators or their negatives, so that $b - |\sigma|b\sigma = b + |\sigma'|b\sigma' + |\sigma''|b\sigma'' \equiv -3b + b + |\sigma'|b\sigma' + |\sigma''|b\sigma''$ modulo $B_2 \equiv (-b + |\sigma'|b\sigma') + (-b + |\sigma''|b\sigma'') \equiv 0$ modulo B_2.

LEMMA 4.7. *Let* $\mathfrak{B} = \mathfrak{A}\mathfrak{N}_2 \wedge \mathfrak{N}_6$. *Then* $\mathfrak{B}^{(4)} = \mathfrak{B}^{(5)}$.

Proof. Since $\mathfrak{B} \leqq \mathfrak{N}_6$, it suffices to show that in R_6 as defined in Lemma 4.6 the additive subgroup B_3 generated by all elements of the form

$$\langle \rho_1, \rho_2, \rho_2; \rho_4, \rho_5, \rho_5 \rangle \tag{4.1}$$

$$\langle \rho_1, \rho_2, \rho_3; \rho_1, \rho_4, \rho_2 \rangle, \text{ and} \tag{4.2}$$

$$\langle \rho_1, \rho_2, \rho_2; \rho_1, \rho_4, \rho_5 \rangle, \text{ where } \rho_i \in R_6 \tag{4.3}$$

contains the commutators

$$\langle x_1, x_2, x_3; x_4, x_5, x_5 \rangle = b_1, \tag{4.4}$$

$$\langle x_1, x_2, x_3; x_1, x_4, x_5 \rangle = b_2, \tag{4.5}$$

$$\langle x_1, x_2, x_3; x_4, x_5, x_1 \rangle = b_3, \text{ and} \tag{4.6}$$

$$\langle x_2, x_3, x_1; x_4, x_5, x_1 \rangle = b_4. \tag{4.7}$$

In (4.3) replacing ρ_2 by $x_2 + x_3$ and ρ_i by x_i for $i \neq 2$, give after a suitable change of variables,

$$\langle x_1, x_2, x_3; x_1, x_4, x_5 \rangle + \langle x_1, x_3, x_2; x_1, x_4, x_5 \rangle = 0 \text{ mod } B_3. \tag{4.8}$$

Similarly, in (4.2) replacing ρ_2 by $x_2 + x_4$ and ρ_2 by $x_3 + x_5$ give respectively,

$$\langle x_1, x_3, x_2; x_1, x_4, x_5 \rangle + \langle x_1, x_3, x_4; x_1, x_2, x_5 \rangle = 0 \tag{4.9}$$

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and

$$(4.10) \qquad \langle x_1, x_3, x_4; x_1, x_2, x_5\rangle + \langle x_1, x_5, x_4; x_1, x_2, x_3\rangle ;=0.$$

Adding (4.8) and (4.10), and using (4.9) gives

$$(4.11) \qquad \langle x_1, x_2, x_3; x_1, x_4, x_5\rangle + \langle x_1, x_2, x_4; x_5, x_1, x_4\rangle = 0$$

so that by the Jacobi identity $b_3 = 0 \bmod B_3$. By Jacobi identity b_4 can be written as a sum of two elements of the form b_3, hence $b_4 = 0$. In b_4, replacing x_1 by $x_1 + x_5$ and using $b_3 = 0$ shows that $b_1 = 0$. And, finally in b_1 replacing x_5 by $x_1 + x_5$ and using $b_3 = 0$ shows that $b_2 = 0$. This completes the proof of the lemma.

From Lemmas 4.4, 4.6 and 4.7 we deduce the following theorem.

THEOREM 4.8. *Let* $\mathfrak{B} = \mathfrak{A}\mathfrak{N}_2 \wedge \mathfrak{N}_6$. *Then*

$$\mathfrak{B}^{(1)} > \mathfrak{B}^{(2)} > \mathfrak{B}^{(3)} > \mathfrak{B}^{(4)} = \mathfrak{B}^{(5)} > \mathfrak{B}^{(6)} = \mathfrak{B}.$$

5. The variety $\mathfrak{N}_{n+k}^{(n)}$ (lemmas). In this section we list some preliminary results required for the investigation of some general properties of $\mathfrak{N}_{n+k}^{(n)}$-groups to be undertaken in the next section.

LEMMA 5.1 (Levi [**7**]). *The law* $[x, y, y] = 1$ *in a group implies the laws* (i) $[x, y, z]^3 = 1$ *and* (ii) $[x, y, z, u] = 1$.

LEMMA 5.2 (Heineken [**5**], Macdonald [**10**]). *The law* $[x_1, \ldots, x_n, x_1] = 1$ $(n \geqq 3)$ *in a group implies the law* $[x_1, \ldots. x_{n+1}] = 1$.

LEMMA 5.3 (Kappe [**6**]). *If z is a fixed element of a group G such that* $[z, x, x] = 1$ *for all* $x \in G$, *then* (i) $[z, x, y] = [z, y, x]^{-1}$ *and* (ii) $[z, x, y, u]^2 = 1$ *for all* $x, y, u \in G$.

LEMMA 5.4 *If z is a fixed element of a group G such that* $[z, x, x] = 1$ *for all* $x \in G$ *then* (i) $[z, u; x, y] = 1$, *and* (ii) $[z; x, y; u] = 1$ *for all* $x, y, u \in G$.

Proof. Since $1 = [z, zx, zx] = [z, x, zx] = [z, x, z]$, it follows that $\langle z^G\rangle$ is abelian. By Lemma 5.3 (i), $[z, x^{u^{-1}}, y^{u^{-1}}] = [z, y^{u^{-1}}, x^{u^{-1}}]^{-1}$ so that $[z^u, x, y] = [z^u, y, x]^{-1}$. Since $\langle z^G\rangle$ is abelian this gives $[z, u, x, y] = [z, u, y, x]^{-1} = [z, u, y, x]$ by Lemma 5.3 (ii). By a theorem of Levin [**9**], this gives $[z, u; x, y] = 1$. Similarly commuting both sides of 5.3 (i) by u gives $[z, x, y, u] = [z, y, x, u]^{-1}$ since $\langle z^G\rangle$ is abelian and as above, $[z, x, y, u] = [z, y, x, u]$ and again Levin's theorem gives $[z; x, y; u] = 1$. This completes the proof of the Lemma.

LEMMA 5.5. *In any group* G, $[x_1, x_2, x_3, x_4][x_2, x_4, x_1, x_3][x_3, x_4, x_1, x_2] \times [x_4, x_3, x_2, x_1][x_4, x_1, x_2, x_3] = 1$ *modulo* $\gamma_5(G)$.

Proof. Modulo $\gamma_5(G)$, $[x_1, x_2; x_3, x_4] = [x_1, x_2, x_3, x_4][x_1, x_2, x_4, x_3]^{-1} = [x_1, x_2, x_3, x_4][x_4, x_1, x_2, x_3][x_2, x_4, x_1, x_3]$; and $[x_3, x_4; x_1, x_2] = [x_3, x_4, x_1, x_2] \times [x_4, x_3, x_2, x_1]$. The Lemma follows on multiplying these two identities.

For the rest of this section $n \geqq 2$ and $k \geqq 1$, unless otherwise stated.

LEMMA 5.6. *Let* $G \in \mathfrak{N}_{n+k}^{(n)} \wedge \mathfrak{N}_{n+k+1}^{(n+1)}$. *Then G satisfies the law*

$$[x_{i(1)}, \ldots, x_{i(\lambda-1)}, x_{i(\lambda)}, x_{i(\lambda+1)}, \ldots, x_{i(\mu-1)}, x_{i(\mu)}, x_{i(\mu+1)}, \ldots, x_{i(n+k+1)}]$$
$$[x_{i(1)}, \ldots, x_{i(\lambda-1)}, x_{i(\mu)}, x_{i(\lambda+1)}, \ldots, x_{i(\mu-1)}, x_{i(\lambda)}, x_{i(\mu+1)}, \ldots, x_{i(n+k+1)}] = 1,$$

where $|\{i(1), \ldots, i(\lambda - 1), i(\lambda + 1), \ldots, i(\mu - 1), i(\mu + 1), \ldots, i(n + k + 1)\}| \leqq n - 1.$

Proof. Since $G \in \mathfrak{N}_{n+k}^{(n)}$, it satisfies the law

$$[x_{i(1)}, \ldots, x_{i(\lambda-1)}, x_{i(\lambda)}x_{i(\mu)}, x_{i(\lambda+1)}, \ldots, x_{i(\mu-1)}, x_{i(\lambda)}x_{i(\mu)}, x_{i(\mu+1)}, \ldots, x_{i(n+k+1)}] = 1,$$

which on expansion (and using $G \in \mathfrak{N}_{n+k+1}^{n+1}$) gives the desired result.

LEMMA 5.7. *If* $G \in \mathfrak{N}_{n+k}^{(n)} \wedge \mathfrak{N}_{n+k+1}^{n+1}$, *then G satisfies the law*

$$[x_{i(1)}, \ldots, x_{i(n+k+1)}] = 1$$

where

$$0 < |\{i(4), \ldots, i(n + k + 1)\}| \leqq n - 2.$$

Proof. By Lemma 5.5, modulo $\gamma_{n+k+2}(G)$ we have

$$\begin{aligned} 1 = {} & [x_{i(1)}, x_{i(2)}, x_{i(3)}, x_{i(4)}, x_{i(5)}, \ldots, x_{i(n+k+1)}] \\ & [x_{i(2)}, x_{i(4)}, x_{i(1)}, x_{i(3)}, x_{i(5)}, \ldots, x_{i(n+k+1)}] \\ & [x_{i(3)}, x_{i(4)}, x_{i(1)}, x_{i(2)}, x_{i(5)}, \ldots, x_{i(n+k+1)}] \\ & [x_{i(4)}, x_{i(3)}, x_{i(2)}, x_{i(1)}, x_{i(5)}, \ldots, x_{i(n+k+1)}] \\ & [x_{i(4)}, x_{i(1)}, x_{i(2)}, x_{i(3)}, x_{i(5)}, \ldots, x_{i(n+k+1)}]. \end{aligned}$$

Since $|\{i(1), i(4), i(5), \ldots, i(n + k + 1)\}| \leqq n - 1$, by Lemma 5.6 the product of second and third commutator is trivial. Similarly the product of fourth and fifth commutator is trivial and we conclude that

$$[x_{i(1)}, x_{i(2)}, \ldots, x_{i(n+k+1)}] = 1.$$

LEMMA 5.8. *Let* $G \in \mathfrak{N}_{n+k}^{(n)}$ $(n \geqq k + 1)$ *and let u be a commutator of weight exceeding k. Then*

$$\left[\prod_i u_i, x_1, \ldots, x_m\right] = \prod_i [u_i, x_1, \ldots, x_m] \text{ for } m \geqslant n,$$

where u_i *is a commutator having u as one of its entries.*

Proof. Any commutator of weight $n + k + 1$ in which u occurs twice is a commutator in at most $n + k + 1 - (k + 1) = n$ variables and so is trivial.

6. $\mathfrak{N}_{n+1}^{(n)}$**-groups.** If $G = C_2 \operatorname{wr} (C_2 \times C_2 \times \ldots)$ then $G \in \mathfrak{N}_{n+1}^{(n)}$ for each $n \geqq 2$ (c.f. [**15**, 34.54]), so that $\mathfrak{N}_{n+1}^{(n)}$-groups are not in general nilpotent. In [**12**], Macdonald and Neumann have shown that if $G \in \mathfrak{N}_{n+1}^{(n)}$ $(n \geqq 3)$ then G is locally nilpotent and $\gamma_{n+3}(G)$ is a 2-group. In this section we investigate in detail the commutator structure of $\mathfrak{N}_{n+1}^{(n)}$-groups, starting with the following Theorem.

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THEOREM 6.1. *Let* $G = F_\infty(\mathfrak{N}_{n+1}^{(n)})$ $(n \geqq 3)$. *Then*
(i) $[\gamma_{m_1}(G), \gamma_{m_2}(G)] = \{1\}$ $(m_1, m_2 \geqq 2$ *and* $m_1 + m_2 = n + 3)$;
(ii) $[\gamma_n(G), \gamma_2(G)] \neq \{1\}$;
(iii) $[\gamma_{m_1}(G), \gamma_{m_2}(G)] = \{1\}$ $(m_1, m_2 \geqq 3$ *and* $m_1 + m_2 = n + 2)$.

Proof. For the proof of (i) it is enough to show that $[\gamma_m(G), \gamma_2(G)] \leqq \zeta_{n-m+1}(G)$ for $m = 3, \ldots, n + 1$. For the result then follows by using P. Hall's three subgroup lemma.

Since G satisfies the law

$$[x_1, x_2, x_2, x_4, \ldots, x_m, x_{m+1}, x_{m+1}, x_{m+4}, \ldots, x_{n+3}] = 1,$$

$G_1 = G/\zeta_{n-m}(G)$ satisfies the law

$$[x_1, x_2, x_2, x_4, \ldots, x_m, x_{m+1}, x_{m+1}] = 1,$$

which in turn implies the law

$$[x_1, x_2, x_2, x_4, \ldots, x_m; x_{m+1}, x_{m+2}; x_{m+3}] = 1 \quad \text{(by Lemma 5.4).}$$

Thus $G_2 = G_1/\zeta(G_1)$ satisfies the law

$$[x_1, x_2, x_2, x_4, \ldots, x_m; x_{m+1}, x_{m+2}] = 1,$$

$G_3 = G_2/\zeta(\gamma_2(G_2))$ satisfies the law $[x_1, x_2, x_2, x_4, \ldots, x_m] = 1$, and $G_4 = G_3/\zeta_{m-3}(G_3)$ satisfies $[x_1, x_2, x_2] = 1$ which implies the law $[x_1, x_2, x_3]^3 = 1$ by Lemma 5.1. Thus by Lemma 5.8 we conclude that $[\gamma_m(G), \gamma_2(G),_{(n-m+1)}G]$ is a 3-group which is also a 2-group by the Macdonald-Neumann result.

For the proof of (ii) we consider the group $G(n, 1)$ of Section 3, which is a homomorphic image of G and note that in $R(n, 1)$,

$$t = \langle y_1, x_1, \ldots, x_{n-1}; x_n, x_{n+1}\rangle \neq 0$$

since the coefficient of $y_1x_1 \ldots x_{n+1}$ is 2 in the expansion of t. Indeed, we observe that if n is even then $\langle y_1, x_1; x_2, x_3; \ldots; x_n, x_{n+1}\rangle \neq 0$ and if n is odd then $\langle y_1, x_1; x_2, x_3; \ldots; x_{n-1}, x_n; x_n\rangle \neq 0$.

For the proof of (iii) we anticipate the result of Theorem 6.2 (proved independently of (iii)) which states that $\mathfrak{N}_{n+1}^{(n)} < \mathfrak{N}_{n+2}^{(n+1)}$ $(n \geqq 3)$. Thus by Lemma 5.7, G satisfies the law $[x_1, x_2, x_3, y_1, \ldots, y_{n-1}] = 1$ where

$$|\{y_1, \ldots, y_{n-1}\}| \leqq n - 2.$$

For $m \geqq 3$, we have

$$\begin{aligned}
[[x_1, x_2, \ldots, x_m], [y_1, y_2, y_3], z_1, \ldots, z_{n-m-1}] = \\
& [x_1, \ldots, x_m, y_1, y_2, y_3, z_1, \ldots, z_{n-m-1}] \\
& [x_1, x_2, \ldots, x_m, y_3, y_2, y_1, z_1, \ldots, z_{n-m-1}] \\
& [x_1, x_2, \ldots, x_m, y_2, y_1, y_3, z_1, \ldots, z_{n-m-1}]^{-1} \\
& [x_1, x_2, \ldots, x_m, y_3, y_1, y_2, z_1, \ldots, z_{n-m-1}]^{-1} \\
& \quad \text{(by Jacobi identity)} \\
& = 1 \quad \text{(by Lemma 5.6).}
\end{aligned}$$

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Using the above result, we can strengthen statements (i) and (iii) as follows.

THEOREM 6.1*. *Let* $G = F_\infty(\mathfrak{N}_{n+1}^{(n)})$ $n \geqq 3$. *Then*
(i) $\gamma_{n+3}(G) \cap \gamma_2(\gamma_2(G)) = \{1\}$;
(ii) $\gamma_{n+2}(G) \cap \gamma_2(\gamma_3(G)) = \{1\}$.

Proof. (i). Let $K = \gamma_{n+3}(G)$, $L = \gamma_2(\gamma_2(G))$. If $K \cap L \neq \{1\}$, then let

$$1 \neq w = \prod_{i=1}^{r} u_i^{\epsilon_i} = \prod_{j=1}^{s} v_j^{\delta_j},$$

where $\epsilon_i, \delta_j = \pm 1$, each u_i is a commutator of weight $\geqq n + 3$ and each v_j is a commutator lying in G''. If w involves m variables, then since G is a relatively free group, each of u_i's and v_j's is a commutator involving all of these m variables. By Theorem 6.2 (proved independently of Theorem 6.1*) $G \in \mathfrak{N}_{n+1}^{(n)} < \mathfrak{N}_{n+2}^{(n+1)} < \mathfrak{N}_{n+3}^{(n+2)}$, so that $m \geqq n + 2$ and every u_i is of weight $n + 3$. By Theorem 6.1 (i), each v_j is of weight $\leqq n + 2$ so that $m = n + 2$. Also by Theorem 6.1 (i), $[y_1, \ldots, y_{n+3}] = [y_1, y_2, y_{3\sigma}, \ldots, y_{(n+3)\sigma}]$ for any permutation σ of $\{3, \ldots, n + 3\}$. In particular, every u_i is a left-normed commutator of the form $[x_{i_1}, x_{i_2}, \ldots, x_{i_{n+3}}]$ with $|\{i_1, \ldots, i_{n+3}\}| = n + 2$. By Lemma 5.7 no two of $x_{i_4}, \ldots, x_{i_{n+3}}$ are the same. This together with conditions implied by Lemma 5.6 enables us to write w as follows:

$$w = \prod_{i=1}^{n} w_i^{\alpha_i}$$

where $w_1 = [x_1, x_2, \ldots, x_{n+2}, x_1]$ and for $i > 1$,

$$w_i = [x_i, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+2}, x_i]$$

and $\alpha_i \in Z$. Without loss of generality assume that $\alpha_1 \neq 0$. Replace x_1 by x_1x_2 to obtain w' from w. By looking at w as $\prod_{j=1}^{s} v_j^{\delta_j}$ and making use of Theorem 6.1 (i), we get $w = w'$. By looking at $w = \prod_{i=1}^{n} w_i^{\alpha_i}$, we get $w^{-1}w' = [x_1, x_2, \ldots, x_{n+2}, x_1]^{\alpha_1} = 1$ in G. Thus $[x_1, x_2, \ldots, x_{n+2}, x_1]^{\alpha_1}$ is a law in G. Interchange x_1 and x_2 to obtain $[x_1, x_2, \ldots, x_{n+2}, x_1]^{-\alpha_1} = 1$ in G. Thus $w_i^{\alpha_i} = 1$ for all i and $w = 1$ in G.

(ii) Let $H = \gamma_{n+2}(G)$, $K = \gamma_2(\gamma_3(G))$. If $H \cap K \neq \{1\}$, then let $1 \neq w \in H \cap K$.

$$w = \prod_{i=1}^{r} u_i^{\epsilon_i} = \prod_{i=1}^{s} v_j^{\delta_j}$$

where $\epsilon_i, \delta_j = \pm 1$, each u_i is a commutator of weight $\geqq n + 2$ and each v_j is a commutator in $\gamma_2(\gamma_3(G))$. As in (i), each u_i and v_j involves m variables where m is number of variables in the expression of w. Since $G \in \mathfrak{N}_{n+1}^{(n)}$, $m \geqq n + 1$. If $m = n + 2$, then the right hand side is trivial by Theorem 6.1 (iii). Thus $m = n + 1$. Let $w = w_1 \ldots w_{n+1}$ where w_i is the product of those $u_i^{\epsilon_i}$ in which x_i is repeated. By interchanging the variables, if necessary,

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assume that $w_1 \neq 1$ in G. By making use of Lemma 5.7 we can assume that

$$w_1 = [x_1, x_2, x_1, x_3, \ldots, x_{n+1}]^\alpha [x_1, x_2, x_3, x_1, x_4, \ldots, x_{n+1}]^\beta$$

where α, β are not both zero. Let $z(x_1, \ldots, x_{n+1}) = w_1 \ldots w_{n+1} v_s^{-\delta_s} \ldots v_1^{-\delta_1}$. Then $z(x_1, \ldots, x_{n+1})$ is a law in G. Now

$$z(x_1, \ldots, x_{n+1}) z^{-1}(x_1 x_2, x_2, , \ldots, x_{n+1})$$
$$= [x_1, x_2, x_2, x_3, \ldots, x_{n+1}]^{-\alpha} [x_1, x_2, x_3, x_2, x_4, \ldots, x_{n+1}]^{-\beta}$$
$$= [x_2, x_1, x_2, x_3, \ldots, x_{n+1}]^\alpha [x_2, x_1, x_3, x_2, x_4, \ldots, x_{n+1}]^\beta$$

is a law in G. Interchange x_1, x_2 to get $w_1 = 1$ in G. This completes the proof.

THEOREM 6.2 (c.f. [**15**, 34.52]). $\mathfrak{N}_{n+1}^{(n)} < \mathfrak{N}_{n+2}^{(n+1)}$ $(n \geqq 3)$.

Proof. Let $c = [x_{i(1)}, \ldots, x_{i(n+3)}]$ be any left-normed commutator in $G \in \mathfrak{N}_{n+1}^{(n)}$ with $|\{i(1), \ldots, i(n+3)\}| = n + 1$. By Theorem 6.1 (i), c is unchanged if we interchange the positions of any two variables appearing after the second entry. Thus we may write

$$c = [x_{i(1)}, x_{i(2)}, x_{j(3)}, \ldots, x_{j(n+3)}],$$

where $j(n + 3) \notin \{i(1), i(2), j(3), \ldots, j(n + 2)\}$. But $G \in \mathfrak{N}_{n+1}^{(n)}$ implies that $[x_{1(1)}, x_{i(2)}, x_{j(3)}, \ldots, x_{j(n+2)}] = 1$ and hence $c = 1$. To see that the inclusion is proper consider $F_n(\mathfrak{N}_{n+2})$ which is not in $\mathfrak{N}_{n+1}^{(n)}$.

Remark 3. In [**12**], Macdonald and Neumann have constructed a $\mathfrak{N}_3^{(2)}$-group which is not a $\mathfrak{N}_4^{(3)}$-group. Thus Theorem 6.2 cannot be improved to include $n = 2$.

7. The variety $\mathfrak{N}_{n+k}^{(n)}$ (continued).

We first prove an analogue of Theorem 6.2.

THEOREM 7.1. $\mathfrak{N}_{n+k}^{(n)} < \mathfrak{N}_{n+k+1}^{(n+1)}$ *for* $k \geqq 1$ *and* $n \geqq 3k + 2$.

Proof. Let $G \in \mathfrak{N}_{n+k}^{(n)}$ and let $c(x) = [x_{i(n+k+2)}, \ldots, x_{i(2)}, x_{i(1)}]$ be a commutator in $n + 1$ variables. Since G is locally nilpotent (see [**4**]), it is sufficient to show that $c(x) = 1$ modulo $\gamma_{n+k+3}(G)$. Term $x_{i(j)}$ free if it occurs precisely once in $c(x)$. If $x_{i(1)}$ is free then since $G \in \mathfrak{N}_{n+k}^{n}$, $c(x) = 1$. We may therefore assume that $x_{i(1)}$ is not free. Among the entries of $c(x)$ we note that there are at least $n - k$ free variables and since $n - k \geqq (n + k + 2)/2$, there is a least integer j such that $x_{i(j)}$ and $x_{i(j+1)}$ are both free in either $c(x)$ or in $c(x)^{-1}$. Moreover $j + 1 < n + k + 2$ for otherwise we could consider

$$[x_{i(n+k+2)}, x_{i(n+k+1)}]$$

as one variable. Since $G \in \mathfrak{N}_{n+k}^{(n)}$, we have

$$[u, x_{i(j+1)} x_{i(j)}, x_{i(j-1)}, \ldots, x_{i(1)}] = 1$$

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where $u = [x_{i(n+k+2)}, \ldots, x_{i(j+2)}]$. This, on expansion, shows that $c(x) = 1$ modulo $\gamma_{n+k+3}(G)$.

THEOREM 7.2. *Let*

$$G \in \bigwedge_{j=0}^{k} \mathfrak{N}_{n+k+j}^{(n+j)} \ (n \geqslant 2k+3).$$

Then $\gamma_{n-2k+1}(G) \leqq \Phi_k(G)$ *where* $\Phi_1(G) = \zeta_1(\gamma_3(G))$ and $\Phi_{s+1}(G)/\Phi_s(G) = \Phi_1(G/\Phi_s(G))$.

Proof. Since $G \in \mathfrak{N}_{n+k}^{(n)} \wedge \mathfrak{N}_{n+k+1}^{n+1}$, by Lemma 5.7 G satisfies the law $[x_1, x_2, x_3, x_{i(1)}, \ldots, x_{i(n+k-2)}] = 1$ where $|\{i(1), \ldots, i(n+k-2)\}| \leqq n-2$ and in particular the law $[[x_{i(1)}, \ldots, x_{i(n+k-2)}], [x_1, x_2, x_3]] = 1$ (see Macdonald [**10**, Lemma, p. 272]). More generally since $G \in \mathfrak{N}_{n+k+t}^{n+t} \wedge \mathfrak{N}_{n+k+t+1}^{n+t+1}$, G satisfies the law $[[x_{i(1)}, \ldots, x_{i(n+k+t-2)}], [x_1, x_2, x_3]] = 1$ where

$$|\{i(1), \ldots, i(n+k+t-2)\}| \leqq n+t-2.$$

From these identities it follows that $G/\Phi_1(G) \in \wedge_{j=0}^{k-1}\mathfrak{N}_{n+k-3+j}^{(n-2+j)}$ and inductively

$$G/\Phi_s(G) \in \bigwedge_{j=0}^{k-s} \mathfrak{N}_{n+k-3s+j}^{(n-2s+j)}.$$

Hence taking $s = k$ we obtain $G/\Phi_k(G) \in \mathfrak{N}_{n+k-3k}^{(n-2k)} = \mathfrak{N}_{n-2k}$ (since $n - 2k \geqq 3$). Thus $\gamma_{n-2k+1}(G) \leqq \Phi_k(G)$.

By Theorem 7.1 if $n \geqq 3k+2$, then $\wedge_{j=0}^{k}\mathfrak{N}_{n+k+j}^{(n+j)} = \mathfrak{N}_{n+k}^{(n)}$ and since $3k+2 \geqq 2k+3$, we obtain the following Theorem as a corollary to Theorem 7.2.

THEOREM 7.3. *If* $G \in \mathfrak{N}_{n+k}^{(n)}$ $(n \geqq 3k+2)$ *then* $[\gamma_{n-2k+1}(G),_{(k)}\gamma_3(G)] = \{1\}$.

The following result shows that Theorem 7.3 is best possible in the following sense.

THEOREM 7.4. *Let* $G = F_\infty(\mathfrak{N}_{n+k}^{(n)})$ $n \geqq 2k-3$. *Then* $[\gamma_{n-2k+4}(G),_{(k-1)}\gamma_3(G)] \neq \{1\}$.

Proof. Consider $G(n, k)$ which is a homomorphic image of G. In $R(n, k)$,

$$\langle y_1, x_1, \ldots, x_{n-2k+3}; y_2, x_{n-2k+4}, x_{n-2k+5}; y_3 \ldots; \ldots; y_k, x_n, x_{n+1}\rangle \neq 0$$

since the coefficient of $y_1x_2 \ldots x_{n-2k+3}y_2x_{n-2k+4}x_{n-2k+5}y_3 \ldots y_kx_nx_{n+1}$ is 1 in the expansion of the commutator as a linear combination of monomials in canonical form.

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